

## A CONVOLUTION RELATED TO THE INVERSE KONTOROVICH-LEBEDEV TRANSFORM

S. B. YAKUBOVICH AND L. E. BRITVINA

ABSTRACT. We establish the mapping properties in the space  $L_2(\mathbb{R}_+; \frac{dt}{t \sinh \nu t})$ ,  $0 < \nu \leq \pi$  for a convolution related to the transformation

$$F(x) = \int_0^{\infty} f(t)K_{it}(x) dt, \quad x \in \mathbb{R}_+$$

involving the modified Bessel function  $K_{it}(x)$  as a kernel. It is shown that the convolution operator of two  $L_2$ -functions exists as a Lebesgue integral and represents a continuous function on  $\mathbb{R}_+$ . As a consequence, we get the multiplication theorem for two modified Bessel functions of different subscripts. Further applications to the corresponding class of convolution integral equations are obtained.

### 1. INTRODUCTION

In this paper we consider the following Kontorovich-Lebedev transformation [1, 2]

$$F(x) = \text{l.i.m.}_{N \rightarrow \infty} \int_0^N f(t)K_{it}(x) dt, \quad x \in \mathbb{R}_+, \quad (1)$$

where the integration is realized in the mean convergence with respect to an index of the modified Bessel function of the second kind  $K_{it}(x)$  [3], which is real-valued and even with respect to  $t$ . A function  $f(t)$  is supposed to be from the space  $L_2^\nu \equiv L_2(\mathbb{R}_+; \frac{dt}{t \sinh \nu t})$ ,  $0 < \nu \leq \pi$ , i.e.

$$L_2^\nu := \left\{ f : \int_0^{\infty} |f(t)|^2 \frac{dt}{t \sinh \nu t} = \|f\|_{L_2^\nu}^2 < \infty \right\}.$$

---

2000 *Mathematics Subject Classification.* 44A15, 44A05, 44A35, 33C10, 45A05.

*Key words and phrases.* Kontorovich-Lebedev transform, modified Bessel function, convolution integral equations, Plancherel theorem, multiplication theorem.

For  $0 < \nu_1 \leq \nu_2 \leq \pi$  it is easily seen the embeddings

$$L_2^{\nu_1} \subseteq L_2^{\nu_2} \subseteq L_2^\pi = L_2\left(\mathbb{R}_+; \frac{dt}{t \sinh \pi t}\right).$$

As it is known [1], the operator (1) maps the space  $L_2^\pi$  onto the space  $L_2(\mathbb{R}_+; \frac{dx}{x})$  and its inverse is described by

$$f(t) = \frac{2}{\pi^2} \text{l.i.m.}_{N \rightarrow \infty} t \sinh \pi t \int_{1/N}^N F(x) K_{it}(x) \frac{dx}{x} \quad (2)$$

in the mean convergence sense with respect to the norm in  $L_2^\pi$ . Moreover, the Parseval equality

$$\int_0^\infty |f(t)|^2 \frac{dt}{t \sinh \pi t} = \frac{2}{\pi^2} \int_0^\infty |F(x)|^2 \frac{dx}{x} \quad (3)$$

is valid.

Using the Macdonald formula [1, 4] for the product of the modified Bessel functions of different arguments

$$K_{it}(x) K_{it}(y) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}\left(u \frac{x^2+y^2}{xy} + \frac{xy}{u}\right)} K_{it}(u) \frac{du}{u}, \quad (4)$$

one can introduce a convolution for the transform (2) defined by the following double integral

$$(F \overset{\gamma}{*} G)(x) = \frac{1}{2} \int_0^\infty \int_0^\infty F(y) G(u) e^{-\frac{1}{2}\left(x \frac{u^2+y^2}{uy} + \frac{uy}{x}\right)} \frac{dy du}{yu}, \quad (5)$$

where  $F(x)$  and  $G(x)$  are two functions from a suitable function space of originals. Here  $\gamma(t) = \frac{\pi^2}{2t \sinh t}$  is the weight function in a sense of the factorization equality

$$\gamma^2(t) f(t) g(t) = \int_0^\infty (F \overset{\gamma}{*} G)(x) K_{it}(x) \frac{dx}{x},$$

and  $f, g$  are transforms (2) of  $F, G$  respectively, written in the corresponding space of images. Operator (5) was first introduced formally in Kakichev [5] as an example of integral nonstandard convolution. Later this operator was considered in detail in [1, 2, 6, 7, 8, 9].

The main goal of this paper is to establish mapping properties of the following convolution

$$(f *^\alpha g)(t) = \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty \int_0^\infty f(\tau) g(\theta) \Omega_\alpha(t, \tau, \theta) d\tau d\theta, \tag{6}$$

where

$$\Omega_\alpha(t, \tau, \theta) = \int_0^\infty x^{\alpha-1} K_{i\tau}(x) K_{i\theta}(x) K_{it}(x) dx, \quad \alpha, t, \tau, \theta \in \mathbb{R}_+. \tag{7}$$

Formula (6) was first introduced in Yakubovich [1, p. 142] and announced later in [10]. The kernel (7) can be calculated employing relation 2.16.46.6 from [4] and we have

$$\begin{aligned} \Omega_\alpha(t, \tau, \theta) = & 2^{\alpha-3} \operatorname{Re} \left[ \Gamma(it) \Gamma(\alpha - it) B \left( \frac{\alpha}{2} + i \frac{\tau + \theta - t}{2}, \frac{\alpha}{2} - i \frac{\tau + \theta + t}{2} \right) \right. \\ & \times B \left( \frac{\alpha}{2} - i \frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + i \frac{\tau - \theta - t}{2} \right) {}_4F_3 \left( \frac{\alpha}{2} + i \frac{\tau + \theta - t}{2}, \frac{\alpha}{2} - i \frac{\tau + \theta + t}{2}, \right. \\ & \left. \left. \frac{\alpha}{2} - i \frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + i \frac{\tau - \theta - t}{2}; 1 - it, \frac{\alpha - it}{2}, \frac{1 + \alpha - it}{2}; \frac{1}{4} \right) \right]. \end{aligned} \tag{8}$$

It contains the generalized hypergeometric function  ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z)$  at the point  $z = 1/4$  and as usual  $\Gamma(z)$ ,  $B(x, y)$  stand for Euler's gamma- and beta-functions [3].

In the sequel we will give some conditions that guarantee the convolution (6) of two  $L_2$ -functions exists as a Lebesgue integral and represents a continuous function. Various applications of the convolution (6) will be considered. In particular, we will prove an analog of the multiplication theorem for two modified Bessel functions with different subscripts. Finally, we will exhibit certain convolution integral equations related to (6).

## 2. MAPPING PROPERTIES OF THE CONVOLUTION $(f *^\alpha g)(t)$

We first present a lemma which gives a boundedness and the norm estimation for the Kontorovich-Lebedev transform (1) as an operator from  $L_2^\nu$ ,  $0 < \nu < \pi$  into the space  $C(\mathbb{R}_+)$  of bounded continuous functions on  $\mathbb{R}_+$ .

**Lemma 1.** *Let  $f(t) \in L_2^\nu$  with  $0 < \nu < \pi$ . Then transformation (1) exists as a Lebesgue integral for all  $x > 0$  and represents a continuous function on  $\mathbb{R}_+$ . Moreover, it is a bounded operator  $F : L_2^\nu \rightarrow C(\mathbb{R}_+)$ , namely*

$$\sup_{x>0} |F(x)| \leq C \|f\|_{L_2^\nu}, \tag{9}$$

with

$$C = \frac{\sqrt{\pi}}{2 \cos^{1/2}(\frac{\nu}{2})}.$$

*Proof.* In fact, appealing to Schwarz's inequality we obtain

$$\begin{aligned} |F(x)| &\leq \int_0^\infty |f(t)| |K_{it}(x)| dt \leq \left( \int_0^\infty |f(t)|^2 \frac{dt}{t \sinh \nu t} \right)^{1/2} \\ &\times \left( \int_0^\infty t \sinh \nu t K_{it}^2(x) dt \right)^{1/2} = \|f\|_{L_2^\nu} \left( \int_0^\infty t \sinh \nu t K_{it}^2(x) dt \right)^{1/2}. \end{aligned} \quad (10)$$

The latter integral in (10) is calculated in terms of the modified Bessel function  $K_1$  (see [4], relation 2.16.52.8), namely

$$\int_0^\infty t \sinh(\nu t) K_{it}^2(x) dt = \frac{\pi}{2} x \sin \frac{\nu}{2} K_1 \left( 2x \cos \frac{\nu}{2} \right), \quad 0 < \nu < \pi. \quad (11)$$

Moreover, since  $xK_1(x) \leq 1$ , when  $x > 0$ , it yields (9)

$$\begin{aligned} \sup_{x>0} |F(x)| &\leq \sup_{x>0} \int_0^\infty |f(t)| |K_{it}(x)| dt \leq \sup_{x>0} \left( \frac{\pi}{2} x \sin \frac{\nu}{2} K_1 \left( 2x \cos \frac{\nu}{2} \right) \right)^{1/2} \|f\|_{L_2^\nu} \\ &\leq \frac{\sqrt{\pi}}{2 \cos^{1/2}(\frac{\nu}{2})} \|f\|_{L_2^\nu} < \infty. \end{aligned}$$

In order to prove that  $F(x)$  is continuous on  $\mathbb{R}_+$  we will establish the uniform convergence of the integral (1) with respect to  $x \geq x_0 > 0$ . To do this we use the following inequality for the modified Bessel function (cf. [1], formula (1.100))

$$|K_{it}(x)| \leq e^{-\delta t} K_0(x \cos \delta),$$

where we choose  $\nu/2 < \delta < \pi/2$ . Then for sufficiently big  $A > 0$  we have

$$\begin{aligned} \int_A^\infty |f(t)| |K_{it}(x)| dt &\leq \|f\|_{L_2^\nu} \left( \int_A^\infty t \sinh \nu t K_{it}^2(x) dt \right)^{1/2} \leq K_0^2(x_0 \cos \delta) \\ &\times \left( \int_A^\infty t \sinh \nu t e^{-2\delta t} dt \right)^{1/2} \rightarrow 0, \quad A \rightarrow \infty. \end{aligned}$$

□

Applying the Parseval equality (3) we get

**Lemma 2.** *The equality*

$$\frac{2}{\pi^2} \int_0^\infty t \sinh \pi t |\Omega_\alpha(t, \tau, \theta)|^2 dt = \int_0^\infty x^{2\alpha-1} K_{i\tau}^2(x) K_{i\theta}^2(x) dx. \quad (12)$$

holds for any  $\alpha, \tau, \theta \in \mathbb{R}_+$ .

*Proof.* Indeed by the asymptotic properties of the modified Bessel functions [3] we observe that the right-hand side of (12) is finite. Therefore  $x^\alpha K_{i\tau}(x) K_{i\theta}(x) \in L_2(\mathbb{R}_+, \frac{dx}{x})$ . Moreover, transformation (1) of this function exists as a Lebesgue integral and is equal to  $\frac{2}{\pi^2} t \sinh \pi t \Omega_\alpha(t, \tau, \theta)$  (see (7)). Thus via Parseval equality (3) we arrive at (12).  $\square$

As a corollary we immediately obtain an analogue of the multiplication theorem for the modified Bessel functions (cf. (4)) of different subscripts.

**Corollary 1.** *Let  $0 < \alpha < 2$  and  $\tau, \theta \in \mathbb{R}_+$  be fixed. Then for any  $x > 0$  the following equality is true*

$$\begin{aligned} x^\alpha K_{i\tau}(x) K_{i\theta}(x) &= \frac{2^{\alpha-3}}{\pi} \int_{-\infty}^\infty \frac{\Gamma(\alpha - it)}{\Gamma(-it)} B\left(\frac{\alpha}{2} + i\frac{\tau + \theta - t}{2}, \frac{\alpha}{2} - i\frac{\tau + \theta + t}{2}\right) \\ &\times B\left(\frac{\alpha}{2} - i\frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + i\frac{\tau - \theta - t}{2}\right) {}_4F_3\left(\frac{\alpha}{2} + i\frac{\tau + \theta - t}{2}, \frac{\alpha}{2} - i\frac{\tau + \theta + t}{2}, \right. \\ &\left. \frac{\alpha}{2} - i\frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + i\frac{\tau - \theta - t}{2}; 1 - it, \frac{\alpha - it}{2}, \frac{1 + \alpha - it}{2}; \frac{1}{4}\right) K_{it}(x) dt. \end{aligned} \quad (13)$$

*Proof.* Using the definition of the generalized hypergeometric function [3] we can write the right-hand side of (13) as

$$I_\alpha(x) = \int_{-\infty}^\infty \omega_\alpha(t, \tau, \theta) K_{it}(x) dt, \quad (14)$$

where

$$\begin{aligned} \omega_\alpha(t, \tau, \theta) &= \frac{2^{\alpha-3}t}{\pi i} \sum_{n=0}^\infty B\left(\frac{\alpha}{2} + n + i\frac{\tau + \theta - t}{2}, \frac{\alpha}{2} + n - i\frac{\tau + \theta + t}{2}\right) \\ &\times B\left(\frac{\alpha}{2} + n - i\frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + n + i\frac{\tau - \theta - t}{2}\right) \frac{\Gamma(\alpha - it + 2n)}{n! \Gamma(1 - it + n)}. \end{aligned} \quad (15)$$

Then by an elementary inequality for the beta-function

$$|B(s, t)| \leq B(\operatorname{Re} s, \operatorname{Re} t), \quad \operatorname{Re} s > 0, \operatorname{Re} t > 0, \quad (16)$$

and the duplication formula for the gamma-function [3] we derive the following estimations

$$\begin{aligned}
\frac{\pi}{2^{\alpha-3}} |\omega_\alpha(t, \tau, \theta)| &\leq \left| B\left(\frac{\alpha}{2} + i\frac{\tau + \theta - t}{2}, \frac{\alpha}{2} - i\frac{\tau + \theta + t}{2}\right) \right| \\
&\times \left| B\left(\frac{\alpha}{2} - i\frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + i\frac{\tau - \theta - t}{2}\right) \right| \frac{|\Gamma(\alpha - it)|}{|\Gamma(-it)|} \\
&+ |t| \sum_{n=1}^{\infty} B^2\left(\frac{\alpha}{2} + n, \frac{\alpha}{2} + n\right) \frac{|\Gamma(\alpha - it + 2n)|}{n!|\Gamma(1 - it + n)|} \\
&\leq \left| B\left(\frac{\alpha}{2} + i\frac{\tau + \theta - t}{2}, \frac{\alpha}{2} - i\frac{\tau + \theta + t}{2}\right) \right| \\
&\times \left| B\left(\frac{\alpha}{2} - i\frac{\tau - \theta + t}{2}, \frac{\alpha}{2} + i\frac{\tau - \theta - t}{2}\right) \right| \frac{|\Gamma(\alpha - it)|}{|\Gamma(-it)|} + \frac{2^{\alpha-1}|t|}{\sqrt{\pi}|\Gamma(1 - \frac{\alpha+it}{2})|} \\
&\times \sum_{n=1}^{\infty} \frac{4^n}{n!} B^2\left(\frac{\alpha}{2} + n, \frac{\alpha}{2} + n\right) B\left(\frac{\alpha}{2} + n, 1 - \frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+1}{2} + n\right) \\
&= \left| \Gamma\left(\frac{\alpha}{2} + i\frac{\tau + \theta - t}{2}\right) \Gamma\left(\frac{\alpha}{2} - i\frac{\tau + \theta + t}{2}\right) \right| \\
&\times \left| \Gamma\left(\frac{\alpha}{2} - i\frac{\tau - \theta + t}{2}\right) \Gamma\left(\frac{\alpha}{2} + i\frac{\tau - \theta - t}{2}\right) \right| \frac{1}{|\Gamma(\alpha - it)\Gamma(-it)|} \\
&+ \frac{\Gamma(1 - \frac{\alpha}{2})|t|}{2^{\alpha-1}|\Gamma(1 - \frac{\alpha+it}{2})|} \sum_{n=1}^{\infty} \frac{\Gamma^3(\frac{\alpha}{2} + n)}{4^n(n!)^2\Gamma(\frac{\alpha+1}{2} + n)}, \quad 0 < \alpha < 2.
\end{aligned}$$

Consequently via Stirling's asymptotic formulas for gamma-functions and factorials [3] the general term of the latter series is  $O(n^{\alpha-5/2}4^{-n})$ ,  $n \rightarrow \infty$ . Therefore it converges and accordingly

$$|\omega_\alpha(t, \tau, \theta)| = O\left(|t|^{\frac{\alpha+1}{2}} e^{\pi|t|/4}\right), \quad |t| \rightarrow \infty. \quad (17)$$

Further appealing to the asymptotic formula for the modified Bessel function with respect to an index (see [1], formula (1.148)) we find

$$|K_{it}(x)| = \sqrt{\frac{2\pi}{|t|}} e^{-\pi|t|/2} (1 + O(1/|t|)), \quad |t| \rightarrow \infty. \quad (18)$$

Therefore  $I_\alpha(x)$  exists as an absolutely convergent integral for all  $x \in \mathbb{R}_+$  and  $0 < \alpha < 2$ .

On the other hand, by virtue of Lemma 2 and formulas (2), (7), (8), (14) we derive

$$\begin{aligned} x^\alpha K_{i\tau}(x)K_{i\theta}(x) &= \frac{2}{\pi^2} \text{l.i.m.}_{N \rightarrow \infty} \int_0^N t \sinh \pi t \Omega_\alpha(t, \tau, \theta) K_{it}(x) dt \\ &= 2 \text{l.i.m.}_{N \rightarrow \infty} \int_0^N K_{it}(x) \text{Re} \omega_\alpha(t, \tau, \theta) dt \\ &= \text{l.i.m.}_{N \rightarrow \infty} \int_{-N}^N \omega_\alpha(t, \tau, \theta) K_{it}(x) dt. \end{aligned} \tag{19}$$

This means that the latter limit in mean coincides with  $I_\alpha(x)$ . Thus we prove (13) and end the proof of Corollary 1.  $\square$

The mapping properties for convolution (6) are given by

**Theorem 1.** *Let  $f, g \in L_2^\nu$  with  $0 < \nu < \pi$ . Then convolution  $(f \overset{\alpha}{*} g)(t)$  exists as a Lebesgue integral and is continuous on  $\mathbb{R}_+$ . Besides it belongs to  $L_2^\pi$ .*

*Proof.* Using Schwarz's inequality for double integrals we deduce

$$\begin{aligned} |(f \overset{\alpha}{*} g)(t)| &\leq \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty \int_0^\infty |f(\tau)| |g(\theta)| |\Omega_\alpha(t, \tau, \theta)| d\tau d\theta \\ &\leq \frac{2}{\pi^2} t \sinh \pi t \left( \int_0^\infty \int_0^\infty |f(\tau)|^2 |g(\theta)|^2 \frac{d\tau d\theta}{\tau \theta \sinh \nu \tau \sinh \nu \theta} \right)^{1/2} J^{1/2}(t) \\ &= \frac{2}{\pi^2} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} t \sinh \pi t J^{1/2}(t), \end{aligned} \tag{20}$$

where by

$$J(t) = \int_0^\infty \int_0^\infty \tau \theta \sinh \nu \tau \sinh \nu \theta |\Omega_\alpha(t, \tau, \theta)|^2 d\tau d\theta. \tag{21}$$

Meanwhile (see (7))

$$\begin{aligned} |\Omega_\alpha(t, \tau, \theta)|^2 &\leq \left[ \int_0^\infty x^{\alpha-1} |K_{i\tau}(x) K_{i\theta}(x) K_{it}(x)| dx \right]^2 \\ &\leq \int_0^\infty x^{\alpha-1} K_{i\tau}^2(x) K_{i\theta}^2(x) dx \int_0^\infty y^{\alpha-1} K_{it}^2(y) dy. \end{aligned}$$

Hence plainly

$$\begin{aligned} J(t) &\leq \int_0^\infty y^{\alpha-1} K_{it}^2(y) dy \int_0^\infty \int_0^\infty \tau \theta \sinh \nu \tau \sinh \nu \theta \int_0^\infty x^{\alpha-1} K_{i\tau}^2(x) K_{i\theta}^2(x) dx d\tau d\theta \\ &= \int_0^\infty y^{\alpha-1} K_{it}^2(y) dy \int_0^\infty x^{\alpha-1} \left[ \int_0^\infty \tau \sinh \nu \tau K_{i\tau}^2(x) d\tau \right]^2 dx. \end{aligned} \quad (22)$$

Calculating the inner integral with respect to  $\tau$  by (11) and integrals with respect to  $y$  and  $x$  via [4], relation 2.16.33.2

$$\int_0^\infty y^{\alpha-1} K_\mu^2(y) dy = 2^{\alpha-3} B\left(\frac{\alpha}{2} + \mu, \frac{\alpha}{2} - \mu\right) \Gamma^2\left(\frac{\alpha}{2}\right), \quad \alpha > 2|\operatorname{Re}\mu| \quad (23)$$

we arrive at the estimate

$$J^{1/2}(t) \leq \frac{C_\alpha}{\cos^{\alpha/2+1} \nu/2} \left| \Gamma\left(\frac{\alpha}{2} + it\right) \right|, \quad 0 < \nu < \pi,$$

where  $C_\alpha > 0$  is a constant depending only on  $\alpha$ . Therefore, for all  $t > 0$

$$\left| (f *^\alpha g)(t) \right| \leq \operatorname{const.} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \frac{|\Gamma(\frac{\alpha}{2} + it)|}{|\Gamma(it)|^2}$$

and clearly  $(f *^\alpha g)(t) = O\left(t^{\frac{\alpha+1}{2}} e^{\pi t/2}\right)$ ,  $t \rightarrow +\infty$ . Thus we obtain that under conditions of the theorem the convolution  $(f *^\alpha g)(t)$  exists as a Lebesgue integral for all  $t > 0$ . To establish its continuity on  $\mathbb{R}_+$  we proceed in a way similar to the approach in Lemma 1. Indeed, choosing a big  $A > 0$ ,  $t \in [0, T]$  and  $\delta \in (\nu/2, \pi/2)$  we invoke the inequality [1]  $|K_{it}(x)| \leq K_0(x)$  and we derive (see (22))

$$\begin{aligned} &\frac{2}{\pi^2} t \sinh \pi t \int_A^\infty \int_A^\infty |f(\tau)| |g(\theta)| |\Omega_\alpha(t, \tau, \theta)| d\tau d\theta \leq \frac{2}{\pi^2} t \sinh \pi t \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \\ &\times \left( \int_A^\infty \int_A^\infty \tau \theta \sinh \nu \tau \sinh \nu \theta |\Omega_\alpha(t, \tau, \theta)|^2 d\tau d\theta \right)^{1/2} \leq \frac{2}{\pi^2} T \sinh \pi T \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \\ &\times \left( \int_0^\infty y^{\alpha-1} K_0^2(y) dy \int_0^\infty x^{\alpha-1} K_0^4(x \cos \delta) dx \right)^{1/2} \int_A^\infty \tau \sinh \nu \tau e^{-2\delta\tau} d\tau \\ &= \operatorname{const.} \int_A^\infty \tau \sinh \nu \tau e^{-2\delta\tau} d\tau \rightarrow 0, \quad A \rightarrow \infty. \end{aligned}$$

Therefore integral (6) converges uniformly by  $t \in [0, T]$  for any  $T > 0$ .



Now we are ready to prove that  $(f \overset{\alpha}{*} g)(t) \in L_2^\pi$ . In fact, in view of (20)

$$\left| (f \overset{\alpha}{*} g)(t) \right|^2 \leq \frac{4}{\pi^4} t^2 \sinh^2 \pi t \|f\|_{L_2^\nu}^2 \|g\|_{L_2^\nu}^2 J(t),$$

where  $J(t)$  is defined by (21). Then

$$\begin{aligned} \|f \overset{\alpha}{*} g\|_{L_2^\pi} &\leq \frac{2}{\pi^2} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \left( \int_0^\infty t \sinh \pi t J(t) dt \right)^{1/2} \\ &= \frac{2}{\pi^2} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \\ &\quad \times \left( \int_0^\infty \int_0^\infty \tau \theta \sinh \nu \tau \sinh \nu \theta \int_0^\infty t \sinh \pi t |\Omega_\alpha(t, \tau, \theta)|^2 dt d\tau d\theta \right)^{1/2}. \end{aligned}$$

Using (12) we have

$$\begin{aligned} \|f \overset{\alpha}{*} g\|_{L_2^\pi} &\leq \frac{\sqrt{2}}{\pi} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \\ &\quad \times \left( \int_0^\infty \int_0^\infty \tau \theta \sinh \nu \tau \sinh \nu \theta \int_0^\infty x^{2\alpha-1} K_{i\tau}^2(x) K_{i\theta}^2(x) dx d\tau d\theta \right)^{1/2} \\ &= \frac{\sqrt{2}}{\pi} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \\ &\quad \times \left( \int_0^\infty x^{2\alpha-1} \int_0^\infty \tau \sinh \nu \tau K_{i\tau}^2(x) d\tau \int_0^\infty \theta \sinh \nu \theta K_{i\theta}^2(x) d\theta dx \right)^{1/2} \\ &= \frac{\sqrt{2}}{\pi} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu} \left( \int_0^\infty x^{2\alpha-1} \left[ \int_0^\infty \tau \sinh \nu \tau K_{i\tau}^2(x) d\tau \right]^2 dx \right)^{1/2}. \end{aligned}$$

The latter iterated integral can be easily calculated explicitly invoking again (11) and (23). So the final inequality for the  $L_2^\pi$ -norm of the convolution (6) can be written as

$$\|f \overset{\alpha}{*} g\|_{L_2^\pi} \leq C_{\alpha,\nu} \|f\|_{L_2^\nu} \|g\|_{L_2^\nu},$$

where  $C_{\alpha,\nu} > 0$  is a constant. □

**Remark 1.** It is clear via Fubini's theorem that convolution (6) is a commutative operation, i.e.  $f \overset{\alpha}{*} g = g \overset{\alpha}{*} f$ .

**Theorem 2.** *Under the conditions of Theorem 1 the following factorization equality for convolution (6) is valid*

$$x^\alpha F(x)G(x) = \text{l.i.m.}_{N \rightarrow \infty} \int_0^N (f *^\alpha g)(t) K_{it}(x) dt, \quad (24)$$

where  $F, G$  are transforms (1) of  $f$  and  $g$ , respectively. Besides, the Parseval equality of type

$$\int_0^\infty \left| (f *^\alpha g)(t) \right|^2 \frac{dt}{t \sinh \pi t} = \frac{2}{\pi^2} \int_0^\infty x^{2\alpha-1} |F(x)G(x)|^2 dx \quad (25)$$

is valid.

*Proof.* By appealing to Lemma 1 and Fubini's theorem we derive the chain of equalities

$$\begin{aligned} (f *^\alpha g)(t) &= \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty \int_0^\infty f(\tau) g(\theta) \Omega_\alpha(t, \tau, \theta) d\tau d\theta \\ &= \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty \int_0^\infty f(\tau) g(\theta) \int_0^\infty x^{\alpha-1} K_{i\tau}(x) K_{i\theta}(x) K_{it}(x) dx d\tau d\theta \\ &= \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty x^{\alpha-1} \left( \int_0^\infty f(\tau) K_{i\tau}(x) d\tau \right) \left( \int_0^\infty g(\theta) K_{i\theta}(x) d\theta \right) K_{it}(x) dx \\ &= \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty x^{\alpha-1} F(x) G(x) K_{it}(x) dx \end{aligned}$$

and all inner integrals are absolutely convergent. But since  $(f *^\alpha g)(t) \in L_2^\pi$  we have  $x^\alpha F(x)G(x) \in L_2(\mathbb{R}_+, \frac{dx}{x})$ . Therefore, equalities (24), (25) are direct consequences of the reciprocities (1), (2) and the Parseval identity (3) for the Kontorovich-Lebedev transformation.  $\square$

Theorems 1-2 can be extended if one of the functions under convolution (6) belongs to  $L_2^\pi$ . Precisely, we prove

**Theorem 3.** *Let  $f \in L_2^\pi$  and  $g \in L_2^\nu$ ,  $0 < \nu < \pi$ . Then convolution  $(f *^\alpha g)(t)$  exists as a Lebesgue integral and is continuous on  $\mathbb{R}_+$ . Moreover, it belongs to  $L_2^\pi$ .*

*Proof.* Similar estimations as in Theorem 1 lead us at the chain of inequalities

$$\begin{aligned} \left| \left( f \overset{\alpha}{*} g \right) (t) \right| &\leq \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty \int_0^\infty |f(\tau)| |g(\theta)| |\Omega_\alpha(t, \tau, \theta)| d\tau d\theta \\ &\leq \frac{2}{\pi^2} t \sinh \pi t \left( \int_0^\infty \int_0^\infty |f(\tau)|^2 |g(\theta)|^2 \frac{d\tau d\theta}{\tau \theta \sinh \pi \tau \sinh \nu \theta} \right)^{1/2} \Psi^{1/2}(t) \\ &= \frac{2}{\pi^2} \|f\|_{L_2^\pi} \|g\|_{L_2^\nu} t \sinh \pi t \Psi^{1/2}(t), \end{aligned} \tag{26}$$

with

$$\Psi(t) = \int_0^\infty \int_0^\infty \tau \theta \sinh \pi \tau \sinh \nu \theta |\Omega_\alpha(t, \tau, \theta)|^2 d\tau d\theta. \tag{27}$$

However, the inner integral with respect to  $\tau$  in (27) can be expressed by using the Parseval equality (12). Hence employing (11), (23) and again the inequality  $x K_1(x) \leq 1, x > 0$  we find

$$\begin{aligned} \Psi(t) &= \frac{\pi^2}{2} \int_0^\infty \int_0^\infty x^{2\alpha-1} \theta \sinh \nu \theta K_{i\theta}^2(x) K_{it}^2(x) dx d\theta \\ &= \frac{\pi^3}{4} \sin \frac{\nu}{2} \int_0^\infty x^{2\alpha} K_{it}^2(x) K_1 \left( 2x \cos \frac{\nu}{2} \right) dx \leq \text{const.} \int_0^\infty x^{2\alpha-1} K_{it}^2(x) dx \\ &= \text{const.} |\Gamma(\alpha + it)|^2. \end{aligned}$$

Thus combining with (26) and taking into account the asymptotic behavior of the gamma function at infinity we get that the convolution (6) exists as a Lebesgue integral and satisfies the following estimate

$$\left| \left( f \overset{\alpha}{*} g \right) (t) \right| = O(e^{\pi t/2} t^{\alpha+1/2}), \quad t \rightarrow +\infty.$$

In the same manner as in Theorem 1 we get its continuity. In fact,

$$\begin{aligned} \frac{2}{\pi^2} t \sinh \pi t \int_A^\infty \int_A^\infty |f(\tau)| |g(\theta)| |\Omega_\alpha(t, \tau, \theta)| d\tau d\theta &\leq \frac{2}{\pi^2} T \sinh \pi T \|f\|_{L_2^\pi} \|g\|_{L_2^\nu} \\ &\times \left( \int_A^\infty \int_A^\infty \tau \theta \sinh \pi \tau \sinh \nu \theta |\Omega_\alpha(t, \tau, \theta)|^2 d\tau d\theta \right)^{1/2} \\ &\leq \text{const.} \left( \int_0^\infty x^{2\alpha-1} K_0^2(x) K_0^2(x \cos \delta) dx \right)^{1/2} \left( \int_A^\infty \theta \sinh \nu \theta e^{-2\delta\theta} d\theta \right)^{1/2} \end{aligned}$$

$$= \text{const.} \int_A^\infty \theta \sinh \nu \theta e^{-2\delta\theta} d\theta \rightarrow 0, \quad \frac{\nu}{2} < \delta < \frac{\pi}{2}, \quad A \rightarrow \infty.$$

Therefore under the conditions of the theorem integral (6) converges uniformly on  $[0, T]$  for any  $T > 0$ .

Further, let us estimate the norm of the convolution (6) in the space  $L_2^\pi$  and show that it is finite. First we observe that since integral (6) is in the sense of Lebesgue one, we can write it for all  $t > 0$  as a usual limit

$$(f \overset{\alpha}{*} g)(t) = \frac{2}{\pi^2} t \sinh \pi t \lim_{n \rightarrow \infty} \int_0^n \int_0^\infty f(\tau) g(\theta) \Omega_\alpha(t, \tau, \theta) d\theta d\tau.$$

Hence denoting by  $f_n(\tau) = f(\tau)$ ,  $\tau \in [0, n]$ , which vanishes outside of the interval  $[0, n]$  we easily find that  $f_n \in L_2^\nu$  for all  $n \in \mathbb{N}$ . Therefore appealing to the Parseval equality (25) via the Fatou lemma we obtain

$$\begin{aligned} \int_0^\infty |(f \overset{\alpha}{*} g)(t)|^2 \frac{dt}{t \sinh \pi t} &\leq \liminf_{n \rightarrow \infty} \int_0^\infty |(f_n \overset{\alpha}{*} g)(t)|^2 \frac{dt}{t \sinh \pi t} \\ &= \frac{2}{\pi^2} \liminf_{n \rightarrow \infty} \int_0^\infty x^{2\alpha-1} |F_n(x)G(x)|^2 dx, \end{aligned} \tag{28}$$

where  $F_n(x)$  is the Kontorovich-Lebedev transformation (1) of the function  $f_n$  and the corresponding integral converges absolutely. Moreover, since  $g \in L_2^\nu$  then making similar estimates as in the proof of Lemma 1 (see (10)) and invoking equality (11) we derive for all  $\alpha > 0$

$$x^\alpha |G(x)| \leq \text{const.} \sup_{x>0} \left[ x^{\alpha+1/2} K_1^{1/2} \left( 2x \cos \frac{\nu}{2} \right) \right] < \text{const.}$$

The latter estimate yields that  $x^\alpha G(x)$  is bounded. Consequently, recalling again equality (3) we return to (28) and it becomes

$$\begin{aligned} \int_0^\infty |(f \overset{\alpha}{*} g)(t)|^2 \frac{dt}{t \sinh \pi t} &\leq \text{const.} \liminf_{n \rightarrow \infty} \int_0^\infty |F_n(x)|^2 \frac{dx}{x} \\ &= \text{const.} \liminf_{n \rightarrow \infty} \int_0^\infty |f_n(\tau)|^2 \frac{d\tau}{\tau \sinh \pi \tau} = \text{const.} \|f\|_{L_2^\pi}^2 < \infty. \end{aligned}$$

□

**Corollary 2.** *Formulas (24), (25) also hold under the conditions of Theorem 3.*

*Proof.* Similar to the proof of Theorem 2 we derive by straightforward calculations for all  $t > 0$

$$\begin{aligned}
 (f *^\alpha g)(t) &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n \rightarrow \infty} \int_0^n \int_0^\infty f(\tau) g(\theta) \Omega_\alpha(t, \tau, \theta) d\tau d\theta \\
 &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n \rightarrow \infty} \int_0^n \int_0^\infty f(\tau) g(\theta) \int_0^\infty x^{\alpha-1} K_{i\tau}(x) K_{i\theta}(x) K_{it}(x) dx d\tau d\theta \\
 &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n \rightarrow \infty} \int_0^\infty x^{\alpha-1} \left( \int_0^n f(\tau) K_{i\tau}(x) d\tau \right) \left( \int_0^\infty g(\theta) K_{i\theta}(x) d\theta \right) K_{it}(x) dx \\
 &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n \rightarrow \infty} \int_0^\infty x^{\alpha-1} F_n(x) G(x) K_{it}(x) dx \\
 &= \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty x^{\alpha-1} F(x) G(x) K_{it}(x) dx.
 \end{aligned}$$

The latter equality is true since for all  $t > 0$  we have

$$\begin{aligned}
 &\int_0^\infty x^{\alpha-1} |F(x) - F_n(x)| |G(x) K_{it}(x)| dx \\
 &\leq \left( \int_0^\infty |F(x) - F_n(x)|^2 \frac{dx}{x} \right)^{1/2} \left( \int_0^\infty x^{2\alpha-1} |G(x) K_0(x)|^2 dx \right)^{1/2} \\
 &\leq \sup_{x>0} [x^\alpha K_0(x)] \|G\|_{L_2(\mathbb{R}_+; x^{-1} dx)} \left( \int_0^\infty |F(x) - F_n(x)|^2 \frac{dx}{x} \right)^{1/2} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

Hence the statement in the corollary follows as in Theorem 2. □

**Remark 2.** It is not difficult now to establish the associativity of the convolution (6), which is guaranteed by Theorems 1-3 and Corollary 2. Precisely, it has the property

$$(f *^\alpha g) *^\alpha h = f *^\alpha (g *^\alpha h) = g *^\alpha (f *^\alpha h),$$

for any  $f, g, h \in L_2^\nu$ ,  $0 < \nu < \pi$ .

### 3. CONVOLUTION INTEGRAL EQUATIONS

As applications we consider in this last section the  $L_2^\pi$ -solvability of integral equations related to convolution (6). Precisely, we exhibit the following integral

equations of the first and second kind, respectively.

$$g(x) = (k \overset{\alpha}{*} f)(x), \quad x \in \mathbb{R}_+, \quad (29)$$

$$f(x) = h(x) + \lambda(k \overset{\alpha}{*} f)(x), \quad x \in \mathbb{R}_+, \lambda \in \mathbb{C}, \quad (30)$$

where  $g, h \in L_2^\pi$ ,  $k \in L_2^\nu$ ,  $0 < \nu < \pi$  are given functions and  $f(x) \in L_2^\pi$  is to be determined. We will prove two theorems, which will guarantee the existence and uniqueness of  $L_2^\pi$ -solutions and give them in the closed form. Certain examples of the function  $k$  will be considered. Similar questions for the convolution integral equations related to (5) were investigated in [1, Ch. 4].

Denoting by

$$\mathcal{K}_\alpha(x, y) = \int_0^\infty k(\theta) \Omega_\alpha(x, y, \theta) d\theta \quad (31)$$

the integral equation (29) can be written in the form

$$g(x) = \frac{2}{\pi^2} x \sinh \pi x \int_0^\infty \mathcal{K}_\alpha(x, y) f(y) dy. \quad (32)$$

**Theorem 4.** *Let  $\alpha > 0$ ,  $g \in L_2^\pi$  and  $k \in L_2^\nu$ ,  $0 < \nu < \pi$ . Then for the existence of a  $L_2^\pi$ -solution of the equation (32) it is necessary and sufficient that  $\frac{G(u)}{u^\alpha \hat{k}(u)} \in L_2(\mathbb{R}_+; \frac{du}{u})$ , where  $G, \hat{k}$  are the Kontorovich-Lebedev transformations (1) of the functions  $g, k$ , respectively. Moreover the solution is unique and is given by the formula*

$$f(x) = \frac{2}{\pi^2} \text{l.i.m.}_{N \rightarrow \infty} x \sinh \pi x \int_{1/N}^N \frac{G(u)}{\hat{k}(u)} K_{ix}(u) \frac{du}{u^{\alpha+1}}, \quad (33)$$

where the convergence is with respect to the norm in  $L_2^\pi$ .

*Proof. Necessity.* If we assume that  $g, k, f$  belong to the corresponding  $L$ -classes and the equation (32) is satisfied, then via Corollary 2 we have the equality

$$G(u) = u^\alpha \hat{k}(u) F(u),$$

where  $F$  is the transformation (1) of the function  $f$ . Hence since  $F \in L_2(\mathbb{R}_+; \frac{du}{u})$  we get that  $\frac{G(u)}{u^\alpha \hat{k}(u)} \in L_2(\mathbb{R}_+; \frac{du}{u})$  and the  $L_2^\pi$ -solution is given reciprocally by formula (33).

*Sufficiency.* If conversely,  $\frac{G(u)}{u^\alpha \hat{k}(u)} \in L_2(\mathbb{R}_+; \frac{du}{u})$ , then  $f$  being defined by (33) belongs to  $L_2^\pi$  and by virtue of Theorem 3 the right-hand side of (32) belongs to  $L_2^\pi$ .

Therefore by Corollary 2 the Kontorovich-Lebedev transform (1) of the right-hand side of (32) is equal to

$$u^\alpha \hat{k}(u)F(u) = u^\alpha \hat{k}(u) \frac{G(u)}{u^\alpha \hat{k}(u)} = G(u).$$

By the reciprocity (2) we see that equation (32) is satisfied and (33) is a unique  $L_2^\pi$ -solution.  $\square$

Let us consider the convolution integral equation (30), which can be written as

$$f(x) = h(x) + \frac{2\lambda}{\pi^2} x \sinh \pi x \int_0^\infty \mathcal{K}_\alpha(x, y) f(y) dy. \tag{34}$$

We have

**Theorem 5.** *Let  $\alpha > 0, \lambda \in \mathbb{C} \setminus \{0\}$ ,  $h \in L_2^\pi$ ,  $k \in L_2^\nu$ ,  $0 < \nu < \pi$  and  $H(u)$  is the Kontorovich-Lebedev transformation (1) of the function  $h$ . Let also  $\sup_{u>0} |u^\alpha \hat{k}(u)| < \frac{1}{|\lambda|}$ . Then*

$$f(x) = \frac{2}{\pi^2} \text{l.i.m.}_{N \rightarrow \infty} x \sinh \pi x \int_{1/N}^N \frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} K_{ix}(u) \frac{du}{u}, \tag{35}$$

is a solution of (34) belonging to  $L_2^\pi$  and any another solution from  $L_2^\pi$  coincides with (35) for almost all  $x \in \mathbb{R}_+$ .

*Proof.* Since  $[1 - \lambda u^\alpha \hat{k}(u)]^{-1}$  is bounded we have  $\frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} \in L_2(\mathbb{R}_+; \frac{du}{u})$ . Consequently, (35) exists in the mean sense and defines a function  $f$  from  $L_2^\pi$ . At the same time the convolution in the right-hand side of (34) belongs to  $L_2^\pi$  via Theorem 3. The transformation (1) of the right-hand side of (34) gives

$$H(u) + \lambda u^\alpha \hat{k}(u) \frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} = \frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} = F(u).$$

Therefore  $f$  by formula (35) satisfies equation (34) for almost all  $x \in \mathbb{R}_+$ .

Conversely, if  $f, h \in L_2^\pi$ ,  $k \in L_2^\nu$  and equality (34) holds then by Corollary 2 we have

$$F(u) = H(u) + \lambda u^\alpha \hat{k}(u)F(u),$$

which implies (35).  $\square$

**Remark 3.** Solution (35) can be written in terms of the resolvent. Indeed, denoting by

$$M(u) = \frac{\hat{k}(u)}{1 - \lambda u^\alpha \hat{k}(u)},$$

and by  $m(\theta) \in L_2^\pi$  the transform (2) of  $M \in L_2(\mathbb{R}_+; \frac{du}{u})$  we can write the solution (35) in the form

$$f(x) = h(x) + \frac{2\lambda}{\pi^2} x \sinh \pi x \int_0^\infty \mathcal{M}_\alpha(x, y) h(y) dy, \quad (36)$$

where

$$\mathcal{M}_\alpha(x, y) = \int_0^\infty m(\theta) \Omega_\alpha(x, y, \theta) d\theta \quad (37)$$

Finally we give certain concrete examples of the kernel (31), the corresponding convolution integral equations (32), (34) and their  $L_2^\pi$ -solutions (33), (35).

**Example 1.** Let  $k(\theta) = \theta \tanh \frac{\pi\theta}{2}$ . It evidently belongs to  $L_2^\nu$  for any  $\nu > 0$ . Recalling relation 2.16.48.14 from [4] in order to calculate the kernel (31) we write it in terms of the notation in (7). Precisely, we have  $\mathcal{K}_\alpha(x, y) = \Omega_{\alpha+1}(x, y, 0)$ . Thus we arrive at the following integral equations

$$g(x) = \frac{2}{\pi^2} x \sinh \pi x \int_0^\infty \Omega_{\alpha+1}(x, y, 0) f(y) dy, \quad (38)$$

$$f(x) = h(x) + \frac{2\lambda}{\pi^2} x \sinh \pi x \int_0^\infty \Omega_{\alpha+1}(x, y, 0) f(y) dy. \quad (39)$$

Hence a unique  $L_2^\pi$ -solution of the equation (38) can be obtained via Theorem 4 by the formula

$$f(x) = \frac{2}{\pi^2} \text{l.i.m.}_{N \rightarrow \infty} x \sinh \pi x \int_{1/N}^N \frac{G(u)}{K_0(u)} K_{ix}(u) \frac{du}{u^{\alpha+2}}$$

under the condition  $\frac{G(u)}{u^{\alpha+1}K_0(u)} \in L_2(\mathbb{R}_+; \frac{du}{u})$ . Even, if  $|\lambda| < [\sup_{u>0} u^{\alpha+1} K_0(u)]^{-1}$ , then (see Theorem 5) we have a unique  $L_2^\pi$ -solution of the equation (39) written in the form (36), where  $\mathcal{M}_\alpha(x, y)$  is defined by (37) with

$$m(\theta) = \frac{2}{\pi^2} \theta \sinh \pi\theta \int_0^\infty \frac{K_0(u)}{1 - \lambda u^{\alpha+1} K_0(u)} K_{i\theta}(u) du.$$

**Example 2.** Let  $k(t) = t \sinh \pi t \left| \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \right|^4 \in L_2^\nu$ . By relation 2.16.49.2 from [4] we arrive at the following convolution integral equations

$$g(x) = 4\sqrt{2\pi} x \sinh \pi x \int_0^\infty \Omega_{\alpha+1/2}(x, y, 0) f(y) dy, \quad (40)$$



$$f(x) = h(x) + 4\lambda\sqrt{2\pi} x \sinh \pi x \int_0^\infty \Omega_{\alpha+1/2}(x, y, 0) f(y) dy. \quad (41)$$

Hence a unique  $L_2^\pi$ -solution of the equation (40) is represented by the formula

$$f(x) = \frac{1}{\pi^4\sqrt{2\pi}} \text{l.i.m.}_{N \rightarrow \infty} x \sinh \pi x \int_{1/N}^N \frac{G(u)}{K_0(u)} K_{ix}(u) \frac{du}{u^{\alpha+3/2}}$$

under condition  $\frac{G(u)}{u^{\alpha+1/2}K_0(u)} \in L_2(\mathbb{R}_+; \frac{du}{u})$ . Moreover, if  $|\lambda| < [2\pi^2\sqrt{2\pi} \sup_{u>0} u^{\alpha+1/2}K_0(u)]^{-1}$  then we have a unique  $L_2^\pi$ -solution of the equation (41) written in the form (36), where  $\mathcal{M}_\alpha(x, y)$  is defined by (37) with

$$m(\theta) = 4\sqrt{2\pi} \theta \sinh \pi \theta \int_0^\infty \frac{u^{-1/2}K_0(u)}{1 - 2\pi^2\sqrt{2\pi}\lambda u^{\alpha+1/2}K_0(u)} K_{i\theta}(u) du.$$

**Example 3.** Let  $k(t) = t \sin at$ ,  $a \neq 0$ . It is easily seen that  $k(t) \in L_2^\nu$ , when  $|\text{Im } a| < \frac{\nu}{2}$ . Appealing in [4] to relation 2.16.48.19 we get the following equations

$$g(x) = \frac{\sinh a}{\pi} x \sinh \pi x \int_0^\infty \mathcal{K}_\alpha(x, y) f(y) dy, \quad (42)$$

$$f(x) = h(x) + \frac{\lambda \sinh a}{\pi} x \sinh \pi x \int_0^\infty \mathcal{K}_\alpha(x, y) f(y) dy, \quad (43)$$

where

$$\mathcal{K}_\alpha(x, y) = \int_0^\infty u^\alpha e^{-u \cosh \alpha} K_{ix}(u) K_{iy}(u) du.$$

A unique  $L_2^\pi$ -solution of the equation (42) is given by the formula

$$f(x) = \frac{4}{\pi^3 \sinh a} \text{l.i.m.}_{N \rightarrow \infty} x \sinh \pi x \int_{1/N}^N e^{u \cosh \alpha} G(u) K_{ix}(u) \frac{du}{u^{\alpha+2}}$$

under the condition  $u^{-(\alpha+1)}e^{u \cosh \alpha}G(u) \in L_2(\mathbb{R}_+; \frac{du}{u})$ . Moreover, if  $|\lambda| < [\frac{\pi \sinh a}{2} \sup_{u>0} u^{\alpha+1}e^{-u \cosh \alpha}]^{-1}$  then we have a unique  $L_2^\pi$ -solution of the equation (43) written in the form (36), where  $\mathcal{M}_\alpha(x, y)$  is defined by (37) with

$$m(\theta) = \frac{2 \sinh a}{\pi} \theta \sinh \pi \theta \int_0^\infty \frac{K_{i\theta}(u)}{2 e^{u \cosh \alpha} - \pi \lambda \sinh a u^{\alpha+1}} du.$$

**Acknowledgement.** The present investigation was partially supported by the “Centro de Matemática” of the University of Porto and Federal Agency on Science and Innovations of Russia (grant 2006-RI-111.0/002/054). This work was completed during the second author’s research visit to University of Porto in October and November 2006.

## REFERENCES

- [1] S. B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.
- [2] S. B. Yakubovich, *The Kontorovich-Lebedev transformation on Sobolev type spaces*, Sarajevo J. Math., 1 (2005), 211–234.
- [3] A. Erdélyi et al., *Higher Transcendental Functions*, Vol.1-2, New York, Toronto, London: MC Graw Hill, 1953.
- [4] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Vol. II: Special Functions*, Gordon and Breach, New York and London, 1986.
- [5] V. A. Kakichev, *On the convolutions for integral transforms*, Izv. Acad. Nauk BSSR. Ser. Fiz.-Mat. Nauk., 2 (1967), 48–57 (in Russian).
- [6] S. B. Yakubovich, *Boundedness and inversion properties of certain convolution transforms*, J. Korean Math. Soc., 40 (6) (2003), 999–1014.
- [7] S. B. Yakubovich, *Convolution Hilbert spaces associated with the Kontorovich-Lebedev transformation*, Thai J. Math., 1 (2) (2003), 9–16.
- [8] S. B. Yakubovich, *On the convolution for the Kontorovich-Lebedev transform and its applications to integral equations*, Dokl. Acad. Nauk BSSR., 31 (2) (1987), 101–103 (in Russian).
- [9] S. B. Yakubovich, *Integral transforms of the Kontorovich-Lebedev convolution type*, Collect. Math., 54 (2) (2003), 99–110.
- [10] S. B. Yakubovich and L. E. Britvina, *A convolution for the Kontorovich-Lebedev transform*, Abstr. of the Intern. Conf. “Matematika v vyze”, Velikiy Novgorod, (2005), 98–99 (in Russian).

(Received: December 7, 2006)

(Revised: February 13, 2007)

S. B. Yakubovich  
 Department of Pure Mathematics  
 Faculty of Sciences, University of Porto  
 Campo Alegre st., 687 4169-007 Porto  
 Portugal  
 E-mail: syakubov@fc.up.pt

L. E. Britvina  
 Department of Theoretical and Mathematical Physics  
 Novgorod State University  
 ul.St.Petersburgskaya 41, Novgorod the Great, 173003  
 Russia  
 E-mail: lyubov.britvina@gmail.com