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ON NORMAL SUBGROUPS OF UNITARY GROUPS OF SOME UNITAL *AF*-ALGEBRAS

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ABSTRACT. In the case of von Neumann factors of types II_1 and III, P. de la Harpe proved that, if $\mathcal{N}$ is a normal subgroup of the unitary group which contains a non-trivial self-adjoint unitary, then $\mathcal{N}$ contains all self-adjoint unitaries of the factor. In this paper, we prove that if A is a unital AF-algebra, which is either a UHF-algebra or its dimension group $K_0(A)$ is a 2-divisible, then any normal subgroup of the unitary group contains all self-adjoint unitaries if it contains some certain non-trivial self-adjoint unitary. Afterwards, we prove that if two unitary group automorphisms agree on a normal subgroup $\mathcal{N}$ of the unitaries, which contains some certain non-trivial self-adjoint unitary, then they differ by some character on the unitary group of A.

1. INTRODUCTION

Let A be a unital C^*-algebra and let $\mathcal{U}(A)$ be the group of unitaries of A. Throughout this paper, self-adjoint unitaries of A are called the involutions of A. If we assume that the center of A is the set of scalar multiples of the unity, then the center of $\mathcal{U}(A)$ will be identified with the group circle $\mathbb{S}^1$. P. de la Harpe in [4] called the quotient group $\mathcal{U}(A)/\mathbb{S}^1$ by the projective unitary group of A and is denoted by $P\mathcal{U}(A)$. The main result, that P. de la Harpe proved in [4, Proposition 2 and 3], was that the projective unitary group $P\mathcal{U}(A)$ is a simple group if A is a factor of type II_1 or III (i.e. either finite continuous or purely infinite factors).

In particular, P. de la Harpe proved the following main theorem:

Theorem 1.1. [4] *If A is a factor of type II_1 or III, and $\mathcal{N}$ is any normal subgroup of $\mathcal{U}(A)$, which is not contained in $\mathbb{S}^1$, then $\mathcal{N} = \mathcal{U}(A)$.*

The proof was spliced into two parts, the first part was to prove that $\mathcal{N}$ contains a non-trivial involution and in the second part he proved that $\mathcal{N}$ contains all involutions of A. Afterwards, P. de la Harpe used the fact that,

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in factors, the set of involutions generates the unitary group, and this result was proved by M. Broise in [1, Theorem 1].

In this paper, we study some cases of approximately finite C^*-algebras (AF-algebras). Recall that a C^*-algebra A is an AF-algebra if $A = \overline{\cup A_n}$ such that for every $n \geq 1$, A_n is a finite dimensional C^*-subalgebra of A, and $A_n \subseteq A_{n+1}$. In other words A is an AF-algebra if it is the C^*-algebraic direct limit of a directed system of finite dimensional C^*-algebras and C^*-algebra homomorphisms, see [2, Ch. III] and [5, 5.6] for more details. First we consider the case that A is a unital AF-algebra, its dimension group $K_0(A)$ is a 2-divisible group, i.e., every element can expressed as a product of 2, and we show that if $\mathcal{N}$ is any normal subgroup of A, which contains a non-trivial involution, then $\mathcal{N}$ contains all the involutions of A.

Also, without any condition on the dimension group $K_0(A)$ of an AF-algebra A, we study the case that A is a UHF-algebra. Recall that a UHF-algebra A is an AF-algebra with C^*-subalgebra

$$A_n = \bigotimes_{j=1}^{n} \mathbb{M}_{k_j}(\mathbb{C}) \approx \mathbb{M}_{k_1 k_2 \dots k_n}(\mathbb{C}),$$

where $(k_j)_{j\geq 1}$ is a sequence of integers with $k_j \geq 2$, and the embedding from A_n to A_{n+1} is given by $x \longmapsto x \otimes 1_{k_{n+1}}$ (for more details see [5, §5.9]). If for all n, the size of A_n is even, then it is 2-divisible UHF-algebra. We prove by the same argument as in a 2-divisible AF-algebra's case, i.e., if $\mathcal{N}$ is any normal subgroup of A, which contains some certain non-trivial involution, then $\mathcal{N}$ contains all involutions of A.

As a good application of the above result, we apply it to the unitary group automorphism of an AF-algebra A. If $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(A)$, which contains a non-trivial involution, and φ, ψ are two automorphisms of $\mathcal{U}(A)$, which agree on $\mathcal{N}$, then $\varphi = \lambda\psi$ for some character λ of $\mathcal{U}(A)$. In particular, if ψ is a $*$-automorphism of A, then we are able to say that φ is implemented by ψ (up to some character). The extension problem of a group automorphism of the unitary group to a $*$-automorphism of A was discussed in the case of von Neumann factors by H. Dye [3], and in the case of AF-algebras by the author in [6].

2. Main result

We recall some definitions and results from [4] concerning the case of factors, and we extend these results to some cases of unital AF-algebras. If u is an involution of a unital C^*-algebra A, then $u = 1 - 2p$, where p is a projection of A.

Let $\mathcal{B}$ be a factor and recall the dimension function D on the set of projections of $\mathcal{B}$, P. de la Harpe in [4] defined the type of u to be the pair

(x, y), with $x = D(1-p)$ and $y = D(p)$. Then he used this definition to prove his main result.

Let A be a unital AF-algebra, recall that the scale of A (which generates $K_0^+(A)$, and is denoted by ΣA) is realized as the equivalence classes of projections in A (see [2, Theorem IV.1.6]). Therefore, let us define the type of involutions of unital AF-algebras as follows:

Definition 2.1. *Let A be a unital AF-algebra and let u be an involution in $\mathcal{U}(A)$. We define the type of u to be the element $[\frac{1-u}{2}]$ in the scale ΣA.*

Then we extend [4, Lemma 5] in the following result.

Lemma 2.2. *Let u, v be any two involutions in a unital AF-algebra A. Then u and v are conjugate in $\mathcal{U}(A)$ if and only if they have the same type.*

Proof. If u and v are two conjugate involutions in $\mathcal{U}(A)$, then there exists a unitary w in A such that $u = wvw^*$. But $u = 1-2e$ and $v = 1-2f$ for some projections e, f in A, so $u = w(1-2f)w^* = 1-2wfw^*$, therefore $e = wfw^*$, and hence $[e] = [f]$.

Conversely, assume that the involutions u and v have the same type. This implies that $e \sim_u f$. So for some unitary w, we have that $e = wfw^*$, therefore

$$u = 1 - 2(wfw^*) = 1 - 2(\frac{1 - wvw^*}{2}) = wvw^*,$$

hence the proof of the lemma is completed. □

For a unital C^*-algebra A, let $\mathcal{P}(A)$ denote the set of all projections of A. Now, we prove the following lemma.

Lemma 2.3. *Let A be a unital AF-algebra and let e be a non-trivial projection in A . If $a \in K_0^+(A)$ with $0 < a \leq [e]$, then there exists a projection p in A such that $[p] = a$, $p \leq e'$, and $e' \sim e$ for some $e' \in \mathcal{P}(A)$. Indeed, the projections p and e' are in $A_\infty = \cup_{n=1}^{\infty} A_n$.*

Proof. Given $e \in \mathcal{P}(A)\backslash\{0,1\}$, $A = \overline{A_\infty}$. Then there exists an $e' \in \mathcal{P}(A_\infty)$ $\backslash\{0,1\}$ such that $e' \sim e$, which means that $[e'] = [e]$. As A is an AF-algebra, the scale ΣA of A equals to $\{[p];\ p \in \mathcal{P}(A)\ \}$ indeed, $\Sigma A = [0, [1]]$. Since $[e] \in \Sigma(A)$, we have that $a \in \Sigma A$, therefore $a = [f]$ for some $f \in \mathcal{P}(A)$. Then there exists a projection $f' \in \mathcal{P}(A_\infty)$ such that $f \sim f'$, which means that $a = [f']$.

Now choose an n large enough so that the projections f' and e' belong to A_n. From the hypothesis we have that $[f'] \leq [e']$ the other hand,

$$A_n = \mathbb{M}_{n_1} \oplus \mathbb{M}_{n_2} \oplus \cdots \oplus \mathbb{M}_{n_k}.$$

Therefore,

$$e' = (e_1, e_2, \ldots, e_k) \text{ and } f' = (f_1, f_2, \ldots, f_k),$$

where $e_i, f_i \in \mathcal{P}(\mathbb{M}_{n_i}(\mathbb{C}))$.

Recall that $K_0(A_n) = \mathbb{Z}^k$, and $\mathbb{Z}^k$ has the simplicial ordering. Denote the class of e_i, f_i in $\mathbb{Z}^+$ by $\overline{e_i}$, $\overline{f_i}$, respectively. Therefore,

$$[e'] = (\overline{e_1}, \overline{e_2}, \ldots, \overline{e_k}) \text{ and } [f'] = (\overline{f_1}, \overline{f_2}, \ldots, \overline{f_k})$$

and for every $1 \leq l \leq k$, we have that $\overline{f_l} \leq \overline{e_l}$. But f_l, e_l are in $\mathbb{M}_{n_l}(\mathbb{C})$, so this means that $\mathrm{rank}(f_l) \leq \mathrm{rank}(e_l)$. Then there exists a projection $g_l \in \mathcal{P}(\mathbb{M}_{n_l}(\mathbb{C}))$ such that g_l has the same rank as f_l moreover, $g_l \leq e_l$.

Finally, choose $p = (g_1, g_2, \ldots, g_k) \in \mathcal{P}(A_n)$. Therefore, $a = [p]$, moreover $p \leq e' \sim e$, which completes the proof. □

Proposition 2.4. *Let A be a unital AF-algebra. If q, f and e are projections in A such that $q \leq f$ and $[f] = [e]$, then there exists a projection t in A such that $[t] = [q]$ and $t \leq e$.*

Proof. As $e \sim_u f$, there exists a unitary u in A such that $e = ufu^*$. Let $t = uqu^*$. Then $[t] = [q]$, and $t \leq e$, hence the proof is completed. □

Now, we shall prove the following main lemma which is similar to a result in the case of factors of type II, proved by P. de la Harpe ([4, Lemma 6]).

Lemma 2.5. *Let A be a unital AF-algebra. If e is a non-trivial projection in A, and $r \in K_0^+(A)$ such that $0 < r \leq [e]$ and $r \leq [1-e]$, then there exists an involution in A of type $2r$.*

Proof. By Lemma 2.3, there exist projections p and e' in A_∞ such that

$$r = [p], \quad p \leq e' \text{ and } e' \sim e.$$

Also, we have projections t and f' in A_∞ such that

$$r = [t], \quad t \leq f' \text{ and } f' \sim 1 - e.$$

Notice that $[f'] = [1 - e']$, therefore, by Proposition 2.4, there exists a projection q such that

$$[t] = [q] \quad \text{and} \quad q \leq 1 - e'.$$

Therefore, p and q are orthogonal equivalent projections. If s is a partial isometry such that $s^*s = p$ and $ss^* = q$, then $sq = ps = 0$ and $s^2 = (sp)(qs) = 0$.

If $w = e' - p + s + s^* = w^*$, then $w^2 = e' + q$ and if we put $v = w + (1 - e' - q)$, then we have

$$v^2 = 1 + w(1 - e' - q) + (1 - e' - q)w = 1,$$

therefore, v is an involution in A. Moreover,

$$\begin{aligned} ve'v &= (w + (1 - e' - q))e'(w + (1 - e' - q)) \\ &= we'(w + (1 - e' - q)) \end{aligned}$$

$$\begin{aligned}&= wew \\ &= (e' - p + sp)(e' - p + s + s^*) \\ &= e' - p + q.\end{aligned}$$

Let u' be the involution defined by $u' = 1 - 2e'$. Then we have

$$\begin{aligned}u'vu'v &= (1 - 2e')v(1 - 2e')v \\ &= (v - 2e'v)(v - 2e'v) \\ &= 1 - 2ve'v - 2e' + 4e've'v \\ &= 1 - 2(e' - p + q) - 2e' + 4e' - 4p \\ &= 1 - 2(p + q).\end{aligned}$$

Therefore, $u'vu'v$ is an involution in A of type $2r$, as desired. □

The following corollary is a consequence of the previous lemma.

Corollary 2.6. *Let A be a unital AF-algebra. If $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(A)$, which contains a non-trivial involution u ($u = 1 - 2e$), and r is the same as in the previous lemma, then $\mathcal{N}$ contains an involution of type $2r$.*

Proof. Let u' be the same involution as in the proof of the previous lemma. Then u' and u are conjugate involutions, and hence $u' \in \mathcal{N}$, therefore, the involution $u'vu'v \in \mathcal{N}$, which is of the type $2r$. □

Now we proceed to prove our main results. We study the case of UHF-algebras, whose K_0 are not 2-divisible groups, for such algebras, we prove that any normal subgroup of $\mathcal{U}(A)$ contains all involutions if it contains at least a single non-trivial involution. Afterwards, we prove the validity of the result for any UHF- algebras.

If A is a UHF-algebra such that its dimension group $K_0(A)$ is not a 2-divisible, then the generalized integer $\bar{n}$ of A equals $(2^{n_2}, 3^{n_3}, 5^{n_5}, \ldots)$, where $0 \leq n_2 < \infty$ and $0 \leq n_p \leq \infty$ for every odd prime $p > 2$. Indeed,

$$K_0(A) \simeq \left\{ \frac{m}{2^{n_2} p_1 p_2 \ldots p_k} \mid m \in \mathbb{Z},\ k \in \mathbb{Z}^+, 1 \leq i \leq k, \right.$$
$$\left. \text{and } p_i \text{ is an odd prime} \right\}.$$

For every $n \geq 0$, put $s_n = 2^{n_2} p_1 \ldots p_n$. Let τ denote the normalized trace of A.

For such C^*-algebras, we have the following lemma:

Lemma 2.7. *Let A be a UHF-algebra such that its $K_0(A)$ is not a 2-divisible group. If $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(A)$ which contains an involution of type $\frac{m}{s_n} \in \tau(K_0(A))$ for some odd integer m and some non-negative integer n, then $\mathcal{N}$ contains all involutions of A.*

Proof. Given any normal subgroup $\mathcal{N}$ of $\mathcal{U}(A)$, first we assume that $\mathcal{N}$ contains an involution of type $\frac{1}{s_n}$ for some $n \geq 0$, and hence any involution of type $\frac{1}{s_n}$ therefore, $\mathcal{N}$ contains any involution of type $\frac{s}{s_n}$ for all $1 \leq s \leq s_n$.

It is enough to show that $\mathcal{N}$ contains an involution of type $\frac{1}{s_{n+1}}$. We know that $\mathcal{N}$ contains any involution of type $\frac{sp_{n+1}}{s_{n+1}}$ for all $1 \leq s \leq s_n$. As

$$\frac{1}{s_{n+1}} < \frac{p_{n+1}}{s_{n+1}} \quad \text{and} \quad \frac{1}{s_{n+1}} < 1 - \frac{p_{n+1}}{s_{n+1}},$$

by Lemma 2.5, $\mathcal{N}$ contains an involution of type $\frac{2}{s_{n+1}}$, and hence any involution of type $\frac{2m}{s_{n+1}}$ such that $1 \leq 2m \leq s_{n+1}$.

Now choose an odd integer s such that $1 \leq s \leq s_n$ therefore, sp_{n+1} is an odd integer, $sp_{n+1} - 1 < s_{n+1}$ and

$$\frac{1}{s_{n+1}} = \frac{sp_{n+1}}{s_{n+1}} - \frac{sp_{n+1} - 1}{s_{n+1}}.$$

Let $u = 1 - 2p$, and $v = 1 - 2q$ indeed,

$$[p] = \frac{sp_{n+1}}{s_{n+1}}, \ [q] = \frac{sp_{n+1} - 1}{s_{n+1}}$$

in $\tau(K_0(A))$ and $q < p$, then $uv \in \mathcal{N}$, moreover, type of uv is $\frac{1}{s_{n+1}}$.

In the general case, assume that $\mathcal{N}$ contains an involution of type $\frac{m}{s_n}$ for some odd integer $m > 1$ and some non-negative integer n. As $\frac{1}{s_n} \leq \frac{m}{s_n}$ and $\frac{1}{s_n} \leq 1 - \frac{m}{s_n}$, then $\mathcal{N}$ contains any involution of type $\frac{2s}{s_n}$ such that $2 \leq 2s \leq s_n$. Let $u = 1 - 2q$ and $v = 1 - 2r$ such that $[q] = \frac{m}{s_n}$, $[r] = \frac{m-1}{s_n}$ and $r \leq q$. Then $uv \in \mathcal{N}$ and its type is $\frac{1}{s_n}$, hence the proof is completed. □

A similar result to the preceding one is now given in the case of unital AF-algebras, whose K_0-dimension groups are 2-divisible.

Recall that if A is a unital C^*-algebra, then a set $\{e^r_{i,j}\}$, $1 \leq i, j \leq n$ and $1 \leq r \leq m$ of elements of A is said to be a system of matrix units in A if it satisfies the followings:

$$e^r_{i,j} e^r_{j,k} = e^r_{i,k}, e^r_{i,j} e^s_{k,l} = 0 \text{ if } r \neq s \text{ or } j \neq k, \ (e^r_{i,j})^* = e^r_{j,i}, \ \sum_{i,r}^{n,m} e^r_{i,i} = 1,$$

and $e_{i,i}$ is a projection in A for every i, this projection is denoted by p_i.

In [6, Corollary 2.5.3.2], the author proved that if A is a unital AF-algebra and if for some projection p of A, $n[p] = [1]$ in $K_0(A)$ for some $n > 1$, then A has a system of matrix units of dimension n. Also in [6, Proposition 2.5.3.3], the author proved that if A is a unital AF-algebra such that $K_0(A)$ is a 2-divisible ordered group, then for any $k > 1$, A has a system of matrix units of dimension 2^k. Using the idea that in such C^*-algebras, we can express elements of K_0 as a product of 2, then we have the following lemma.

Lemma 2.8. *Let A be a unital AF-algebra such that the dimension group $K_0(A)$ is a 2-divisible group. If $\mathcal{N}$ is any normal subgroup of $\mathcal{U}(A)$ which contains a non-trivial involution of type $[p_1]$, then $\mathcal{N}$ contains all involutions of A.*

Proof. As $\{e_{i,j}\}_{i,j=1}^{2^k}$ is a system of matrix units of A for some $k > 1$, we have $[1] = 2^k[p_1]$. We will do the proof for the case $k = 2$, and the general case is similar. From the assumption $[p_1]$ is a type of a non-trivial involution in $\mathcal{N}$. Therefore for every r belongs to the scale ΣA of A such that $0 < r \leq [p_1]$, apply Lemma 2.5, to get an involution in $\mathcal{N}$ of type $2r$. Indeed, an involution of type $2[p_1]$.

Again, for every r such that $0 < r \leq 2[p_1]$, we get an involution in $\mathcal{N}$ of type $2r$, indeed an involution of type $[1]$. For a general case, continue applying the lemma to get involutions in $\mathcal{N}$ of types: $8[p_1], \ldots, 2^{k-1}[p_1], [1]$.

Now for any $[p]$ in the scale of A, therefore $0 \leq [p] \leq 4[p_1]$ and $[p] = 2[t]$, for some projection t in A. As $K_0(A)$ is unperforated, we have $2[p_1] - \frac{1}{2}[p] \in K_0^+(A)$, hence $[t] \leq 2[p_1]$, therefore by the first part of the proof, we get an involution in $\mathcal{N}$ of type $[p]$. So we have proved that $\mathcal{N}$ contains an involution of any given type.

Now to finish the proof, if u is any involution of A, then by Lemma 2.2, u is a conjugate to some involution in $\mathcal{N}$ therefore, $u \in \mathcal{N}$. □

More generally, for the case of unital AF-algebras, whose dimension groups are not necessary 2-divisible groups, we have the following.

Proposition 2.9. *Let A be a unital AF-algebra with a set of matrix units $\{e_{i,j}\}_{i,j}^n$, and m_0 be the largest positive integer such that $2^{m_0+1} \leq n$. If $\mathcal{N}$ is any normal subgroup of $\mathcal{U}(A)$ which contains an involution of type $[p_1]$, then $\mathcal{N}$ contains any involution of the following types: $s[p_1]$ for all $1 \leq s \leq n$ and $2r$ for all $0 < r \leq 2^{m_0}[p_1]$.*

Proof. Recall that for all $1 \leq i \leq n$, $u_i = 1 - 2p_i$. As $\mathcal{N}$ contains an involution of type $[p_1]$, by Lemma 2.2, we have that $\mathcal{N}$ contains all the $u_i's$, and hence for all $1 \leq s \leq n$, the product $\prod_{i=1}^{s} u_i$ belongs to $\mathcal{N}$, which is of type $s[p_1]$ in particular, $-1 \in \mathcal{N}$.

By Lemma 2.5, there is an involution in $\mathcal{N}$ of type $2r$ for all $0 < r \leq 2^{m_0}[p_1]$, hence $\mathcal{N}$ contains any involution of such type. □

Therefore, we now have the following main result in this paper.

Theorem 2.10. *If A is a unital AF-algebra such that either*

(i) *The dimension group $K_0(A)$ is a 2-divisible, or*

(ii) *A is any UHF-algebra,*

and $\mathcal{N}$ is a normal subgroup of the unitary group $\mathcal{U}(A)$, which contains an involution of type $[p_1]$, then $\mathcal{N}$ contains all involutions of A.

Proof. Lemma 2.8 proves (i). Lemma 2.7 together with (i) proves (ii), and hence the proof of the theorem is completed. □

Consequently, we have the following result concerning the group automorphisms of $\mathcal{U}(A)$.

Corollary 2.11. *Let A and $\mathcal{N}$ be as in Theorem 2.10. If φ and ψ are automorphisms of $\mathcal{U}(A)$ such that $\varphi = \psi$ on $\mathcal{N}$, then $\varphi = \lambda\psi$ for some possible character λ of $\mathcal{U}(A)$.*

Proof. By Theorem 2.10, we have that $\mathcal{N}$ contains all involutions of A, hence $\varphi = \psi$ on the normal subgroup K which is generated by all involutions of A. Therefore, as in [6, Theorem 6.3.0.8], we have that $\varphi = \lambda\psi$ for some possible character λ of $\mathcal{U}(A)$. □

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