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GENERALIZED QUASILINEARIZATION FOR NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEMS WITH NONLOCAL CONDITIONS

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Abstract. We apply the generalized quasilinearization technique to obtain a monotone sequence of iterates converging quadratically to the unique solution of a general second order nonlinear differential equation with nonlinear nonlocal mixed three-point boundary conditions. The convergence of order n ($n \geq 2$) of the sequence of iterates has also been established.

1. Introduction

The subject of multi-point nonlocal boundary value problems, initiated by Ilin and Moiseev [1,2], has been addressed by many authors, for instance, [3-9]. In particular, Eloe and Gao [10] discussed the quasilinearization method for a three-point nonlinear boundary value problem. The quasilinearization technique [11] is quite fruitful as it not only proves the existence of the solutions of the problem but also provides an iterative scheme for approximating the solutions. The nineties brought new dimensions to this technique. The most interesting new idea was introduced by Lakshmikantham [12-13] who generalized the method of quasilinearizaion by relaxing the convexity assumption. This development was so significant that it attracted the attention of many researchers and the method was extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations, see [14-22] and references therein.

In this paper, we develop the method of generalized quasilinearization for the following second order three-point boundary value problem with mixed nonlinear nonlocal boundary conditions

$$x''(t) = f(t, x(t), x'(t)), \quad t \in [0, 1], \tag{1.1}$$

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$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = g(x(\gamma)), \;\; 0 < \gamma < 1, \tag{1.2}$$

where $f \in C([0,1] \times R^2)$, p and q are positive constants, $a \in R$ and $g : R \to R$ is continuous. The importance of the work lies in the fact that the boundary conditions of the type (1.2) appear in certain problems of thermodynamics and wave propagation where the controller at the end $t = 1$ dissipates or adds energy according to a censor located at a position $t = \gamma$ where as the other end $t = 0$ is maintained at a constant level of energy. A sequence of approximate solutions converging monotonically and quadratically to a unique solution of (1.1) and (1.2) will be obtained in Theorem 3.1 and the convergence of order $n (n \geq 2)$ has been established in Theorem 3.2.

2. Preliminaries

It is well known that the solution, $x(t)$ of (1.1) and (1.2) can be written as

$$x(t) = a\left(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}\right) + g(x(\gamma))\left[\frac{t}{p+2q} + \frac{q}{p^2+2pq}\right] + \int_0^1 G(t,s) f(s, x(s), x'(s)) ds, \tag{2.1}$$

where $G(t,s)$ is the Green's function for the mixed three-point boundary value problem and is given by

$$G(t,s) = \frac{1}{(p^2+2pq)} \begin{cases} (pt+q)(p(s-1)-q), & \text{if } \; 0 \leq t \leq s \leq 1, \\ (p(t-1)-q)(ps+q), & \text{if } \; 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

Note that $G(t,s) < 0$ on $[0,1] \times [0,1]$.

We say that $\alpha \in C^2[0,1]$ is a lower solution of the boundary value problem (1.1) and (1.2) if

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), \;\; t \in [0,1],$$
$$p\alpha(0) - q\alpha'(0) \leq a, \quad p\alpha(1) + q\alpha'(1) \leq g(\alpha(\gamma)),$$

and $\beta \in C^2[0,1]$ is an upper solution of the boundary value problem (1.1) and (1.2) if

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)), \;\; t \in [0,1],$$
$$p\beta(0) - q\beta'(0) \geq a, \quad p\beta(1) + q\beta'(1) \geq g(\beta(\gamma)).$$

We need the following version of Kamke's convergence theorem (Page 14 [23]) to prove the existence theorem.

Theorem 2.1. *Assume that each solution of* (1.1) *extends to the interval* $(0,1)$ *or becomes unbounded on its maximal interval of convergence. Let* $\{x_m(t)\}$ *be a sequence of solutions of* (1.1) *such that there exists a sequence* $\{t_m\} \subset (0,1)$ *with* $\lim_{m\to\infty} t_m = t_0 \in (0,1)$ *and* $\lim_{m\to\infty} x_m^{(i)}(t_m) = x_i, \;\; i =$

$0,1$. *Then there is a solution* $x(t)$ *of* (1.1) *such that* $x^{(i)}(t_0) = x_i,\ i = 0,1$ *and a subsequence* $\{x_{m_j}(t)\}$ *of* $\{x_m(t)\}$ *such that* $\lim_{j\to\infty} x^{(i)}_{m_j}(t) = x^{(i)}(t),\ i = 0,1$, *uniformly on each compact subinterval of* $(0,1)$.

Remark 2.2. In the study of boundary value problems involving an $n'th$ order differential equation of the form

$$x^n = f(t, x, x', \ldots, x^{n-1}), \tag{2.3}$$

the following proposition concerning the convergence of the sequence of solutions of the problem has attracted several mathematicians. For the study of different criteria equivalent to this proposition, we refer the reader to a detailed survey article by Agarwal [24].

Proposition 2.3. *If* $[c,d]$ *is a compact subinterval of* (a,b) *and* $\{x_m(t)\}$ *is a sequence of solutions of* (2.2) *which is uniformly bounded, that is,* $|x_m(t)| \le M$ *on* $[c,d]$ *for some* $M > 0$ *and for all* $m = 1,2,\ldots$, *then there is a subsequence* $\{x_{m_j}(t)\}$ *such that* $\{x^{(i)}_{m_j}\}$ *converges uniformly on* $[c,d]$ *for each* $i = 0,1,\ldots,n-1$.

Theorem 2.4. (Existence Theorem) *Assume that* f *is continuous on* $[0,1]\times R^2$, g *is continuous on* R *satisfying a one-sided Lipschitz condition (* $g(x) - g(y) \le L(x-y),\ 0 \le L < p$*) and each solution of* $x''(t) = f(t, x(t), x'(t))$ *extends to* $[0,1]$ *or becomes unbounded on its maximal interval of convergence. Let* α, β *be lower and upper solutions of* (1.1) *and* (1.2) *respectively such that* $\alpha(t) \le \beta(t)$. *Then there exists a solution,* $x(t)$ *of* (1.1) *and* (1.2) *such that* $\alpha(t) \le x(t) \le \beta(t)$.

Proof. We define $\overline{f}$ and $\overline{g}$ by

$$\overline{f}(t,x,y) = \begin{cases} f(t,\beta(t),y) + \frac{x-\beta(t)}{1+x-\beta(t)}, & \text{if } x(t) > \beta(t), \\ f(t,x,y), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ f(t,\alpha(t),y) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x(t) < \alpha(t), \end{cases}$$

$$\overline{g}(x) = \begin{cases} g(\beta(\gamma)), & \text{if } x > \beta(\gamma), \\ g(x), & \text{if } \alpha(\gamma) \le x \le \beta(\gamma), \\ g(\alpha(\gamma)), & \text{if } x < \alpha(\gamma). \end{cases}$$

Let $N = \max\{|\alpha'(t)|,\ |\beta'(t)|,\ |g(\alpha(\gamma))|,\ |g(\beta(\gamma))|\}$. For each positive integer l, we set

$$f_l(t,x,y) = \begin{cases} \overline{f}(t,x,N+l), & \text{if } y > N+l, \\ \overline{f}(t,x,y), & \text{if } |y| \le N+l, \\ \overline{f}(t,x,-(N+l)), & \text{if } y < -(N+l). \end{cases}$$

Observe that f_l is bounded and continuous on $[0,1]\times R^2$, $\overline{g}$ is bounded and continuous on R. By a standard application of Schauder's fixed point theorem to the operator defined by (2.1), it follows that (1.1) and (1.2) has a solution x_l with $f=f_l$ and $g=\overline{g}$.

Now, we show that each solution x_l satisfies $\alpha(t)\le x_l(t)\le\beta(t)$ $t\in[0,1]$. For the sake of contradiction, let us suppose that $\alpha(t)\le x_l(t)$ does not hold and set $r(t)=\alpha(t)-x_l(t),\ t\in[0,1]$. By the standard arguments [25], let $r(t)$ have a positive maximum at $\tau\in(0,1)$. Then $r''(\tau)\le 0$ and $|\alpha'(\tau)|=|x_l'(\tau)|\le N<N+l$. On the other hand, $r''(\tau)=\alpha''(\tau)-x_l''(\tau)\ge r(\tau)/(1+r(\tau))>0$, which is a contradiction. For $\tau=1$, we have

$$pr(1)+qr'(1)\le g(\alpha(\gamma))-g(x_l(\gamma))\le Lr(\gamma).$$

Thus, $pr(1)\le Lr(\gamma)$ or $r(1)<r(\gamma)$, which is a contradiction. Similarly, we get a contradiction for $\tau=0$. Hence we conclude that $\alpha\le x_l$ on $[0,1]$. Similarly, one can prove that $x_l\le\beta$ on $[0,1]$. Thus, it follows that $\alpha(t)\le x_l(t)\le\beta(t),\ t\in[0,1]$. Moreover, for each l, there exists $t_l\in[0,t_1]$ such that

$$t_1|x_{kl}'(t_l)|=|x_{kl}(t_1)-x_{kl}(0)|\le\max\{\beta(0)-\alpha(t_1),\beta(t_1)-\alpha(0)\}.$$

Thus, each of the subsequences $\{x_{kl}(t_l)\}$ and $\{x_{kl}'(t_l)\}$ is bounded. Hence, by Kamke convergence theorem, there exists a subsequence of $\{x_{kl}\}$ which converges to a solution of $x''(t)=\overline{f}(t,x(t),x'(t))$ on a maximal subinterval of $[0,t_1]$. Clearly, $\alpha(t)\le x(t)\le\beta(t)$ and all solutions of $x''(t)=f(t,x(t),x'(t))$ extend to all of $[0,1]$ or become unbounded. Hence $\alpha(t)\le x(t)\le\beta(t)$ and $\overline{f}(t,x(t),x'(t))=f(t,x(t),x'(t))$. This completes the proof. □

Corollary 2.5. *Assume that f is continuous with $f_x>0$ on $[0,1]\times R$ and g is continuous with $0\le g'\le 1$ on R. Then the solution of (1.1) and (1.2) is unique.*

Remark 2.6. The simplified version of the condition that each solution of $x''(t)=f(t,x(t),x'(t))$ extends to $[0,1]$ or becomes unbounded on its maximal interval of convergence is that f satisfies a Nagumo condition [16, 22], that is, for each $M>0$, there exists a positive continuous function h_M on $[0,\infty]$ such that $|f(t,x,x')|\le h_M(|x'|)$ for all $(t,x,x')\in[0,1]\times[-M,M]\times R$ and

$$\int_0^\infty s[h_M(s)]^{-1}ds=\infty.$$

3. Main Results

Theorem 3.1. *Assume that*

($\mathbf{A_1}$) *α,β are lower and upper solutions of (1.1) and* (1.2) *respectively.*

(A_2) *$f(t,x,y) \in C([0,1]\times R^2)$ such that $\frac{\partial f}{\partial x}(t,x,y) > 0$, $\frac{\partial^2}{\partial x^2}(f(t,x,y) + \phi(t,x,y)) \le 0$, where $\phi \in C^2[J \times R^2, R]$ with $\frac{\partial^2}{\partial x^2}\phi(t,x,y) \le 0$. Moreover, f satisfies a Nagumo condition in y.*

(A_3) *g, g' are continuous on R and g'' exists with $0 \le g' \le 1$, $g'' \ge 0$.*

Then there exists a monotone sequence of solutions converging quadratically to the unique solution, $x(t)$ of (1.1) and (1.2).

Proof. Define

$$f(t,x,y) = F(t,x) - \phi(t,x,y),\ t \in [0,1], \tag{3.1}$$

where $F(t,x) : [0,1]\times R \to R$ is such that F, F_x, F_{xx} are continuous on $[0,1]\times R$ and in view of (A_2), it follows that $F_{xx}(t,x) \le 0$. Thus, applying the generalized mean value theorem on $F(t,x)$ yields

$$F(t,x) \le F(t,x_1) + F_x(t,x_1)(x - x_1), \tag{3.2}$$

which together with (3.1) takes the form

$$f(t,x,y) \le f(t,x_1,y) + F_x(t,x_1)(x - x_1) - (\phi(t,x,y) - \phi(t,x_1,y)).$$

Let us define

$$G(t,x,x_1,y) = f(t,x_1,y) + F_x(t,x_1)(x - x_1) - (\phi(t,x,y) - \phi(t,x_1,y)). \tag{3.3}$$

Observe that

$$G(t,x,x_1,y) \ge f(t,x,y), \quad G(t,x,x,y) = f(t,x,y). \tag{3.4}$$

Moreover, using (3.3) together with (A_2) yields

$$G_x(t,x,x_1,y) = F_x(t,x_1) - \phi_x(t,x,y) \ge F_x(t,x) - \phi_x(t,x,y) = f_x(t,x,y) > 0, \tag{3.5}$$

which implies that $G(t,x,x_1,y)$ is increasing in x for each fixed $(t,x_1,y) \in J\times R^2$. In view of ($A_3$), we get

$$g(x) \ge g(y) + g'(y)(x - y).$$

Letting

$$g^*(x,y) = g(y) + g'(y)(x - y),$$

we notice that

$$g(x) = \max_y g^*(x,y), \quad g(x) = g^*(x,x), \tag{3.6}$$

and $0 \le g^*_x(x,y) = g'(y) \le 1$. Now, we set $x_1 = \alpha_0(t) = \alpha(t),\ y = x'(t)$ in (3.3) and consider the BVP

$$x''(t) = G(t, x(t), \alpha_0(t), x'(t)),\ t \in [0,1], \tag{3.7}$$

$$px(0) - qx'(0) = a, \quad px(1) + qx'(1) = g^*(x(\gamma), \alpha_0(\gamma)). \tag{3.8}$$

In view of (A_1), (3.4) and (3.6), we have

$$\alpha_0''(t) \geq f(t, \alpha_0(t), \alpha_0'(t)) = G(t, \alpha_0(t), \alpha_0(t), \alpha_0'(t)),\ t \in [0,1],$$
$$p\alpha_0(0) - q\alpha_0'(0) \leq a,\ \ p\alpha_0(1) + q\alpha_0'(1) \leq g^*(\alpha_0(\gamma), \alpha_0(\gamma)),$$

and

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)) \leq G(t, \beta(t), \alpha_0(t), \beta'(t)),\ t \in [0,1],$$
$$p\beta(0) - q\beta'(0) \geq a,\ \ p\beta(1) + q\beta'(1) \geq g^*(\beta(\gamma), \alpha_0(\gamma)),$$

which imply that α_0 and β are lower and upper solutions of (3.7) and (3.8) respectively. Thus, by Theorem 2.4 and Corollary 2.5, there exists a unique solution α_1 of (3.7) and (3.8) such that

$$\alpha_0 \leq \alpha_1 \leq \beta.$$

Next, we consider

$$x''(t) = G(t, x(t), \alpha_1(t), x'(t)),\ t \in [0,1], \tag{3.9}$$
$$px(0) - qx'(0) = a, \quad px(1) + qx'(1) = g^*(x(\gamma), \alpha_1(\gamma)). \tag{3.10}$$

Employing the earlier arguments, we find that α_1 is a lower solution of (3.9) and (3.10), that is,

$$\alpha_1''(t) = G(t, \alpha_1(t), \alpha_0(t), \alpha_1'(t)) \geq G(t, \alpha_1(t), \alpha_1(t), \alpha_1'(t)),\ t \in [0,1],$$
$$p\alpha_1(0) - q\alpha_1'(0) \leq,\ \ p\alpha_1(1) + q\alpha_1'(1) = g^*(\alpha_1(\gamma), \alpha_0(\gamma)) \leq g^*(\alpha_1(\gamma), \alpha_1(\gamma)).$$

Similarly, β is an upper solution of (3.9) and (3.10) as we have

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)) \leq G(t, \beta(t), \alpha_1(t), \beta'(t)),\ t \in [0,1],$$
$$p\beta(0) - q\beta'(0) \geq a,\ \ p\beta(1) + q\beta'(1) \geq g^*(\beta(\gamma), \alpha_1(\gamma)).$$

Again by Theorem 2.4 and Corollary 2.5, it follows that there exists a unique solution α_2 of (3.9) and (3.10) such that

$$\alpha_1 \leq \alpha_2 \leq \beta.$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_j\}$ satisfying

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_j \leq \beta,$$

where the element α_j of the sequence $\{\alpha_j\}$ is a solution of the problem

$$x''(t) = G(t, x(t), \alpha_{j-1}(t), x'(t)),\ t \in [0,1],$$
$$px(0) - qx'(0) = a,\ \ px(1) + qx'(1) = g^*(x(\gamma), \alpha_{j-1}(\gamma)).$$

Applying Theorem 2.1 (Kamke's convergence theorem) on the sequence $\{\alpha_j\}$ of solutions of the above problem, there exists a function $x(t) \in C^2[0,1]$ and a subsequence $\{\alpha_{j_k}\}$ of $\{\alpha_j\}$ such that $\lim_{j\to\infty} \alpha_{j_k}^{(i)}(t) = x^i(t),\ i = 0, 1,$

uniformly on the compact interval $[0,1]$. Thus, the sequence $\{\alpha_j\}$ converges uniformly in $C^1[0,1]$ to $x(t)$, the unique solution of (1.1) and (1.2).

Now, we prove the quadratic convergence. For that we set $e_j(t) = x(t) - \alpha_j(t),\ a_j = \alpha_j(t) - \alpha_{j-1}(t)$ and note that $e_j(t) \geq 0,\ a_j(t) \geq 0$. Further

$$pe_j(0) - qe_j'(0) = 0, \quad pe_j(1) + qe_j'(1) = g(x(\gamma)) - g^*(\alpha_j(\gamma), \alpha_{j-1}(\gamma)).$$

Using the generalized mean value theorem, (A_2), (3.1) and (3.3), we have

$$\begin{aligned}
e_j''(t) &= x''(t) - \alpha_j''(t)\\
&= F(t,x) - \phi(t,x,x') - G(t,\alpha_j(t),\alpha_{j-1}(t),\alpha_j'(t))\\
&= F(t,x) - \phi(t,x,x') - \big\{F(t,\alpha_{j-1})\\
&\qquad\qquad + F_x(t,\alpha_{j-1})(\alpha_j - \alpha_{j-1}) - \phi(t,\alpha_j,\alpha_j')\big\}\\
&= F_x(t,c_1)(x-\alpha_{j-1}) - F_x(t,\alpha_{j-1})(\alpha_j - \alpha_{j-1})\\
&\qquad\qquad - (\phi(t,x,x') - \phi(t,\alpha_j,\alpha_j'))\\
&= [F_x(t,c_1) - F_x(t,\alpha_{j-1})]e_{j-1}(t) + F_x(t,\alpha_{j-1})e_j(t)\\
&\qquad\qquad - \phi_x(t,c_2,c_3)e_j(t) - (\phi_{x'}(t,c_2,c_3)e_j'(t)\\
&= F_{xx}(t,c_4)(c_1 - \alpha_{j-1})e_{j-1}(t) + [F_x(t,\alpha_{j-1}) - \phi_x(t,c_2,c_3)]e_j(t)\\
&\qquad\qquad - \phi_{x'}(t,c_2,c_3)e_j'(t).\\
&\geq -F_{xx}(t,c_4)e_{j-1}^2(t) + f_{x'}(t,c_2,c_3)e_j'(t),\ t \in [0,1],
\end{aligned}$$

where $\alpha_{j-1} \leq c_1 \leq x,\ \alpha_j \leq c_2 \leq x,\ \alpha_j' \leq c_3 \leq x',\ \alpha_{j-1} \leq c_4 \leq c_1$. In particular, we can write

$$e_j''(t) - f_{x'}(t,c_2,c_3)e_j'(t) \geq -M_1 e_{j-1}^2(t), \tag{3.11}$$

where $M_1 > \max_{(t,x)\in D} F_{xx}(t,x)$ for

$$D = \{(t,x) : 0 < t < 1,\ \alpha(t) \leq x(t) \leq \beta(t)\}.$$

Let $\mu(t) = \exp\{-\int_0^t f_{x'}(s,c_2(s),c_3(s))ds\}$ be the integrating factor associated with (3.11). Then

$$e_j'(t)\mu(t) - e_j'(0) \geq -M_1 e_{j-1}^2 \int_0^t \mu(s)\,ds, \tag{3.12}$$

and in view of $e_j'(0) \geq 0$, it follows that

$$e_j'(t) \geq -M_1 e_{j-1}^2 \int_0^t \mu(s)ds/\mu(t).$$

Thus, there exists $N_1 > 0$ for sufficiently large j such that

$$e_j'(t) \geq -N_1 \|e_{j-1}\|^2,\ 0 \leq t \leq 1.$$

Consequently, we have

$$e_j''(t) \geq -M\|e_{j-1}\|^2,\ M > 0.$$

From (2.1), we have

$$e_j(t) = [g(x(\gamma)) - g^*(\alpha_j(\gamma), \alpha_{j-1}(\gamma))]\left(\frac{t}{p+2q} + \frac{q}{p^2+2pq}\right) + \int_0^1 G(t,s)e_j''(s)\,ds. \tag{3.13}$$

Observe that

$$\begin{aligned} & g(x(\gamma)) - g^*(\alpha_j(\gamma), \alpha_{j-1}(\gamma)) \\ &= g(x(\gamma)) - g(\alpha_{j-1}(\gamma)) - g'(\alpha_{j-1}(\gamma))(\alpha_j(\gamma) - \alpha_{j-1}(\gamma)) \\ &= g'(c_o)e_{j-1}(\gamma) - g'(\alpha_{j-1}(\gamma))(e_{j-1}(\gamma) - e_j(\gamma)) \\ &= g''(c_1)e_{j-1}^2(\gamma) + g'(\alpha_{j-1})e_j(\gamma). \end{aligned} \tag{3.14}$$

Substituting (3.14) in (3.13) and taking the maximum over the interval $[0,1]$, we obtain

$$\|e_j\| \leq M_3\|e_{j-1}\|^2 + \lambda_1\|e_j\| + M_1\|e_{j-1}\|^2, \tag{3.15}$$

where $M_3 = M_2\zeta,\ \zeta = (\frac{1}{p+2q} + \frac{q}{p^2+2pq}),\ M_2$ provides a bound for $\|g''\|$ on $[\alpha_{j-1}(\gamma), x(\gamma)],\ \lambda_1 = \lambda\zeta,\ \|g'\| \leq \lambda < 1$ and $M_1 = \max\int_0^1 M|G(t,s)|ds$. Solving (3.15) algebraically, we get

$$\|e_j\| \leq \delta\|e_{j-1}\|^2,$$

where $\delta = (M_3 + M_1)/(1 - \lambda_1)$ and $\|e_j\| = \max\{|e_j(t)| : t \in [0,1]\}$ is the usual uniform norm. This establishes the quadratic convergence. $\square$

Theorem 3.2. *Assume that*

- $(\mathbf{B_1})$ *α, β are lower and upper solutions of* (1.1) *and* (1.2) *respectively.*
- $(\mathbf{B_2})$ *$\frac{\partial^i}{\partial x^i}f(t,x,y) \in C([0,1] \times R^2)$ for $i = 0,1,2,\ldots,n$ such that $\frac{\partial^i}{\partial x^i}f(t,x,y) > 0$ for $i = 1,2,\ldots,n-1$, $\frac{\partial}{\partial y}(\frac{\partial^i}{\partial x^i}f(t,x,y)) \geq 0$, $\frac{\partial^n}{\partial x^n}(f(t,x,y) + \phi(t,x,y)) \leq 0$, where $\phi \in C^{0,n}[J \times R^2, R]$ with $\frac{\partial^n}{\partial x^n}\phi(t,x,y) \leq 0$. Moreover, f satisfies a Nagumo condition in y.*
- $(\mathbf{B_3})$ *$\frac{d^i}{dx^i}g(x) \in C(R)$ for $i = 0,1,2,\ldots,n$ satisfying $0 \leq \frac{d^i}{dx^i}g(x) < \frac{M}{(\beta-\alpha)^{i-1}}$ for $i = 1,2,\ldots,n-1$ with $0 < M < \frac{1}{3}$ and $\frac{d^n}{dx^n}g(x) \geq 0$.*

Then there exists a monotone sequence of solutions converging monotonically to the unique solution of (1.1) *and* (1.2) *with the order of convergence n $(n \geq 2)$.*

Proof. Let us define

$$f(t,x,y) = F(t,x) - \phi(t,x,y),\ t \in [0,1]. \tag{3.16}$$

In view of (B_2), we note that $F \in C^{0,n}([0,1] \times R)$ and $\frac{\partial^n}{\partial x^n} F(t,x) \leq 0$. Applying the generalized mean value theorem on $F(t,x)$, we get

$$F(t,x) \leq \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} F(t,x_1) \frac{(x-x_1)^i}{i!},$$

which together with (3.16) takes the form

$$f(t,x,y) \leq \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t,x_1,y) \frac{(x-x_1)^i}{i!} - \frac{\partial^n}{\partial x^n} \phi(t,\xi,y) \frac{(x-x_1)^n}{n!}, \quad (3.17)$$

where $x_1 \leq \xi \leq x$. Define

$$G^*(t,x,x_1,y) = \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t,x_1,y) \frac{(x-x_1)^i}{i!} - \frac{\partial^n}{\partial x^n} \phi(t,\xi,y) \frac{(x-x_1)^n}{n!}, \quad (3.18)$$

where ξ depends on x_1. Observe that

$$G^*(t,x,x_1,y) \geq f(t,x,y), \quad G^*(t,x,x,y) = f(t,x,y). \quad (3.19)$$

Moreover, using (B_2), we find that

$$G_x^*(t,x,x_1,y) \geq \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t,x_1,y) \frac{(x-x_1)^{i-1}}{(i-1)!} > 0,$$

which implies that $G^*(t,x,x_1,y)$ is increasing in x for each fixed $(t,x_1,y) \in J \times R^2$. Further, the generalized mean value theorem together with (B_3) gives

$$g(x) \geq \sum_{i=0}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}.$$

Setting

$$g^{**}(x,y) = \sum_{i=0}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!},$$

we notice that

$$g(x) = \max_y g^{**}(x,y), \quad g(x) = g^{**}(x,x). \quad (3.20)$$

Clearly $g_x^{**}(x,y) \geq 0$ and

$$\begin{aligned} g_x^{**}(x,y) &= \sum_{i=1}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!} \leq \sum_{i=1}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \\ &\leq \sum_{i=1}^{n-1} \frac{M}{(i-1)!} < M(3 - \frac{1}{2^{n-3}}) < 3M < 1. \end{aligned}$$

Now, we consider the BVP

$$x''(t) = G^*(t, x(t), \alpha_0(t), x'(t)),\ t \in [0,1], \tag{3.21}$$

$$px(0) - qx'(0) = a, \quad px(1) + qx'(1) = g^{**}(x(\gamma), \alpha_0(\gamma)). \tag{3.22}$$

In view of (B_1), (3.19) and (3.20) yield

$$\alpha_0''(t) \geq f(t, \alpha_0(t), \alpha_0'(t)) = G^*(t, \alpha_0(t), \alpha_0(t), \alpha_0'(t)),\ t \in [0,1],$$
$$p\alpha_0(0) - q\alpha_0'(0) \leq a, \quad p\alpha_0(1) + q\alpha_0'(1) \leq g^{**}(\alpha_0(\gamma), \alpha_0(\gamma)),$$

and

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)) \leq G^*(t, \beta(t), \alpha_0(t), \beta'(t)),\ t \in [0,1],$$
$$p\beta(0) - q\beta'(0) \geq a, \quad p\beta(1) + q\beta'(1) \geq g^{**}(\beta(\gamma), \alpha_0(\gamma)),$$

which imply that α_0 and β are lower and upper solutions of (3.21) and (3.22). Thus, by Theorem 2.4 and Corollary 2.5, there exists a unique solution α_1 of (3.21) and (3.22) such that

$$\alpha_0 \leq \alpha_1 \leq \beta.$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_j\}$ satisfying

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_j \leq \beta,$$

where the element α_j of the sequence $\{\alpha_j\}$ is a solution of the problem

$$x''(t) = G^*(t, x(t), \alpha_{j-1}(t), x'(t)),\ t \in [0,1],$$
$$px(0) - qx'(0) = a, \quad px(1) + qx'(1) = g^{**}(x(\gamma), \alpha_{j-1}(\gamma)),$$

where ξ in G^* (given by (3.18)) depends on α_{j-1}. Employing the arguments used in the preceding theorem, it follows that the sequence $\{\alpha_j\}$ converges in $C^1[0,1]$ to x, the unique solution of (1.1) and (1.2).

Now, we prove the convergence of order $n \geq 2$. For that we set $e_j(t) = x(t) - \alpha_j(t),\ a_{j-1} = \alpha_j(t) - \alpha_{j-1}(t)$ and note that

$$pe_j(0) - qe_j'(0) = 0, \quad pe_j(1) + qe_j'(1) = g(x(\gamma)) - g^{**}(\alpha_j(\gamma), \alpha_{j-1}(\gamma)).$$

Using the generalized mean value theorem, (B_2), (3.16) and (3.18), we can find $\alpha_{j-1} \leq \xi_1 \leq x$ (ξ_1 depends on α_{j-1}) such that

$$\begin{aligned} e_j''(t) &= x''(t) - \alpha_j''(t) \\ &= \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, x') \frac{(x - \alpha_{j-1})^i}{i!} + \frac{\partial^n}{\partial x^n} f(t, \xi_1, x') \frac{(x - \alpha_{j-1})^n}{n!} \\ &- \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{(\alpha_j - \alpha_{j-1})^i}{i!} + \frac{\partial^n}{\partial x^n} \phi(t, \xi, \alpha_j') \frac{(\alpha_j - \alpha_{j-1})^n}{n!} \end{aligned}$$

$$\geq \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha'_j) \frac{e_{j-1}^i}{i!} - \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha'_j) \frac{a_{j-1}^i}{i!}$$
$$+ \left[\frac{\partial^n}{\partial x^n} f(t, \xi_1, x') + \frac{\partial^n}{\partial x^n} \phi(t, \xi, \alpha'_j) \right] \frac{e_{j-1}^n}{n!}$$
$$\geq \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha'_j) \frac{1}{i!} [e_{j-1}^i - a_{j-1}^i] + \left[\frac{\partial^n}{\partial x^n} f(t, \xi, \alpha'_j) \right.$$
$$\left. + \frac{\partial^n}{\partial x^n} \phi(t, \xi, \alpha'_j) \right] \frac{e_{j-1}^n}{n!}$$
$$= \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha'_j) \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-r-1} a_{j-1}^r (e_{j-1} - a_{j-1})$$
$$+ \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha'_j) \frac{e_{j-1}^n}{n!}$$
$$= \left[\sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha'_j) \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-r-1} a_{j-1}^r \right] e_j + \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha'_j) \frac{e_{j-1}^n}{n!}$$
$$= \omega(t) e_j + \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha'_j) \frac{e_{j-1}^n}{n!} \geq \omega(t) e_j - \epsilon_1 e_{j-1}^n, \; t \in [0, 1],$$

where $\omega(t) = \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha'_j) \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-r-1} a_{j-1}^r > 0$ and $\frac{1}{n!} \frac{\partial^n}{\partial x^n} F (t, \xi, \alpha'_j) \geq -\epsilon_1$, for some $\epsilon_1 > 0$. Here, we have chosen $\xi_1 = \xi$ (in general ξ_1 and ξ, depending on α_{j-1}, are independent) as $\alpha_{j-1} \leq \xi, \; \xi_1 \leq x$. Thus, for each j, we have

$$e''_j(t) \geq -\epsilon_1 e_{j-1}^n, \; t \in [0, 1]. \tag{3.23}$$

As before, from (2.1), we have

$$e_j(t) = [g(x(\gamma)) - g^{**}(\alpha_j(\gamma), \alpha_{j-1}(\gamma))] \left(\frac{t}{p+2q} + \frac{q}{p^2 + 2pq} \right) + \int_0^1 G(t, s) e''_j(s) ds. \tag{3.24}$$

Clearly

$$g(x(\gamma)) - g^{**}(\alpha_j(\gamma), \alpha_{j-1}(\gamma)) = \sum_{i=0}^{n-1} \frac{d^i}{dx^i} g(\alpha_{j-1}(\gamma)) \frac{(x(\gamma) - \alpha_{j-1}(\gamma))^i}{i!}$$
$$+ \frac{d^n}{dx^n} g(\xi(\gamma)) \frac{(x(\gamma) - \alpha_{j-1}(\gamma))^n}{n!} - \sum_{i=0}^{n-1} \frac{d^i}{dx^i} g(\alpha_{j-1}(\gamma)) \frac{(\alpha_j(\gamma) - \alpha_{j-1}(\gamma))^i}{i!}$$
$$= \sum_{i=1}^{n-1} \frac{d^i}{dx^i} g(\alpha_{j-1}(\gamma)) \frac{(e_{j-1}^i(\gamma) - a_{j-1}^i(\gamma))}{i!} + \frac{d^n}{dx^n} g(\xi(\gamma)) \frac{(e_{j-1}(\gamma))^n}{n!}$$

$$= \sum_{i=1}^{n-1} \frac{d^i}{dx^i} g(\alpha_{j-1}(\gamma)) \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{r}(\gamma) a_{j-1}^{i-1-r}(\gamma) e_j(\gamma) + \frac{d^n}{dx^n} g(\xi(\gamma)) \frac{(e_{j-1}(\gamma))^n}{n!}$$

$$\leq \left[\sum_{i=0}^{n-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-1-r}(\gamma) a_{j-1}^{r}(\gamma) \right] e_j(\gamma) + \epsilon_3 e_{j-1}^n, \tag{3.25}$$

where ϵ_3 provides a bound for $\frac{1}{n!}\frac{d^n}{dx^n} g(\xi(\gamma))$. Letting

$$P_j(t) = \sum_{i=0}^{n-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-1-r}(\gamma) a_{j-1}^{r}(\gamma),$$

we observe that

$$\lim_{j\to\infty} P_j(t) = \lim_{j\to\infty} \sum_{i=0}^{n-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-1-r}(\gamma) a_{j-1}^{r}(\gamma) = M < \frac{1}{3}.$$

Therefore, we can choose $\lambda_1 < \frac{1}{3}$ and $j_0 \in N$ such that for $n \geq j_0$, we have $P_j(t) < \lambda_2$. Thus, using (3.23) and (3.25) in (3.24) and taking the maximum over the interval $[0, 1]$, we obtain

$$\|e_j\| \leq \epsilon_4 \|e_{j-1}\|^n + \lambda_3 \|e_j\| + \epsilon_5 \|e_{j-1}\|^n, \tag{3.26}$$

where $\epsilon_4 = \epsilon_3 \zeta$, $\lambda_3 = \lambda_2 \zeta$, $\zeta = (\frac{1}{p+2q} + \frac{q}{p^2+2pq})$ and $\epsilon_5 = \max \int_0^1 \epsilon_1 |G(t,s)| ds$. Solving (3.26) algebraically, we get

$$\|e_j\| \leq \epsilon \|e_{j-1}\|^n,$$

where $\delta = (\epsilon_4 + \epsilon_5)/(1 - \lambda_3)$. This establishes the convergence of order n $(n \geq 2)$. □

Remark 3.3. It is clear that Theorem 3.2 remains valid if we replace the condition $\frac{\partial^i}{\partial x^i} f(t, x, x') > 0$ for $i = 1, 2, \ldots, n-1$ in (B_2) by that of $\Gamma f(t, x, x') > 0$ with $\frac{\partial}{\partial x} f(t, x, x') > 0$, where $\Gamma = \sum_{i=1}^{n-1} \frac{\partial^i(.)}{\partial x^i} \frac{(x-y)^{i-1}}{(i-1)}$.

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