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ON HYERS-ULAM STABILITY OF WILSON'S FUNCTIONAL EQUATION ON P_3-GROUPS

MINGXING LUO

ABSTRACT. The purposes of paper is to obtain the Hyers-Ulam stability of Wilson's equation $\varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x)\phi(y)$ for $\varphi, \phi : G \to K$, where G is a P_3- group and K a field with char$K \neq 2$.

1. INTRODUCTION

In 1989, Aczél, Chung and Ng have solved Wilson's equation,

$$\varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x)\phi(y) \tag{1.1}$$

assuming that the function ϕ satisfies Kannappan's condition $\phi(xyz) = \phi(xzy)$ and $\varphi(xy) = \varphi(yx)$ for all $x, y, z \in G$.

Let $(G;+)$ be a topological abelian group and let K be a compact subgroup of automorphisms of G with the normalized Haar measure μ. Assume that the topologies on K and G are related in such a way that the map $k \mapsto ky \in G$, $k \in K$ is continuous for each fixed $y \in G$, where ky denotes the action of $k \in K$ on $y \in G$. We say that a continuous function $\varphi : G \to C$ is K-spherical if and only if there exists a non-zero continuous function $\phi : G \to C$ such that

$$\int_K \phi(x + ky)d\mu(k) = \phi(x)\varphi(y) \tag{1.2}$$

for all $x, y \in G$. Equivalently, a non-zero continuous function $\varphi : G \to C$ is K-spherical if it satisfies the integral equation $\int_K \varphi(x+ky)d\mu(k) = \varphi(x)\varphi(y)$ for all $x, y \to G$. R. Badora[4] has studied the Hyers-Ulam stability of Wilson's functional equation for spherical functions.

Classical examples of (1.1) are d'Alembert's functional equation $\varphi(x + y) + \varphi(x - y) = 2\varphi(x)\varphi(y)$, where $K = \{Id, -Id\}$ and Cauchy's equation $\varphi(x+y) = \varphi(x)\varphi(y)$ with $K = \{Id\}$. The generalization for (1.2) of Wilson's functional equation (1.1) was considered discussed by W. Chojnacki [6], R.

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Badora [3, 4] and H. Stetkaer [17, 18]. For (1.2) with K finite this problem was solved by W. Förg-Rob and J. Schwaiger in [12], R. Badora in [3], and for d'Alembert's functional equation by J. Baker in [5].

Several papers deal with Wilson's functional equation, see e.g. the monograph [1] by Aczél and Dhombres for references and results. Aczél, Chung and Ng [2], where K is a quadratically closed field of char$K \neq 2$, assuming that the function g satisfies Kannappan's condition, $\phi(xyz) = \phi(xzy)$ for all $x, y, z \in G$ and $\varphi(xy) = \varphi(yx)$. Penney and Rukhin [15] found square integrable solutions of a version of the equation (1.1). Sinopoulos [16] has determined the general solution of (1.1) where G is a 2-divisible abelian group, φ is a vector-valued function and ϕ is a matrix-valued function. Also, Wilson's equation was investigated in the contex of spherical functions on groups by Stetkaer [18]. In this paper we study the problem of the Hyers-Ulam stability of equation (1.1) for K a P_3-Group, if the commutator subgroup K_0 of K, which is generated by all commutators $[x, y] := x^{-1}y^{-1}xy$, has order one or two.

2. Main results on stability

The main results on stability are contained in the following

Theorem 1. *Let $\varphi, \phi; G \to K$ be continuous functions, where G is a P_3-group and K is a quadratically closed field with char$K \neq 2$; also K is Abelian under multiplication. Assume that there exists a $c \geq 0$ such that*

$$\|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)\phi(y)\| \leq c, \quad x, y \in K \tag{2.1}$$

Then either

(i) *φ, ϕ are bounded or*

(ii) *φ is unbounded and*

$$\phi(x) = \frac{1}{2} \lim_{n\to\infty} \varphi(u_n)^{-1}(\varphi(u_n x) + \varphi(u_n x^{-1})), \tag{2.2}$$

satisfies

$$\phi(y) = \phi(y^{-1}), \quad \varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x)\varphi(y), \tag{2.3}$$

or

(iii) *ϕ is unbounded,*

$$\varphi(x) = \frac{1}{2} \lim_{n\to\infty} (\varphi(xu_n) + \varphi(xu_n^{-1}))\phi(u_n)^{-1}, \tag{2.4}$$

and φ, ϕ satisfies (1.1).

Corollary 1. *Let $\varphi : G \to K$ is continuous, if there exists a $c \geq 0$, then φ is bounded or satisfied* (1.1).

Consider the signed Wilson's functional equation

$$\varphi(xy\Lambda(xy)) + \varphi(xy^{-1}\Lambda(xy^{-1})) = 2\varphi(x)\phi(y), \tag{2.5}$$

where $\Lambda : G_0 \to T = \{a : |a| = e\}$ and G_0 is the commutator subgroup of group G, which is generated by all commutators $[x, y] := x^{-1}y^{-1}xy$, has order one or two.

If we set $\Lambda \equiv e$ with e is unit of K, we can obtain (1.1).

Theorem 2. *Let $\varphi, \phi, G \to K$ be continuous functions and Λ be G-even, where G is a P_3-group and K is a quadratically closed field with $charK \neq 2$, also K is Abelian under multiplication. Assume that there exists a $c \geq 0$ such that*

$$\|\varphi(xy\Lambda(xy)) + \varphi(xy^{-1}\Lambda(xy^{-1})) - 2\varphi(x)g(y)\| \leq c, \qquad x, y \in K \tag{2.6}$$

Then either

(i) *φ, ϕ are bounded or*

(ii) *φ is unbounded and*

$$\phi(x) = \frac{1}{2}\lim_{n\to\infty} \varphi(u_n)^{-1}(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))). \tag{2.7}$$

satisfies

$$\phi(x) = \phi(x^{-1}), \quad \phi(xy\Lambda(xy)) + \phi(xy^{-1}\Lambda(xy^{-1})) = 2\phi(x)\phi(y), \tag{2.8}$$

or

(iii) *ϕ is unbounded,*

$$\varphi(x) = \frac{1}{2}\lim_{n\to\infty}(\varphi(xu_n\Lambda(xu_n)) + \varphi(xu_n^{-1}\Lambda(xu_n^{-1})))\phi(u_n)^{-1}, \tag{2.9}$$

and φ, ϕ satisfies (2.6).

3. Proofs of theorems

Proof of Theorem 1. Let

$$f(x, y) = \varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)\phi(y), \quad x, y \in G,$$

then we obtain

$$\|f(x, y)\| \leq c, \quad x, y \in G. \tag{3.10}$$

Furthermore, we get identities

$$\begin{aligned} f(x, y) - f(x, y^{-1}) &= \varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)\phi(y) \\ &\qquad - [\varphi(xy^{-1}) + \varphi(xy) - 2\varphi(x)\phi(y)] \\ &= 2\varphi(x)(\phi(y^{-1}) - \phi(y)). \end{aligned} \tag{3.11}$$

If $\varphi = 0$, φ is solution of (1.1).

If φ is unbounded, from (3.11) we get $\phi(y) = \phi(y^{-1})$, i.e., ϕ is even. We now prove (ii) and (iii). Assuming (ii), there exists a sequence $\{u_n\}, m \in N$ in K such that

$$\varphi(u_n) \neq 0, \quad \text{and} \quad \lim_{n\to\infty} \|\varphi(u_n)\| = +\infty. \tag{3.12}$$

Let $x = u_n, y = x$ in (2.1), we get

$$\|\varphi(u_n x) + \varphi(u_n x^{-1}) - 2\varphi(u_n)\phi(x)\| \le c, \quad x, y \in K. \tag{3.13}$$

Then we obtain

$$\|\varphi(u_n)^{-1}(\varphi(u_n x) + \varphi(u_n x^{-1})) - 2\phi(x)\| \le \frac{c}{\|\varphi(u_n)\|}, \quad x, y \in K.$$

Consequently

$$\lim_{n\to\infty} \varphi(u_n)^{-1}(\varphi(u_n x) + \varphi(u_n x^{-1})) = 2\phi(x). \tag{3.14}$$

Now for each $x, y, z \in K$ and $n \in N$, by setting $x = u_n, y = xz$ in (2.1) we obtain

$$\|\varphi(u_n xz) + \varphi(u_n (xz)^{-1}) - 2\varphi(u_n)\phi(xz)\| \le c, \quad x, y \in K. \tag{3.15}$$

then

$$\lim_{n\to\infty} \varphi(u_n)^{-1}(\varphi(u_n xz) + \varphi(u_n (xz)^{-1})) = 2\phi(xz). \tag{3.16}$$

By the arbitrariness of z, (3.14) converges to a unique function ϕ which satisfies (2.3). In fact,

$$\begin{aligned}
&\|\varphi^{-1}(u_n)(\varphi(u_n xy) + \varphi(u_n (xy)^{-1})) + \varphi^{-1}(u_n)(\varphi(u_n xy^{-1}) \\
&\qquad + \varphi(u_n (xy^{-1})^{-1})) - 2\varphi^{-1}(u_n)(\varphi(u_n x) + \varphi(u_n x^{-1}))\phi(y)\| \\
&\le \|\varphi^{-1}(u_n)(\varphi(u_n xy) + \varphi(u_n xy^{-1}) - 2\varphi(u_n x)\phi(y))\| \\
&\qquad + \|(\varphi(u_n (xy)^{-1}) + \varphi(u_n (xy^{-1})^{-1}) - 2\varphi(u_n x^{-1})\phi(y)))\| \\
&\le 2c\|\varphi^{-1}(u_n)\|,
\end{aligned} \tag{3.17}$$

here we have used Kannappan's condition on φ to get (3.17). Then taking limits in (3.17) we get that ϕ satisfies (2.3). Hence (ii) is proved.

If ϕ is unbounded, there exists a sequence $\{u_n\}, m \in N$ in K such that

$$\phi(u_n) \neq 0, \quad u_n \neq 0, \quad \text{and} \quad \lim_{n\to\infty} \|\phi(u_n)\| = +\infty. \tag{3.18}$$

By setting $y = u_n$ in (2.1) we obtain

$$\|\varphi(xu_n) + \varphi(xu_n^{-1}) - 2\varphi(x)\phi(u_n)\| \le c, \quad x, y \in K. \tag{3.19}$$

Then we obtain

$$\|(\varphi(xu_n) + \varphi(xu_n^{-1}))\phi(u_n)^{-1} - 2\varphi(x)\| \le \frac{c}{\|\phi(u_n)\|}, \quad x, y \in K.$$

Consequently

$$\lim_{n\to\infty}(\varphi(xu_n)+\varphi(xu_n^{-1}))\phi(u_n)^{-1}=2\varphi(x). \tag{3.20}$$

Now for each $x,y,z\in K$ and $n\in N$, by setting $x=xz, y=u_n$, in (2.1) we obtain

$$\|\varphi(xzu_n)+\varphi((xz)u_n^{-1})-2\varphi(xz)\phi(u_n)\|\le c,\quad x,y\in K. \tag{3.21}$$

Hence

$$\lim_{n\to\infty}(\varphi(xzu_n)+\varphi((xz)u_n^{-1}))\phi(u_n)^{-1}=2\varphi(xz). \tag{3.22}$$

By the arbitrariness of z, (3.20) converges to a unique function φ which satisfies (1.1). In fact,

$$\begin{aligned}
&\|(f(u_nxy)+f(u_n(xy)^{-1}))\phi^{-1}(u_n)+(f(u_nxy^{-1})+f(u_n(xy^{-1})^{-1}))\phi^{-1}(u_n)\\
&\qquad\qquad\qquad -2\phi(y)(\varphi(u_nx)+\varphi(u_nx^{-1}))\phi^{-1}(u_n)\|\\
&\le\|(\varphi(u_nxy)+\varphi(u_nxy^{-1}))\phi^{-1}(u_n)-2\phi(y)\varphi(u_nx)\phi^{-1}(u_n)\|\\
&\quad+\|(\varphi(u_n(xy)^{-1})+\varphi(u_n(xy^{-1})^{-1})\phi^{-1}(u_n)-2\phi(y)\varphi(u_nx^{-1})\phi^{-1}(u_n)\|\\
&\le 2c\|\varphi^{-1}(u_n)\|,
\end{aligned} \tag{3.23}$$

here we have used Kannappan's condition on φ to get (3.23). Then taking limits in (3.23) we get that ϕ satisfies (2.4), and φ,ϕ satisfy (1.1). Then (iii) is proved. Then case (i) is also proved. □

Remark 3.1. If φ is bounded, then in (iii) $\varphi=0$, moreover, φ,ϕ satisfy (2.1).

Proof of Theorem 2. Let

$$f(x,y)=\varphi(xy\Lambda(xy))+\varphi(xy^{-1}\Lambda(xy^{-1}))-2\varphi(x)\phi(y),\quad x,y\in G,$$

then we obtain

$$\|f(x,y)\|\le c,\quad x,y\in G. \tag{3.24}$$

Furthermore, we get identities

$$\begin{aligned}
f(x,y)-f(x,y^{-1})&=\varphi(xy\Lambda(xy))+\varphi(xy^{-1}\Lambda(xy^{-1}))-2\varphi(x)g(y)\\
&\qquad-[\varphi(xy^{-1}\Lambda(xy^{-1}))+\varphi(xy\Lambda(xy))-2\varphi(x)\phi(y)]\\
&=2\varphi(x)(\phi(y^{-1})-\phi(y)).
\end{aligned} \tag{3.25}$$

If $\varphi=0$, φ is solution of (2.5). If φ is unbounded, from (3.25) we get $\phi(y)=\phi(y^{-1})$, i.e., ϕ is even.

Also there exists a sequence $\{u_n\}, m\in N$ in K such that

$$\varphi(u_n)\neq 0,\quad\text{and}\quad\lim_{n\to\infty}\|\varphi(u_n)\|=+\infty. \tag{3.26}$$

Let $x = u_n, y = x$ in (2.6), we get

$$\|\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1})) - 2\varphi(u_n)g(x)\| \le c, \quad x, y \in K. \tag{3.27}$$

Then we obtain

$$\|\varphi(u_n)^{-1}(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) - 2g(x)\| \le \frac{c}{\|\varphi(u_n)\|}, \ x, y \in K.$$

Consequently

$$\lim_{n\to\infty} \varphi(u_n)^{-1}(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) = 2\phi(x). \tag{3.28}$$

Now for each $x, y, z \in K$ and $n \in N$, by setting $x = u_n, y = xz$ in (2.6) we obtain

$$\|\varphi(u_n xz\Lambda(u_n xz)) + \varphi(u_n (xz)^{-1}\Lambda(u_n xz^{-1})) - 2\varphi(u_n)\phi(xz)\| \le c, \quad x, y \in K, \tag{3.29}$$

then

$$\lim_{n\to\infty} \varphi(u_n)^{-1}(\varphi(u_n xz\Lambda(u_n xz)) + \varphi(u_n (xz)^{-1}\Lambda(u_n (xz)^{-1}))) = 2\phi(xz). \tag{3.30}$$

By the arbitrariness of z, (3.28) converges to a unique function ϕ which satisfies (2.6). In fact,

$$\begin{aligned}
&\|\varphi^{-1}(u_n)(\varphi(u_n xy\Lambda(u_n xy)) + \varphi(u_n (xy)^{-1}\Lambda(u_n (xy)^{-1})))\\
&\qquad + \varphi^{-1}(u_n)(\varphi(u_n xy^{-1}\Lambda(u_n xy^{-1})) + \varphi(u_n (xy^{-1})^{-1}\Lambda(u_n y^{-1}x)))\\
&\qquad - 2\varphi^{-1}(u_n)(\varphi(u_n x) + \varphi(u_n x^{-1}))\phi(y)\|\\
&\le \|\varphi^{-1}(u_n)(\varphi(u_n xy\Lambda(u_n xy)) + \varphi(u_n (xy)^{-1}\Lambda(u_n (xy)^{-1})) - 2\varphi(u_n x)\phi(y))\|\\
&\quad + \|\varphi^{-1}(u_n)(\varphi(u_n xy^{-1}\Lambda(u_n xy^{-1})) + \varphi(u_n y^{-1}x\Lambda(u_n y^{-1}x))\\
&\quad - 2\varphi(u_n x^{-1})\phi(y))\| \le 2c\|\varphi^{-1}(u_n)\|,
\end{aligned} \tag{3.31}$$

where we have used Kannappan's condition on φ and (2.6) to get (3.31). Then by taking limits in (3.31), we get that ϕ satisfies (2.8). Then (ii) is proved.

If ϕ is unbounded, there exists a sequence $\{u_n\}, m \in N$ in K such that

$$\phi(u_n) \ne 0, \quad u_n \ne 0, \quad \text{and} \quad \lim_{n\to\infty} \|\phi(u_n)\| = +\infty. \tag{3.32}$$

By setting $y = u_n$ in (2.1) we obtain

$$\|\varphi(xu_n\Lambda(xu_n)) + \varphi(xu_n^{-1}\Lambda(xu_n^{-1})) - 2\varphi(x)\phi(u_n)\| \le c, \quad x, y \in K. \tag{3.33}$$

Then we obtain

$$\|(\varphi(xu_n\Lambda(xu_n)) + \varphi(xu_n^{-1}\Lambda(xu_n^{-1})))\phi(u_n)^{-1} - 2\varphi(x)\| \le \frac{c}{\|\phi(u_n)\|}, \ x, y \in K.$$

Consequently

$$\lim_{n\to\infty}(\varphi(xu_n\Lambda(xu_n))+\varphi(xu_n^{-1}\Lambda(xu_n^{-1})))\phi(u_n)^{-1}=2\varphi(x). \tag{3.34}$$

Now for each $x, y, z \in K$ and $n \in N$, by setting $x = xy, y = u_n$, in (2.6) we obtain

$$\|\varphi(xyu_n\Lambda(xyu_n))+\varphi(xyu_n^{-1}\Lambda(xyu_n^{-1}))-2\varphi(xy)\phi(u_n)\|\le c,\quad x,y\in K. \tag{3.35}$$

then

$$\lim_{n\to\infty}(\varphi(xyu_n\Lambda(xyu_n))+\varphi(xyu_n^{-1}\Lambda(xyu_n^{-1})))\phi(u_n)^{-1}=2\varphi(xy). \tag{3.36}$$

By the arbitrariness of y, (3.34) converges to a unique function φ which satisfies (2.6). In fact,

$$\begin{aligned}&\|(\varphi(u_nxy\Lambda(u_nxy))+\varphi(u_n(xy)^{-1}\Lambda(u_n(xy)^{-1})))\phi(u_n)^{-1}\\&\qquad+(\varphi(u_nxy^{-1}\Lambda(u_nxy^{-1}))+\varphi(u_nyx^{-1})\Lambda(\varphi(u_nyx^{-1})))\phi(u_n)^{-1}\\&\qquad-2\phi(y)(\varphi(u_nx)+\varphi(u_nx^{-1}))\phi(u_n)^{-1}\|\\&\le\|(\varphi(u_nxy\Lambda(u_nxy))+\varphi(u_nxy^{-1}\Lambda(u_nxy^{-1})-2\phi(y)\varphi(u_nx))\phi(u_n)^{-1}\|\\&\quad+\|(\varphi(u_nx^{-1}y^{-1}\Lambda(u_nx^{-1}y^{-1}))+\varphi(u_nx^{-1}y\Lambda(u_nx^{-1}y)\\&\quad-2\phi(y)\varphi(u_nx^{-1}))\phi(u_n)^{-1}\|\le 2c\|\varphi^{-1}(u_n)\|,\end{aligned} \tag{3.37}$$

where we have used Kannappan's condition on φ, Λ and (2.6) to get (3.37). Then taking limits in (3.37), we get that φ satisfies (2.5). Therefore (iii) is proved. Then the case (i) is also proved. □

4. Example

Example. Let C the field of complex numbers with the complex unit $i = \sqrt{-1}$, and G be the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$. The center of G is $G_0 = \{\pm 1\}$ and G is a P_3-group. Take $\Lambda = Id$ or $-Id$, $\varphi, \phi : G \to C$, $\varphi \neq 0$. If ϕ is unbounded and φ, ϕ satisfy (2.3) or (2.6), then φ as defined by (2.4) or (2.9) and ϕ are solutions of (1.1) or (2.5) respectively.

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College of Mathematics
Sichuan University
Chengdu, Sichuan 610064
China
E–mail: luom858@yahoo.com.cn