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SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR A CLASS OF THIRD-ORDER FUNCTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. Consider the boundary value problems consisting of the functional difference equation

$$\Delta^3 x(n) = f(n, x(n+2), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))), \quad n \in [0, T]$$

and the following boundary value conditions

$$\begin{cases} x(0) = x(T+3) = x(1) = 0, \\ x(n) = \psi(n), \ n \in [-\tau, -1], \\ x(n) = \phi(n), \ n \in [T+4, T+\delta]. \end{cases}$$

Sufficient conditions for the existence of at least one solution of this problem are established. We allow f to be at most linear, superlinear or sublinear in the obtained results.

1. INTRODUCTION

In this paper, we consider the existence of solutions of the problem

$$\begin{cases} \Delta^3 x(n) = f(n, x(n+2), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))), \quad n \in [0, T], \\ x(0) = x(T+3) = x(1) = 0, \\ x(n) = \psi(n), \quad n \in [-\tau, -1], \\ x(n) = \phi(n), \quad n \in [T+4, T+\delta] \end{cases} \tag{1}$$

where $\tau_i : [0, T] \to N$, $i = 1, \dots, m$, are sequences with $T \geq 1$,

$$\tau = \max \left\{ \max_{n \in [0,T]} \{0, \tau_i(n)\} : \quad i = 1, \dots, m \right\},$$

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$$\delta = -\min\left\{0, \min_{n\in[0,T]}\{0, \tau_i(n)\} : \quad i = 1, \ldots, m\right\},$$

$f(n, u)$ is continuous about $u = (x_0, \ldots, x_m)$ for each n. The motivation of this paper is as follows:

In [1], Agarwal and Henderson considered the BVP

$$\begin{cases} \Delta^3 x(n) + a(n)f(x(n)) = 0, & n \in [0, T], \\ x(0) = x(T + 3) = x(1) = 0, \end{cases}$$

where $f : [0, +\infty) \to [0, +\infty)$ is continuous and a is a positive sequence. Let

$$f_0 = \lim_{x\to 0} \frac{f(x)}{x}, \quad f_\infty = \lim_{x\to+|infty} \frac{f(x)}{x}.$$

The authors' obtained results on the existence of positive solutions of the above mentioned BVP based on the limits f_0 and f_∞. Their main result is that if $f_0 = 0, f_\infty = \infty$ or $f_0 = \infty, f_\infty = 0$, then the above BVP has at least one positive solution.

In [2], Kong, Kong and Zhang studied the following BVP

$$\begin{cases} \Delta^3 x(n) + a(n)f(n, x(\omega(n))) = 0, & n \in [0, T], \\ x(n) = \phi(n), & n \in [n_1, 0], \\ x(0) = x(1) = x(T + 3) = 0, \\ x(n) = \psi(n), & n \in [T + 3, n_2], \end{cases}$$

where $f : [0, T]\times[0, +\infty) \to [0, \infty)$ is continuous and a is a positive sequence, ω satisfies

$$p = \min_{n\in[0,T]} \omega(n) \le T + 2, \quad q = \max_{n\in[0,T]} \omega(n) \ge 2,$$

and ψ, ϕ are sequences with $n_1 = \min\{T + 3, p\}$ and $n_2 = \max\{0, q\}$, and $\phi(0) = \psi(T + 3) = 0$. The authors establish existence results for positive solutions of the above mentioned problem under the assumptions:

$$f(n, y) \le M_1\lambda - N_1, \quad (n, y) \in E \times [2\lambda/((T + 1)(T + 2)), \lambda],$$

and

$$f(n, y) \ge M_1\lambda - N_2, \quad (n, y) \in E \times [2\eta/((T + 1)(T + 2)), \eta],$$

where $\lambda, \eta, M_1, N_1, N_2, E$ are defined in paper [2].

To the best of my knowledge, no paper has discussed the existence of solutions of problem (1). The purpose of this paper is to establish sufficient conditions for the existence of at least one solutions of problem (1). The methods of this paper are based upon the coincidence degree theory.

This paper is organized as follows. In Section 2, we give the main result, and in Section 3, an example that illustrates the main result will be presented.

2. Main results

Let $X = R^{T+\tau+\delta+1}$ be endowed with the norm $||x||_X = \max_{n\in[1,T+\tau+\delta+1]} |x(n)|$, $Y = R^{T+1}$ be endowed with the norm $||y||_Y = \max_{n\in[0,T]} |y(n)|$. It is easy to see that X and Y are Banach spaces. Choose

$$\mathrm{Dom}L = \left\{ x = (\overbrace{0,\ldots,0,x_2,\ldots,x_{T+2},0,\ldots,0}^{T+\tau+\delta+1}) \in X \right\}.$$

Set

$$L : \mathrm{Dom}\, L \cap X \to X, \quad L \bullet y(n) = \Delta^3 y(n), \quad n \in [0,T],$$

and $N : X \to Y$ by

$$N \bullet y(n) = f(n, y(n+2) + x_0(n+2), y(n-\tau_1(n)) + x_0(n-\tau_1(n)), \ldots, y(n-\tau_m(n)) + x_0(n-\tau_m(n)))$$

$n \in [0,T]$, for all $y \in X$, where

$$x_0(n) = \begin{cases} \phi(n), & n \in [-\tau,-1], \\ 0, & n \in [0,T+3], \\ \psi(n), & n \in [T+4, T+\delta]. \end{cases}$$

It is easy to show that $\Delta^3 x_0(n) = 0$ for $n \in [0,T]$ and that if $y \in \mathrm{Dom}L$ is a solution of $L \bullet x = N \bullet x$ then $y + x_0$ is a solution of problem (1). It is easy to check the following results.

(i). $\mathrm{Ker}L = \{0\}$.

(ii). L is a Fredholm operator of index zero.

(iii). Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \mathrm{Dom}L \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$.

Suppose

(B_1)**.** There exist numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p(n)$, $p_i(n), r(n) (i = 1,\ldots,m)$, functions $g(n,x_0,\ldots,x_m)$, $h(n,x_0,\ldots,x_m)$ such that

$$f(n,x_0,\ldots,x_m) = g(n,x_0,\ldots,x_m) + h(n,x_0,\ldots,x_m),$$
$$g(n,x_0,x_1,\ldots,x_m)x_0 \geq \beta |x_0|^{\theta+1},$$

and

$$|h(n,x_0,\ldots,x_m)| \leq \sum_{s=0}^{m} p_i(n)|x_i|^\theta + r(n),$$

for all $n \in \{1,\ldots,T\}$, $(x_0,x_1,\ldots,x_m) \in R^{m+1}$.

Lemma 2.1. *Let* $\Omega_1 = \{x : Ly = \lambda Ny,\ (y,\lambda) \in (DomL) \times (0,1)\}$*. Suppose* (B_1) *holds. Then* Ω_1 *is bounded if*

$$||p_0|| + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| < \beta. \tag{2}$$

Proof. For $y \in \Omega_1$, we have $L\bullet y = \lambda N\bullet y$, $\lambda \in (0,1)$. Let $x(n) = y(n)+x_0(n)$. Then

$$\Delta^3 x(n) = \lambda f(n, x(n+2), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))), \quad n \in [0,T].$$

So

$$\begin{aligned}\Delta^3 x(n)x(n+2) = \lambda f(n, x(n+2), x(n-\tau_1(n)), \ldots,&\\ x(n-\tau_m(n)))x(n+2), \quad n \in [0,T].&\end{aligned} \tag{3}$$

It is easy to see from $x(0) = x(1) = x(T+3) = 0$ that

$$\begin{aligned}
&\sum_{n=0}^{T} [\Delta^3 x(n))]x(n+2)\\
&= \sum_{n=0}^{T} [\Delta^2 x(n+1) - \Delta^2 x(n)][x(n+3) - \Delta x(n+2)]]\\
&= \sum_{n=0}^{T} \Big[\ \Delta^2 x(n+1)x(n+3) - \Delta^2 x(n))x(n+2)\Big] - \sum_{n=0}^{T+1} \Delta^2 x(n+1)\Delta x(n+2)\\
&= -\Delta^2 x(0)x(2) - \sum_{n=0}^{T} \Delta^2 x(n+1)\Delta x(n+2)\\
&= -[x(2)]^2 - \sum_{n=0}^{T} [(\Delta x(n+2))^2 - \Delta x(n+2)\Delta x(n+1)]\\
&\le -\frac{1}{2} \sum_{n=0}^{T} [\Delta x(n+2) - \Delta x(n+1)]^2 - \frac{1}{2}[\Delta x(T+2)]^2\\
&= -\frac{1}{2} \sum_{n=0}^{T} [\Delta x(n+2) - \Delta x(n+1)]^2 - \frac{1}{2}[x(T+2)]^2 \le 0.
\end{aligned}$$

So, we get

$$\sum_{n=0}^{T} f(n, x(n+2), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n)))x(n+2) \le 0.$$

It follows from (B_1) that

$$
\begin{aligned}
&\beta\sum_{n=0}^{T}|x(n+2)|^{\theta+1}\\
&\le \sum_{n=0}^{T} g(n,x(n+2),x(n-\tau_1(n)),\dots,x(n-\tau_m(n)))x(n+2)\\
&\le -\sum_{n=0}^{T} h(n,x(n+2),x(n-\tau_1(n)),\dots,x(n-\tau_m(n)))x(n+2)\\
&\le \sum_{n=0}^{T} |h(n,x(n+2),x(n-\tau_1(n)),\dots,x(n-\tau_m(n)))||x(n+2)|\\
&\le \sum_{i=1}^{m}\sum_{n=1}^{T} p_i(n)|x(n-\tau_i(n))|^{\theta}|x(n+2)|\\
&\qquad\qquad + \sum_{n=0}^{T} r(n)|x(n+2)| + \sum_{n=0}^{T} p_0(n)|x(n+2)|^{\theta+1}\\
&\le \sum_{i=1}^{m}||p_i||\sum_{n=0}^{T}|x(n-\tau_i(n))|^{\theta}|x(n+2)|\\
&\qquad\qquad + ||r||\sum_{n=0}^{T}|x(n+2)| + ||p_0||\sum_{n=0}^{T}|x(n+2)|^{\theta+1}.
\end{aligned}
$$

For $x_i \ge 0, y_i \ge 0$, Hölder inequality implies

$$
\sum_{i=1}^{s} x_i y_i \le \left(\sum_{i=1}^{s} x_i^p\right)^{1/p}\left(\sum_{i=1}^{s} y_i^q\right)^{1/q},\ \ 1/p+1/q=1,\ \ q>0,\ p>0.
$$

It follows that

$$
\begin{aligned}
&\beta\sum_{n=0}^{T}|x(n+2)|^{\theta+1}\\
&\le \sum_{i=1}^{m}||p_i||\left(\sum_{n=0}^{T}|x(n-\tau_i(n))|^{\theta+1}\right)^{\frac{\theta}{1+\theta}}\left(\sum_{n=0}^{T}|x(n+2)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\
&+(T+1)^{\frac{\theta}{\theta+1}}||r||\left(\sum_{n=0}^{T}|x(k)|^{\theta+1}\right)^{\frac{1}{\theta+1}} + ||p_0||\sum_{n=0}^{T}|x(n+2)|^{\theta+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} ||p_i|| \Bigg(\sum_{\substack{u \in \{k-\tau_i(k)-2: \\ k \in [0,T]\}}} |x(u+2)|^{\theta+1} \Bigg)^{\frac{\theta}{1+\theta}} \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} \\
&+ (T+1)^{\frac{\theta}{\theta+1}} ||r|| \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} + ||p_0|| \sum_{n=0}^{T} |x(n+2)|^{\theta+1} \\
&\leq (T+1)^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \sum_{n=0}^{T} |x(n+2)|^{\theta+1} \\
&+ (T+1)^{\frac{\theta}{\theta+1}} ||r|| \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^{m} ||p_i|| \Bigg((T+1) \sum_{s=-\tau}^{-1} |\psi(s)|^{\theta+1} \Bigg)^{\frac{\theta}{1+\theta}} \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^{m} ||p_i|| \Bigg((T+1) \sum_{s=T+4,T+\delta]} |\phi(s)|^{\theta+1} \Bigg)^{\frac{\theta}{1+\theta}} \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} \\
&+ ||p_0|| \sum_{n=0}^{T} |x(n+2)|^{\theta+1}.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\Bigg(\beta - ||p_0|| - (T+1)^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \Bigg) \sum_{n=0}^{T} |x(n+2)|^{\theta+1} \\
&\leq (T+1)^{\frac{\theta}{\theta+1}} ||r|| \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^{m} ||p_i|| \Bigg((T+1) \sum_{s=-\tau}^{-1} |\psi(s)|^{\theta+1} \Bigg)^{\frac{\theta}{1+\theta}} \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}} \\
&+ \sum_{i=1}^{m} ||p_i|| \Bigg((T+1) \sum_{s=T+4,T+\delta]} |\phi(s)|^{\theta+1} \Bigg)^{\frac{\theta}{1+\theta}} \Bigg(\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \Bigg)^{\frac{1}{\theta+1}}.
\end{aligned}
$$

It follows from (4) that there is an $M_1 > 0$ such that $\sum_{n=0}^{T} |x(n+2)|^{\theta+1} \leq M_1$.

Hence $|x(n+2)| \leq (M_1/(T+1))^{1/(\theta+1)}$ for all $n \in \{0, \ldots, T\}$. Hence $||x|| \leq (M_1/(T+1))^{1/(\theta+1)}$. It follows that $|y(n)| = |x(n) - x_0(n)| \leq |x(n)| + |x_0(n)| \leq (M_1/(T+1))^{1/(\theta+1)} + ||x_0||$. So Ω_1 is bounded. The proof is complete. $\square$

Let X and Y be Banach spaces, $L: D(L) \subset X \to Y$ be a Fredholm operator of index zero.

If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N: X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \overline{\Omega} \to X$ is compact.

Lemma 2.2. [3] *Let X and Y be Banach spaces. Suppose $L: D(L) \subset X \to Y$ is a Fredholm operator of index zero with* $\mathrm{Ker}L = \{0\}$, *$N: X \to Y$ is L-compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is a open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0,1]$, then there is at least one $x \in \Omega$ so that $Lx = Nx$.*

Theorem L. *Suppose that* (B_1) *holds. Then equation* (1) *has at least one solution if* (2) *holds.*

Proof. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \Omega_1$ is centered at zero. It is easy to see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Thus, from Lemma 2.2, $Lx = Nx$ has at least one solution $x \in D(L) \cap \overline{\Omega}$, So x is a solution of problem (1). The proof is complete. □

3. An example

In this section, we present an example to illustrate the main result in Section 2.

Example 3.1. Consider the problem

$$\begin{cases} \Delta^3 x(n) = \beta[x(n+2)]^{2k+1} + \sum_{i=1}^{m} p_i(n)[x(n-i)]^{2k+1} + r(n), \\ x(0) = x(1) = x(T+3) = 0, \\ x(i) = \psi(i),\ i \in [-m,-1], \end{cases} \tag{4}$$

where k, m are a positive integers, $p_1(n), \ldots, p_m(n), r(n)$ are sequences, $\beta > 0$, $\tau_i(n)$ are sequences. Corresponding to the assumptions of Theorem L, we set

$$g(n, x_0, x_1, \ldots, x_m) = \beta x_0^{2k+1},$$

and

$$h(n, x_0, \ldots, x_m) = \sum_{i=1}^{m} p_i(n) x_i^{2k+1} + r(n)$$

with $\theta = 2k+1$. It is easy to see that (B_1) holds. It follows from Theorem L that (4) has at least one solution if

$$T^{\frac{2k+1}{2k+2}} \sum_{i=1}^{m} ||p_i|| < \beta.$$

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