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NEW CLASSES OF NON-NORMALIZED MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. Making use of certain differential operators, this paper introduces two new classes:

$$\mathcal{M}^{\kappa}_{m,n,\delta}(q;p) \quad \text{and} \quad \mathcal{S}K^{\kappa}_{m,n,\delta}(q;p)$$

which consist of non-normalized meromorphically multivalent functions with complex coefficients in the punctured unit disk. A theorem is established concerning an inclusion property for the above classes, and in the concluding section, several consequences of the main result are pointed out.

1. Introduction and definitions

Recently, Chen et al. ([1], [2]) introduced and studied certain subclasses of the class $\mathcal{M}^{\kappa}_{p,q}$ consisting of (non-normalized) meromorphically multivalent functions $f(z)$ of the form :

$$f(z) = \frac{\kappa}{z^q} + \sum_{k=p}^{\infty} a_k z^k \quad (\kappa \neq 0; q, p \in \mathcal{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic in the punctured unit disk $\mathcal{D} = \mathcal{U} - \{0\}$, where $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$, and which (in the special cases when $q = 1$) have a simple pole at the origin ($z = 0$) with residue κ. See also [6], [9] and [14].

A function $f(z)$ is said to be in the subclass $\mathcal{S}^{\kappa}_{p}(q)$ of meromorphically multivalent starlike functions (or, meromorphically starlike when $q = 1$) if and only if

$$\Re e\left(-\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}). \tag{1.2}$$

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Furthermore, $f(z)$ is said to be in the subclass $\mathcal{K}_p^\kappa(q)$ of meromorphically multivalent convex functions (or, meromorphically convex when $q = 1$) if and only if

$$\Re e\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.3}$$

In addition, also let

$$\mathcal{M}_p^\kappa := \mathcal{M}_{p,1}^\kappa, \quad \mathcal{S}_p^\kappa := \mathcal{S}_p^\kappa(1) \text{ and } \mathcal{K}_p^\kappa := \mathcal{K}_p^\kappa(1).$$

The classes $\mathcal{S}_p^\kappa$ and $\mathcal{K}_p^\kappa$ consist of meromorphically starlike functions and meromorphically convex functions in $\mathcal{U}$, respectively.

It is easily seen that

$$f(z) \in \mathcal{K}_p^\kappa(q) \Longleftrightarrow -\frac{zf'(z)}{q} \in \mathcal{S}_p^\kappa(q)$$

and

$$f(z) \in \mathcal{K}_p^\kappa \Longleftrightarrow -zf'(z) \in \mathcal{S}_p^\kappa.$$

For more details of the above definitions, one may refer to [4], [7], and [14].

We define two differential operators of a function $f(z) \in \mathcal{M}_{p,q}^\kappa$ as follows:

$$D_z^n\{f(z)\} = \frac{(q+n-1)!}{(q-1)!}\cdot\frac{\kappa}{z^{q+n}}\cdot(-1)^n + \sum_{k=p}^{\infty}\frac{k!}{(k-n)!}\,a_k z^{k-n}, \tag{1.4}$$

and

$$\begin{aligned} D_z^m\{z^{2q+n}D_z^n\{f(z)\}\} = {} & \frac{q(q+n-1)!}{(q-m)!}\cdot\frac{\kappa}{z^{m-q}}\cdot(-1)^n \\ & + \sum_{k=p}^{\infty}\frac{k!(k+2q)!}{(k-n)!(k+2q-m)!}\,a_k z^{k+2q-m}, \end{aligned} \tag{1.5}$$

$$(\kappa \neq 0; p \geq n, q \geq m; p, q \in \mathcal{N}; m, n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}).$$

It is clear when $m = n = 0$, then

$$D_z^0\{z^{2q}D_z^0\{f(z)\}\} = D_z^0\{z^{2q}\{f(z)\}\} = z^{2q}f(z), \tag{1.6}$$

and when $m = n = 1$, then

$$D_z^1\{z^{2q+1}D_z^1\{f(z)\}\} = D_z^1\{z^{2q+1}\{f'(z)\}\} = [z^{2q+1}f'(z)]'. \tag{1.7}$$

We note that the differential operator (1.4) has been used in several recent works (see, [3], [5], [8] and [13]).

In the present paper, by making use of the differential operators defined in (1.4) and (1.5), two new subclasses involving functions of the form (1.1) are introduced, and then a theorem exhibiting an inclusion relation between the classes $\mathcal{M}^{\kappa}_{m,n,\delta}\,(q;p)$ and $\mathcal{S}K^{\kappa}_{m,n,\delta}\,(q;p)$ is established. Several consequences of the main result are treated in the concluding section.

Using the hypotheses of differential operators, by making use of operators in (1.4) and (1.5), we define two new classes

$$\mathcal{M}^{\kappa}_{m,n,\delta}\,(q;p) \quad \text{and} \quad \mathcal{S}K^{\kappa}_{m,n,\delta}\,(q;p)$$

consisting of functions $f(z)$ in the class $\mathcal{M}^{\kappa}_{p,q}$, which satisfy the following inequalities, respectively:

$$1+\Re e\left\{\Omega_{1+m}(z)-\Omega_m(z)\right\}\begin{cases} < \frac{1}{2\delta} & \text{when } \delta>0 \\ > \frac{1}{2\delta} & \text{when } \delta<0 \end{cases} \tag{1.8}$$

and

$$\Re e\left\{\ [\Omega_m(z)]^{\delta}\ \right\}>0 \quad (z\in\mathcal{D};\delta\neq 0;m\in\mathcal{N}_0), \tag{1.9}$$

where

$$\mathcal{F}(z)=z^{2q+n}D^n_z\{f(z)\} \quad (q\in\mathcal{N};n\in\mathcal{N}_0;f(z)\in\mathcal{M}^{\kappa}_{p,q}) \tag{1.10}$$

and

$$\Omega_m(z)=\frac{zD^{1+m}_z\{\mathcal{F}(z)\}}{D^m_z\{\mathcal{F}(z)\}} \quad (z\in\mathcal{U};m\in\mathcal{N}_0). \tag{1.11}$$

We assume here and throughout this paper, that the values of the exponential form occurring in (1.9) are taken to be their principal values, and $\mathcal{R}$ will denote the set of real numbers.

2. The main result

In proving our result (contained in Theorem 1 below), we shall need the following result due to Jack [12].

Lemma 1. *Let $w(z)$ be an analytic function in the unit disk $\mathcal{U}$, such that $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathcal{U}$, then $z_0w'(z_0)=cw(z_0)$, where c is real and $c\geq 1$.*

We begin by proving

Theorem 1. *Let $\delta\neq 0$; $\kappa\neq 0$; $q\geq m\geq n$; $p,q\in\mathcal{N}$; $m,n\in\mathcal{N}_0$, and also let $f(z)\in\mathcal{M}^{\kappa}_{p,q}$. Then*

$$f(z)\in\mathcal{M}^{\kappa}_{m,n,\delta}\,(q;p)\Rightarrow f(z)\in\mathcal{S}K^{\kappa}_{m,n,\delta}\,(q;p). \tag{2.1}$$

Proof. To prove (2.1), we show that if a function $f(z)$ is in the class $\mathcal{M}^{\kappa}_{p,q}$, then

$$f(z)\in\mathcal{M}^{\kappa}_{m,n,\delta}\,(q;p), \tag{2.2}$$

implies that

$$f(z)\in\mathcal{S}K^{\kappa}_{m,n,\delta}\,(q;p). \tag{2.3}$$

Let us define a function $w(z)$ by

$$[\Omega_m(z)]^\delta = (q-m)^\delta[1+w(z)], \tag{2.4}$$

where the function $\Omega_m(z)$ is defined by (1.11). Obviously, the function $w(z)$ is either analytic or multivalently meromorphic in $\mathcal{U}$, with $w(0)=0$. Differentiating (2.4), we obtain

$$1+[\Omega_{1+m}(z)-\Omega_m(z)] = \frac{1}{\delta}\cdot\frac{zw'(z)}{1+w(z)}, \tag{2.5}$$

If we now suppose that $\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$ $(z_0, z\in\mathcal{U})$, and apply Lemma 1, we also obtain that $z_0w'(z_0) = cw(z_0)$ $(c\geq 1)$. On setting $z=z_0$, and then putting $w(z_0)=e^{i\theta}$ $(e^{i\theta}\neq -1)$ in (2.5), we finally get

$$1+\Re e\,\{\Omega_{1+m}(z_0)-\Omega_m(z_0)\}$$
$$= \Re e\left(\frac{c}{\delta}\cdot\frac{e^{i\theta}}{1+e^{i\theta}}\right)\begin{cases}\geq \frac{1}{2\delta} & \text{when } \delta>0\\ \leq \frac{1}{2\delta} & \text{when } \delta<0\end{cases}. \tag{2.6}$$

But the inequalities in (2.6) contradict our assumptions stated in (1.8). Hence, we must have $|w(z)|<1$ for all $z\in\mathcal{U}$. Evidently, it follows from (2.4) that

$$\left|[\Omega_m(z)]^\delta-(q-m)^\delta\right| = (q-m)^\delta|w(z)| < (q-m)^\delta,$$

which implies that

$$\Re e\left\{[\Omega_m(z)\}^\delta\right\} > 0 \quad (\delta\neq 0),$$

and so $f(z)\in\mathcal{SK}^\kappa_{m,n,\delta}\,(q;p)$. This completes the proof. □

3. Some consequences of the main result

In view of the definitions of the classes $\mathcal{M}^\kappa_{m,n,\delta}\,(q;p)$ and $\mathcal{SK}^\kappa_{m,n,\delta}\,(q;p)$, we exhibit below several interesting (and useful) subclasses of meromorphically multivalent or meromorphically univalent functions. By specializing the parameters, we obtain the following subclasses:
$\mathcal{V}^\kappa_{\delta_1}(q;p)\equiv\mathcal{M}^\kappa_{0,0,\delta}(q;p)$ $(\delta_1=\delta\neq 0)$, $\mathcal{V}^\kappa_{\delta_2}(q;p)\equiv\mathcal{M}^\kappa_{0,1,\delta}(q;p)$ $(\delta_2=\delta\neq 0)$, $\mathcal{V}^\kappa_{\delta_3}(p)\equiv\mathcal{M}^\kappa_{0,0,\delta}(1;p)$ $(\delta_3=\delta\neq 0)$, $\mathcal{V}^\kappa_{\delta_4}(p)\equiv\mathcal{M}^\kappa_{0,1,\delta}(1;p)$ $(\delta_4=\delta\neq 0)$, $\mathcal{V}^\kappa_1(q;p)\equiv\mathcal{M}^\kappa_{0,0,1}(q;p)$, $\mathcal{V}^\kappa_2(q;p)\equiv\mathcal{M}^\kappa_{0,1,1}(q;p)$, $\mathcal{V}^\kappa_1(p)\equiv\mathcal{M}^\kappa_{0,0,1}(1;p)$, $\mathcal{V}^\kappa_2(p)\equiv\mathcal{M}^\kappa_{0,1,1}(1;p)$, $\mathcal{W}^\kappa_{\delta_1}(q;p)\equiv\mathcal{SK}^\kappa_{0,0,\delta}(q;p)$ $(\delta_1=\delta\neq 0)$, $\mathcal{W}^\kappa_{\delta_2}(q;p)\equiv\mathcal{SK}^\kappa_{0,1,\delta}(q;p)$ $(\delta_2=\delta\neq 0)$, $\mathcal{W}^\kappa_{\delta_3}(p)\equiv\mathcal{SK}^\kappa_{0,0,\delta}(1;p)$ $(\delta_3=\delta\neq 0)$, $\mathcal{W}^\kappa_{\delta_4}(p)\equiv\mathcal{SK}^\kappa_{0,1,\delta}(1;p)$ $(\delta_4=\delta\neq 0)$, $\mathcal{W}^\kappa_1(q;p)\equiv\mathcal{SK}^\kappa_{0,0,1}(q;p)$, $\mathcal{W}^\kappa_2(q;p)\equiv\mathcal{SK}^\kappa_{0,1,1}(q;p)$, $\mathcal{W}^\kappa_1(p)\equiv\mathcal{SK}^\kappa_{0,0,1}(1;p)$ $\mathcal{W}^\kappa_2(p)\equiv\mathcal{SK}^\kappa_{0,1,1}\,(1;p)$.

Also, we observe that

$\mathcal{S}^\kappa_p(q)\equiv\mathcal{W}^\kappa_1(q;p)$, $\mathcal{S}^\kappa_p\equiv\mathcal{W}^\kappa_1\,(p)$, $\mathcal{K}^\kappa_p(q)\equiv\mathcal{W}^\kappa_2\,(q;p)$, and $\mathcal{K}^\kappa_p\equiv\mathcal{W}^\kappa_2\,(p)$.

In view of the above relationships, we conclude this paper by mentioning some of the deducible cases which stem from our main result (Theorem 1). These consequences are contained in the following:

Corollary 1. *Let* $\delta_1 \neq 0$, $\kappa \neq 0$, $p, q \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p,q}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{\delta_1}(q;p)$, *then* $f(z) \in \mathcal{W}^{\kappa}_{\delta_1}(q;p)$.

Corollary 2. *Let* $\delta_2 \neq 0$, $\kappa \neq 0$, $p, q \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p,q}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{\delta_2}(q;p)$, *then* $f(z) \in \mathcal{W}^{\kappa}_{\delta_2}(q;p)$.

Corollary 3. *Let* $\delta_3 \neq 0$, $\kappa \neq 0$, $p \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{\delta_3}(p)$, *then* $f(z) \in \mathcal{W}^{\kappa}_{\delta_3}(p)$.

Corollary 4. *Let* $\delta_4 \neq 0$, $\kappa \neq 0$, $p \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{q}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{\delta_4}(p)$, *then* $f(z) \in \mathcal{W}^{\kappa}_{\delta_4}(p)$.

Corollary 5. *Let* $\kappa \neq 0$, $p, q \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p,q}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{1}(q;p)$, *then* $f(z) \in \mathcal{S}^{\kappa}_{p}(q)$, *i.e.,* $f(z)$ *is meromorphically multivalent in* $\mathcal{D}$.

Corollary 6. *Let* $\kappa \neq 0$, $p, q \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p,q}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{2}(q;p)$, *then* $f(z) \in \mathcal{K}^{\kappa}_{p}(q)$, *i.e.,* $f(z)$ *is meromorphically multivalent convex in* $\mathcal{D}$.

Corollary 7. *Let* $\kappa \neq 0$, $p \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{1}(q;p)$, *then* $f(z) \in \mathcal{S}^{\kappa}_{p}$, *i.e.,* $f(z)$ *is meromorphically starlike in* $\mathcal{D}$.

Corollary 8. *Let* $\kappa \neq 0$, $p \in \mathcal{N}$, *and also let* $f(z) \in \mathcal{M}^{\kappa}_{p}$. *If* $f(z) \in \mathcal{V}^{\kappa}_{2}(p)$, *then* $f(z) \in \mathcal{K}^{\kappa}_{p}$, *i.e.,* $f(z)$ *is meromorphically convex in* $\mathcal{D}$.

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