ON A CONJECTURE OF ZHENG JIANHUA

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ABSTRACT. In this paper, a new proof of the existence of the $T$ direction is given. Furthermore, we prove that for a meromorphic function $f$ with finite lower order, satisfying $\limsup_{r \to \infty} \frac{T(r,f)}{\log^2 r} = \infty$, there exists a $T$ direction concerning small functions.

1. Introduction and results

Let $f(z)$ be a transcendental meromorphic function defined on the whole complex plane. The singular direction for $f$ is one of main objects studied in the theory of value distributions of meromorphic functions. Here, we shall give a brief history of this research (see [6]). In 1919, Julia [1] introduced the concept of Julia direction for meromorphic function $f$, and showed that every transcendental meromorphic function has at least one Julia direction under the condition $\limsup_{r \to \infty} \frac{T(r,f)}{\log^2 r} = \infty$. Ostrowski [2] and Sun [13] gave a simple example of transcendental meromorphic function $f(z)$ such that $T(r,f) = O(\log^2 r)$ and $f(z)$ has no Julia direction, respectively. This shows that the growth condition (1) is sharp for $f(z)$ to have a Julia direction. Valiron [3] introduced a more refined notion of Borel directions as a ray $\arg z = \theta$, called a Borel direction of order $\rho$ for $f$ if for every $0 < \varepsilon < \pi$,

$$\limsup_{r \to \infty} \frac{\log n(r, \theta, \varepsilon, a)}{\log r} \geq \rho,$$

for all $a$ in $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ with at most two exceptions. Note that the definition is only meaningful in the case of $0 < \rho < \infty$. In this case it is well known that $f$ must have at least one Borel direction in [4]. When the order $\rho = 0$ or $\infty$, it is not better to use the order to characterize the growth of

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f. In this case, the Nevanlinna characteristic function $T(r, f)$ is certainly a more appropriate object to consider. Actually, in general, $T(r, f)$ is the most basic function that can be used to describe the growth of meromorphic functions. Zheng [5] noted this fact, that is, it is more natural to use the Nevanlinna characteristic function $T(r, f)$ instead of $\log r$ as a comparison function and introduced a new singular direction, namely, the $T$ direction for $f$. Here, we recall Zheng’s definition as follows.

**Definition 1.1.** A ray $\arg z = \theta$ is called a $T$ direction for a meromorphic function $f(z)$, if for every $0 < \varepsilon < \pi$,

$$\limsup_{r \to \infty} \frac{N(r, \theta, \varepsilon, a)}{T(r, f)} > 0$$

holds for all $a$ in $\mathbb{C}_\infty$ with at most two exceptions.

The existence of a $T$ direction was proved by Guo, Zheng and Ng [6]. They showed the following Theorem 1.1.

**Theorem 1.1.** If $f(z)$ is a meromorphic function defined on the whole complex plane, then $f(z)$ has at least one $T$ direction, suppose that

$$\limsup_{r \to \infty} \frac{T(r, f)}{(\log r)^2} = +\infty. \quad (1)$$

Theorem 1.1 was conjectured by Zheng [5]. Most recently, Wu and Sun [14] proved the existence of $T$ direction of a meromorphic function of order zero. The main purpose of this paper is to give a new proof of Theorem 1.1 and prove the following theorems.

**Theorem 1.2.** If $f(z)$ is a meromorphic function defined on the whole complex plane which satisfies (1), a ray $J : \arg z = \theta_0$ satisfies for any $0 < \delta < \pi/2$,

$$\limsup_{r \to \infty} \frac{T(r, J)}{T(r, f)} > 0,$$

then $J$ is a $T$ direction of $f(z)$.

**Theorem 1.3.** If $f(z)$ is a meromorphic function defined on the whole complex plane and satisfies (1), then there exists a direction $J : \arg z = \theta_0$ which is not only a Nevanlinna direction but also a $T$ direction.

Theorem 1.2 and Theorem 1.3 have been announced by Zheng Jianhua in [12]. We denote by $S(r, f)$ any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f$, if and only if $T(r, a(z)) = S(r, f)$. Concerning small meromorphic functions, Biernacki established the following extending theorem of Valiron [3] for Borel direction of meromorphic function. He proved the following theorem.
Theorem 1.4. (see [7]) Let \( f(z) \) be a meromorphic function on the complex plane \( \mathbb{C} \) with finite positive order \( \lambda \), there exists a direction \( \arg z = \theta \), such that for every \( 0 < \varepsilon < \pi \),
\[
\limsup_{r \to \infty} \frac{\log \sum_{j=1}^{3} n(r, \theta, \varepsilon, f = \varphi_j(z))}{\log r} = \lambda,
\]
holds for any three distinct small meromorphic functions of \( f \).

The existence of a \( T \) direction dealing with small functions was suggested by Zheng Jianhua. In this regard, we will consider this problem and show that

Theorem 1.5. Let \( f(z) \) be a transcendental meromorphic function on \( \mathbb{C} \) of finite lower order and satisfy (1), then there exists a ray \( \arg z = \theta \) such that for every \( 0 < \varepsilon < \pi \),
\[
\limsup_{r \to \infty} \frac{\sum_{j=1}^{3} N(r, \theta, \varepsilon, f = \varphi_j(z))}{T(r, \varphi(z))} > 0,
\]
holds for any three distinct meromorphic functions \( \varphi_j(z)(j = 1, 2, 3) \in \mathcal{A} \), where \( \mathcal{A} \) be the set of meromorphic functions \( \varphi(z) \) on the complex plane which satisfies \( T(r, \varphi(z)) = o\left(\frac{T(r,f)}{(\log r)^2}\right) \).

2. Proof of theorems

We shall prove these theorems by using a fundamental inequality of Ahlfors-Shimizu’s for characteristic functions in the angular domain. For the sake of convenience, we give the following notation (see Tsuji [11]). Denote the angular domain by \( \triangle(\theta, \alpha) = \{z : |\arg z - \theta| \leq \alpha\} \) and \( \triangle(r) \) be the part of \( \triangle(\theta, \alpha) \), which is contained in \( |z| \leq r \) and put
\[
S(r, \triangle(\theta, \alpha), f) = \frac{A(r)}{\pi} = \frac{1}{\pi} \iint_{\triangle(r)} \left( \frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 r d\theta dr, \quad z = re^{i\theta},
\]
\[
T(r, \triangle(\theta, \alpha), f) = \int_{0}^{r} S(r, \triangle(\theta, \alpha), f) dr,
\]
are respectively the Ahlfors characteristic function and Ahlfors-Shimizu’s characteristic function for \( f(z) \) in domain \( \triangle(\theta, \alpha) \). In particular, when \( \theta = 0, \alpha = 2\pi \), then \( T(r, \triangle(\theta, \alpha), f = T(r, f) \), i.e., the Nevanlinna characteristic function for the meromorphic function.
Let \( n(r, \theta, \alpha, a) \) be the number of zero points of \( f(z) - a \), contained in \( \triangle(r) \), multiple zeros being counted, and put

\[
N(r, \theta, \alpha, a) = \int_0^r \frac{n(r, \theta, \alpha, a)}{r} \, dr.
\]

Given a direction \( J : \arg z = \theta \), the for any \( a \in \mathbb{C}_\infty \)

\[
\delta(f, J; a) = 1 - \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \frac{N(r, \theta, \varepsilon, a)}{T(r, \triangle(\theta, \varepsilon), f)}
\]

is called the deficiency of the value \( a \) with respect to direction \( J \), and the value \( a \) is called deficiency with respect to \( J \) if \( \delta(f, J; a) > 0 \). If the sum of deficiency of all values of \( a \) with respect to \( J \) satisfies

\[
\sum a \delta(f, J; a) \leq 2,
\]

we call the direction \( J \) is a Nevanlinna direction for \( f \) (see [9]).

In order to prove our theorems, we first introduce the following main lemmas.

**Lemma 2.1.** (see [8]) Let \( f(z) \) be meromorphic on complex plane, for an angular domain \( \triangle(\theta_0, \delta) \), give different points \( a_1, a_2, \ldots, a_q \in \mathbb{C}_\infty \) \((q > 2)\), if \( 0 < \sigma < \delta \), then

\[
(q - 2)T(r, \triangle(\theta, \sigma), f) \leq \sum_{i=1}^q N(r, \theta, \delta, a_i) + O(\log^2 r)
\]

\[
+ h[2^\delta \pi T(r, \triangle(\theta, \delta), f)]^{1/2} \log T(r, \triangle(\theta, \delta), f).
\]

with at most one exceptional set of \( r \) denote it \( E_3 \), where \( h \) is a constant depending only on \( a_1, a_2, \ldots, a_q \).

**Lemma 2.2.** (see [9]) If \( f(z) \) is a meromorphic function defined on the whole complex plane and satisfies (1), then there exists at least one Nevanlinna direction \( J \) satisfies yet for any \( 0 < \delta < \pi/2 \),

\[
\limsup_{r \to \infty} \frac{T(r, \triangle(\theta_0, \delta), f)}{T(r, f)} > 0, \quad \text{and} \quad \limsup_{r \to \infty} \frac{T(r, \Delta(\theta_0, \delta), f)}{(\log r)^2} = +\infty.
\]

**Lemma 2.3.** (see [10]) Let \( S(r) \) be a positive continuous non-decreasing function of \( r \) in \([0, +\infty)\). Suppose that

\[
\liminf_{r \to \infty} \frac{\log S(r)}{\log r} = \mu < +\infty,
\]

\[
\limsup_{r \to \infty} \frac{S(r)}{\log^2 r} = +\infty.
\]
Then for any $h > 0$, there exists the sequence $\{r_n\}$ and $\{R_n\}$, $R_n^{1 - o(1)} \leq r_n \leq R_n (n \to \infty)$, satisfying
\[
\lim_{n \to \infty} \frac{S(r_n)}{\log^2 r_n} = +\infty, \quad S(e^h R_n) \leq e^{h_n} S(R_n)(1 + o(1))(n \to \infty).
\]

Now we are in a position to prove the theorems.

**Proof of the Theorem 1.1.** Since
\[
\limsup_{r \to \infty} \frac{T(r, f)}{(\log r)^2} = +\infty,
\]
then there exists an increasing sequence $\{r_n\}, r_n \to \infty (n \to \infty)$ such that
\[
\lim_{r_n \to \infty} \frac{T(r_n, f)}{(\log r_n)^2} = +\infty. \tag{2}
\]

Using the finite covering theorem in $[0, 2\pi]$, there surely exists some $\theta_0 \in [0, 2\pi]$ such that
\[
\limsup_{r_n \to \infty} \frac{T(r_n, \Delta(\theta_0, \delta), f)}{T(r_n, f)} > 0. \tag{3}
\]

Now, we prove that $J : \arg z = \theta_0$ is a $T$ direction. If the above statement is false, then there exists three distinct point $a_1, a_2, a_3 \in \mathbb{C}_\infty$ and a $\sigma$ such that
\[
\limsup_{r \to \infty} \frac{\sum_{i=1}^{3} N(r, \theta, \sigma, a_i)}{T(r, f)} = 0.
\]

Therefore, we have
\[
\lim_{r_n \to \infty} \frac{\sum_{i=1}^{3} N(r_n, \theta, \sigma, a_i)}{T(r_n, f)} = 0. \tag{4}
\]

For any $0 < \delta < \sigma$, by using Lemma 2.1, we have the following
\[
T(r, \Delta(\theta, \delta), f) \leq \sum_{i=1}^{3} N(r, \theta, \sigma, a_i) + O(\log^2 r)
\]
\[
+ h[2^\delta \pi T(r, \Delta(\theta, \sigma), f)]^{1/2} \log T(r, \Delta(\theta, \sigma), f),
\]
which holds with except the set $E_\sigma$.

We can suppose that $r_n$ does not belong to $E_\sigma$, thus
\[
T(r_n, \Delta(\theta, \delta), f) \leq \sum_{i=1}^{3} N(r_n, \theta, \sigma, a_i) + O(\log^2 r)
\]
\[
+ h[2^\delta \pi T(r_n, \Delta(\theta, \sigma), f)]^{1/2} \log T(r_n, \Delta(\theta, \sigma), f).
\]
Dividing by $T(r_n, f)$ the both sides of the inequality and taking superior limits, using (2) and (4) we get
\[
\limsup_{r_n \to \infty} \frac{T(r_n, \Delta(\theta_0, \delta), f)}{T(r_n, f)} \leq \limsup_{r_n \to \infty} \frac{\sum_{i=1}^{3} N(r_n, \theta, \sigma, a_i)}{T(r_n, f)} = 0.
\]
This contradicts with (3). Hence $J$ is a $T$ direction. □

**Proof of the Theorem 1.2.** From the proof of Theorem 1.1 we can find Theorem 1.2 is valid. □

**Proof of the Theorem 1.3.** By the Lemma 2.2 and the Theorem 1.2 it is easy to derive there exists a ray $J : \arg z = \theta_0$ which is not only a Nevanlinna direction but also a $T$ direction. □

**Proof of the Theorem 1.5.** By the hypothesis, we have
\[
\liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} = \mu < +\infty,
\]
and
\[
\limsup_{r \to \infty} \frac{T(r, f)}{\log^2 r} = +\infty.
\]
Applying the Lemma 2.3, there exists the sequence $\{r_n\}$ and $\{R_n\}$, such that
\[
\lim_{n \to \infty} \frac{T(r_n, f)}{\log^2 r_n} = +\infty,
\]
\[
T(128R_n, f) = T(e^{\log 128} R_n, f) \leq e^{\mu \log 128} T(R_n, f)(1 + o(1))(n \to \infty),
\]
where $R_n^{1-o(1)} \leq r_n \leq R_n(n \to \infty)$. Hence, we have
\[
\lim_{n \to \infty} \frac{T(R_n, f)}{\log^2 R_n} = +\infty,
\]
and
\[
\lim_{n \to \infty} \frac{T(R_n, f)}{T(128R_n, f)} > 0.
\]
The above expression (6) implies that there is a ray $\arg z = \theta_0$ ($0 < \theta_0 \leq 2\pi$), such that
\[
\lim_{n \to \infty} \frac{T(R_n, \Omega(\theta_0, \varepsilon), f)}{T(128R_n, f)} > 0,
\]
which holds for any $\varepsilon (0 < \varepsilon < \pi)$.

Now we are in a position to prove that the ray $\arg z = \theta_0$ satisfies Theorem 1.5.
For arbitrary $\delta \in (0, \pi)$, and $a_1(z), a_2(z), a_3(z) \in \mathcal{A}$ be any three distinct functions, put
\[ F(z) = \frac{(f(z) - a_1(z))(a_3(z) - a_2(z))}{(f(z) - a_2(z))(a_3(z) - a_1(z))}. \]
Then the function $f$ can be written as
\[ f(z) = \frac{g_1(z)F(z) + g_2(z)}{g_3(z)F(z) + g_4(z)}, \]
where $g_1(z), g_2(z), g_3(z), g_4(z) \in \mathcal{A}$. Using a Lemma of Tsuji [11], we have
\begin{align*}
T(R_n, \Omega(\theta, \frac{\delta}{4}, f)) &\leq 27T(64R_n, \Omega(\theta, \frac{\delta}{2}, F)) + o(T(128R_n, f)) \\
&\leq 81N(128R_n, \Omega(\theta, \delta), F = 0) + 81N(128R_n, \Omega(\theta, \delta), F = 1) \\
&+ 81N(128R_n, \Omega(\theta, \delta), F = \infty) + o(T(128R_n, f)) + O(\log^2 64R_n) \\
&= 81 \sum_{i=1}^{3} N(128R_n, \Omega(\theta, \delta), f = a_i(z)) + o(T(128R_n, f)) + O(\log^2 R_n).
\end{align*}
Dividing the both sides of the above expression by $T(128R_n, f)$ and applying (5) and (7), we can obtain that
\[ \lim_{n \to \infty} \frac{\sum_{i=1}^{3} N(128R_n, \Omega(\theta, \delta), f = a_i(z))}{T(128R_n, f)} > 0. \]
Hence for any three distinct functions $a_1(z), a_2(z), a_3(z) \in \mathcal{A}$ and any $0 < \varepsilon < \pi$, we have
\[ \limsup_{r \to \infty} \frac{\sum_{j=1}^{3} N(r, \theta, \varepsilon, f = a_j(z))}{T(r, f)} > 0. \]
The proof of Theorem 1.5 is complete. \hfill \Box

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**References**


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