SARAJEVO JOURNAL OF MATHEMATICS
Vol.3 (16) (2007), 163–170

ON A CONJECTURE OF ZHENG JIANHUA

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Abstract. In this paper, a new proof of the existence of the T direction is given. Furthermore, we prove that for a meromorphic function f with finite lower order, satisfying $\limsup_{r\to\infty} \frac{T(r,f)}{\log^2 r} = \infty$, there exists a T direction concerning small functions.

1. Introduction and results

Let $f(z)$ be a transcendental meromorphic function defined on the whole complex plane. The singular direction for f is one of main objects studied in the theory of value distributions of meromorphic functions. Here, we shall give a brief history of this research (see [6]). In 1919, Julia [1] introduced the concept of Julia direction for meromorphic function f, and showed that every transcendental meromorphic function has at least one Julia direction under the condition $\limsup_{r\to\infty} \frac{T(r,f)}{\log^2 r} = \infty$. Ostrowskï [2] and Sun [13] gave a simple example of transcendental meromorphic function $f(z)$ such that $T(r,f) = O(\log^2 r)$ and $f(z)$ has no Julia direction, respectively. This shows that the growth condition (1) is sharp for $f(z)$ to have a Julia direction. Valiron [3] introduced a more refined notion of Borel directions as a ray $\arg z = \theta$, called a Borel direction of order ρ for f if for every $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{\log n(r,\theta,\varepsilon,a)}{\log r} \geq \rho,$$

for all a in $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ with at most two exceptions. Note that the definition is only meaningful in the case of $0 < \rho < \infty$. In this case it is well known that f must have at least one Borel direction in [4]. When the order $\rho = 0$ or ∞, it is not better to use the order to characterize the growth of

1991 *Mathematics Subject Classification.* Primary 30D30, Secondary 30D35.

Key words and phrases. T direction, Nevanlinna direction, small meromorphic function, Ahlfors characteristic function.

The work was supported by the NSF of Xianning University (Grant No: KT0623, KZ0629, KY0718).

f. In this case, the Nevanlinna characteristic function $T(r,f)$ is certainly a more appropriate object to consider. Actually, in general, $T(r,f)$ is the most basic function that can be used to describe the growth of meromorphic functions. Zheng [5] noted this fact, that is, it is more natural to use the Nevanlinna characteristic function $T(r,f)$ instead of $\log r$ as a comparison function and introduced a new singular direction, namely, the T direction for f. Here, we recall Zheng's definition as follows.

Definition 1.1. *A ray $\arg z = \theta$ is called a T direction for a meromorphic function $f(z)$, if for every $0 < \varepsilon < \pi$,*

$$\limsup_{r\to\infty} \frac{N(r,\theta,\varepsilon,a)}{T(r,f)} > 0$$

holds for all a in $\mathbb{C}_\infty$ with at most two exceptions.

The existence of a T direction was proved by Guo, Zheng and Ng [6]. They showed the following Theorem 1.1.

Theorem 1.1. *If $f(z)$ is a meromorphic function defined on the whole complex plane, then $f(z)$ has at least one T direction, suppose that*

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = +\infty. \tag{1}$$

Theorem 1.1 was conjectured by Zheng [5]. Most recently, Wu and Sun [14] proved the existence of T direction of a meromorphic function of order zero. The main purpose of this paper is to give a new proof of Theorem 1.1 and prove the following theorems.

Theorem 1.2. *If $f(z)$ is a meromorphic function defined on the whole complex plane which satisfies* (1)*, a ray $J : \arg z = \theta_0$ satisfies for any $0 < \delta < \pi/2$,*

$$\limsup_{r\to\infty} \frac{T(r,\Delta(\theta_0,\delta),f)}{T(r,f)} > 0,$$

then J is a T direction of $f(z)$.

Theorem 1.3. *If $f(z)$ is a meromorphic function defined on the whole complex plane and satisfies* (1)*, Then there exists a direction $J : \arg z = \theta_0$ which is not only a Nevanlinna direction but also a T direction.*

Theorem 1.2 and Theorem 1.3 have been announced by Zheng Jianhua in [12]. We denote by $S(r,f)$ any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function of f, if and only if $T(r,a(z)) = S(r,f)$. Concerning small meromorphic functions, Biernacki established the following extending theorem of Valiron [3] for Borel direction of meromorphic function. He proved the following theorem.

Theorem 1.4. (see [7]) *Let $f(z)$ be a meromorphic function on the complex plane $\mathbb{C}$ with finite positive order λ,there exists a direction* $\arg z = \theta$, *such that for every* $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{\log \sum_{j=1}^{3} n(r,\theta,\varepsilon, f = \varphi_j(z))}{\log r} = \lambda,$$

holds for any three distinct small meromorphic functions of f.

The existence of a T direction dealing with small functions was suggested by Zheng Jianhua. In this regard, we will consider this problem and show that

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$ of finite lower order and satisfy* (1), *then there exists a ray* $\arg z = \theta$ *such that for every* $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{\sum_{j=1}^{3} N(r,\theta,\varepsilon, f = \varphi_j(z))}{T(r,f)} > 0,$$

holds for any three distinct meromorphic functions $\varphi_j(z)(j = 1,2,3) \in \mathcal{A}$, where $\mathcal{A}$ be the set of meromorphic functions $\varphi(z)$ on the complex plane which satisfies $T(r,\varphi(z)) = o(\frac{T(r,f)}{(\log r)^2})$.

2. Proof of theorems

We shall prove these theorems by using a fundamental inequality of Ahlfors-Shimizu's for characteristic functions in the angular domain. For the sake of convenience, we give the following notation (see Tsuji [11]). Denote the angular domain by $\triangle(\theta,\alpha) = \{z : |\arg z - \theta| \le \alpha\}$ and $\triangle(r)$ be the part of $\triangle(\theta,\alpha)$, which is contained in $|z| \le r$ and put

$$S(r,\triangle(\theta,\alpha),f) = \frac{A(r)}{\pi} = \frac{1}{\pi}\iint\limits_{\triangle(r)} \left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 r d\theta dr, \quad z = re^{i\theta},$$

$$T(r,\triangle(\theta,\alpha),f) = \int_0^r \frac{S(r,\triangle(\theta,\alpha),f)}{r} dr,$$

are respectively the Ahlfors characteristic function and Ahlfors-Shimizu's characteristic function for $f(z)$ in domain $\triangle(\theta,\alpha)$. In particular, when $\theta = 0, \alpha = 2\pi$, then $T(r,\triangle(\theta,\alpha), f = T(r,f)$, i.e., the Nevanlinna characteristic function for the meromorphic function.

Let $n(r,\theta,\alpha,a)$ be the number of zero points of $f(z)-a$, contained in $\triangle(r)$, multiple zeros being counted, and put

$$N(r,\theta,\alpha,a)=\int\limits_0^r \frac{n(r,\theta,\alpha,a)}{r}dr.$$

Given a direction $J:\arg z=\theta$, the for any $a\in\mathbb{C}_\infty$

$$\delta(f,J;a)=1-\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{N(r,\theta,\varepsilon,a)}{T(r,\triangle(\theta,\varepsilon),f)}$$

is called the *deficiency* of the value a with respect to direction J, and the value a is called deficiency with respect to J if $\delta(f,J;a)>0$. If the sum of deficiency of all values of a with respect to J satisfies $\sum_a \delta(f,J;a)\le 2$, we call the direction J is a *Nevanlinna direction* for f (see [9]).

In order to prove our theorems, we first introduce the following main lemmas.

Lemma 2.1. (see [8]) *Let $f(z)$ be meromorphic on complex plane, for an angular domain $\triangle(\theta_0,\delta)$, give different points $a_1,a_2,\dots,a_q\in\mathbb{C}_\infty(q>2)$, if $0<\sigma<\delta$, then*

$$\begin{aligned}(q-2)T(r,\triangle(\theta,\sigma),f)\le&\sum_{i=1}^q N(r,\theta,\delta,a_i)+O(\log^2 r)\\&+h[2^\delta\pi T(r,\triangle(\theta,\delta),f)]^{1/2}\log T(r,\triangle(\theta,\delta),f).\end{aligned}$$

with at most one exceptional set of r denote it E_δ, where h is a constant depending only on $a_1,a_2,\cdots,a_q$.

Lemma 2.2. (see [9]) *If $f(z)$ is a meromorphic function defined on the whole complex plane and satisfies* (1)*, then there exists at least one Nevanlinna direction J satisfyies yet for any* $0<\delta<\pi/2$,

$$\limsup_{r\to\infty}\frac{T(r,\Delta(\theta_0,\delta),f)}{T(r,f)}>0,\quad \textit{and}\quad \limsup_{r\to\infty}\frac{T(r,\Delta(\theta_0,\delta),f)}{(\log r)^2}=+\infty.$$

Lemma 2.3. (see [10]) *Let $S(r)$ be a positive continous non-decreasing function of r in $[0,+\infty)$. Suppose that*

$$\liminf_{r\to\infty}\frac{\log S(r)}{\log r}=\mu<+\infty,$$
$$\limsup_{r\to\infty}\frac{S(r)}{\log^2 r}=+\infty.$$

Then for any $h > 0$, there exists the sequence $\{r_n\}$ and $\{R_n\}$, $R_n^{1-o(1)} \leq r_n \leq R_n (n \to \infty)$, satisfying

$$\lim_{n\to\infty} \frac{S(r_n)}{\log^2 r_n} = +\infty, \quad S(e^h R_n) \leq e^{h\mu} S(R_n)(1 + o(1))(n \to \infty).$$

Now we are in a position to prove the theorems.

Proof of the Theorem 1.1. Since

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = +\infty,$$

then there exists an increasing sequence $\{r_n\}, r_n \to \infty(n \to \infty)$ such that

$$\lim_{r_n\to\infty} \frac{T(r_n, f)}{(\log r_n)^2} = +\infty. \tag{2}$$

Using the finite covering theorem in $[0, 2\pi]$, there surely exists some $\theta_0 \in [0, 2\pi]$ such that

$$\limsup_{r_n\to\infty} \frac{T(r_n, \Delta(\theta_0, \delta), f)}{T(r_n, f)} > 0. \tag{3}$$

Now, we prove that $J : \arg z = \theta_0$ is a T direction. If the above statement is false, then there exists three distinct point $a_1, a_2, a_3 \in \mathbb{C}_\infty$ and a σ such that

$$\limsup_{r\to\infty} \frac{\sum_{i=1}^{3} N(r, \theta, \sigma, a_i)}{T(r, f)} = 0.$$

Therefore, we have

$$\lim_{r_n\to\infty} \frac{\sum_{i=1}^{3} N(r_n, \theta, \sigma, a_i)}{T(r_n, f)} = 0. \tag{4}$$

For any $0 < \delta < \sigma$, by using Lemma 2.1, we have the following

$$T(r, \triangle(\theta, \delta), f) \leq \sum_{i=1}^{3} N(r, \theta, \sigma, a_i) + O(\log^2 r) + h[2^\delta \pi T(r, \triangle(\theta, \sigma), f)]^{1/2} \log T(r, \triangle(\theta, \sigma), f),$$

which holds with except the set E_σ.

We can suppose that r_n does not belong to E_σ, thus

$$T(r_n, \triangle(\theta, \delta), f) \leq \sum_{i=1}^{3} N(r_n, \theta, \sigma, a_i) + O(\log^2 r) + h[2^\delta \pi T(r_n, \triangle(\theta, \sigma), f)]^{1/2} \log T(r_n, \triangle(\theta, \sigma), f).$$

Dividing by $T(r_n, f)$ the both sides of the inequality and taking superior limits, using (2) and (4) we get

$$\limsup_{r_n\to\infty} \frac{T(r_n, \Delta(\theta_0, \delta), f)}{T(r_n, f)} \le \limsup_{r_n\to\infty} \frac{\sum_{i=1}^{3} N(r_n, \theta, \sigma, a_i)}{T(r_n, f)} = 0.$$

This contradicts with (3). Hence J is a T direction. □

Proof of the Theorem 1.2. From the proof of Theorem 1.1 we can find Theorem 1.2 is valid. □

Proof of the Theorem 1.3. By the Lemma 2.2 and the Theorem 1.2 it is easy to derive there exists a ray $J : \arg z = \theta_0$ which is not only a Nevanlinna direction but also a T direction. □

Proof of the Theorem 1.5. By the hypothesis, we have

$$\liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} = \mu < +\infty,$$

and

$$\limsup_{r\to\infty} \frac{T(r, f)}{\log^2 r} = +\infty.$$

Applying the Lemma 2.3, there exists the sequence $\{r_n\}$ and $\{R_n\}$, such that

$$\lim_{n\to\infty} \frac{T(r_n, f)}{\log^2 r_n} = +\infty,$$

$$T(128R_n, f) = T(e^{\log 128} R_n, f) \le e^{\mu \log 128} T(R_n, f)(1 + o(1))(n \to \infty).$$

where $R_n^{1-o(1)} \le r_n \le R_n (n \to \infty)$. Hence, we have

$$\lim_{n\to\infty} \frac{T(R_n, f)}{\log^2 R_n} = +\infty, \tag{5}$$

and

$$\lim_{n\to\infty} \frac{T(R_n, f)}{T(128R_n, f)} > 0. \tag{6}$$

The above expression (6) implies that there is a ray $\arg z = \theta_0 \quad (0 < \theta_0 \le 2\pi)$, such that

$$\lim_{n\to\infty} \frac{T(R_n, \Omega(\theta_0, \varepsilon), f)}{T(128R_n, f)} > 0, \tag{7}$$

which holds for any $\varepsilon (0 < \varepsilon < \pi)$.

Now we are in a position to prove that the ray $\arg z = \theta_0$ satisfies Theorem 1.5.

For arbitrary $\delta \in (0,\pi)$, and $a_1(z), a_2(z), a_3(z) \in \mathcal{A}$ be any three distinct functions, put

$$F(z) = \frac{(f(z)-a_1(z))(a_3(z)-a_2(z))}{(f(z)-a_2(z))(a_3(z)-a_1(z))}.$$

Then the function f can be written as

$$f(z) = \frac{g_1(z)F(z)+g_2(z)}{g_3(z)F(z)+g_4(z)},$$

where $g_1(z), g_2(z), g_3(z), g_4(z) \in \mathcal{A}$. Using a Lemma of Tsuji [11], we have

$$\begin{aligned} T(R_n, \Omega(\theta_0, \frac{\delta}{4}), f) &\le 27T(64R_n, \Omega(\theta_0, \frac{\delta}{2}), F) + o(T(128R_n, f)) \\ &\le 81N(128R_n, \Omega(\theta_0,\delta), F=0) + 81N(128R_n, \Omega(\theta_0,\delta), F=1) \\ &+81N(128R_n, \Omega(\theta_0,\delta), F=\infty) + o(T(128R_n, f)) + O(\log^2 64R_n) \\ &= 81\sum_{i=1}^{3} N(128R_n, \Omega(\theta_0,\delta), f=a_i(z)) + o(T(128R_n, f)) + O(\log^2 R_n). \end{aligned}$$

Dividing the both sides of the above expression by $T(128R_n, f)$ and applying (5) and (7), we can obtain that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{3} N(128R_n, \Omega(\theta_0,\delta), f=a_i(z))}{T(128R_n, f)} > 0.$$

Hence for any three distinct functions $a_1(z), a_2(z), a_3(z) \in \mathcal{A}$ and any $0 < \varepsilon < \pi$, we have

$$\limsup_{r\to\infty} \frac{\sum_{j=1}^{3} N(r, \theta_0, \varepsilon, f=a_j(z))}{T(r,f)} > 0.$$

The proof of Theorem 1.5 is complete. □

Acknowledgment. The author is very grateful for the kind help of Prof. J. H. Zheng and D. C. Sun.

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(Received: October 24, 2006)
(Revised: May 4, 2007)

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