ON THE FUNCTIONAL EQUATION $U_t + U_{-t} = V_t + V_{-t}$ IN A BANACH SPACE

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Abstract. In this paper we consider commuting one-parameter groups, 
$\{U_t : t \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ of unitary operators and the functional 
equation $U_t + U_{-t} = V_t + V_{-t}$ on a reflexive strictly convex Banach space 
with Gâteaux differentiable norm.

The operator equation $\alpha + \alpha_1 = \beta + \beta_{-1}$, where $\alpha$ and $\beta$ are $^*$-automor-
phisms on a Von Neumann algebra, has an important role in the geometric 
interpretation of Tomita-Takesaki modular theory and its generalization for 
Jordan algebras. Commuting one-parameter groups $\{U_t : t \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ of unitary operators on Hilbert space $H$ such that $U_t + U_{-t} = V_t + V_{-t}$, 
for all $t \in \mathbb{R}$, are considered in [4].

The following theorem is proved in [4].

Theorem 1. Let $\{U_t : t \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ be two commuting one-
parameter groups of unitary operators on a Hilbert space $H$ such that $U_t + U_{-t} = V_t + V_{-t}$, 
for all $t \in \mathbb{R}$. Then there is projection $P$ on $H$ such that $U_t = V_t$ on $PH$, $U_t = V_{-t}$ on $(I - P)H$ and $P$ commutes with $U_t$ and $V_t$ for 
all $t \in \mathbb{R}$.

We can remark that it is not explicitly stated in Theorem 1 that $U_t$ and $V_t$ are strongly continuous groups. However, in the proof of this theorem in 
[4] it is taken into account that $U_t$ and $V_t$ are strongly continuous groups. In 
this note, we consider one-parameter groups $\{U_t : t \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ of unitary operators on reflexive strictly convex Banach space with Gâteaux 
differentiable norm.

Let $X$ be real normed linear space and $f$ a functional defined on $X$. Recall 
that by the first right-hand Gâteaux derivative of $f$ at $x$ in the direction $h$ 
we mean
$$f'_+(x)(h) = \lim_{t \to +0} \frac{f(x + th) - f(x)}{t}. \quad (1)$$

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We have an analogous definition for first left-hand Gâteaux derivative of $f$ at $x$ in the direction $h$. If $f'_+(x)(h) = f'_-(x)(h)$ we say that $f$ is Gâteaux differentiable at $x$ in the direction $h$. Let $f(x) = \frac{1}{2}\|x\|^2$ and $\langle x, y \rangle = f'_+(x)(y)$. It is easy to prove that this derivative exists.

The proof of the next proposition is given in [3].

**Proposition 2.** Every real normed linear space is a generalized inner product space in the sense that

(a) $\langle x, y \rangle$ is well defined;
(b) $\|x\| = \langle x, x \rangle^{1/2}$;
(c) $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ (Cauchy - Schwarz - Buniakovsky inequality);
(d) If $X$ is an inner product space with inner product $[x, y]$ then $\langle x, y \rangle = [x, y]$.

**Definition.** A normed linear space $X$ is said to be strictly convex if $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|\frac{x+y}{2}\| < 1$.

Some features of strictly convex space are given in [2]. The next theorem is proved in [3].

**Theorem 3.** Suppose $X$ is real Banach space. Then the Riesz representation theorem holds: Given $\delta \in X^*$, there exist $x_\delta \in X$ such that

$$\delta(y) = \langle x_\delta, y \rangle \quad (\forall y \in X) \quad \text{and} \quad \|x\| = \|x_\delta\|$$

if and only if $X$ is reflexive with the Gâteaux differentiable norm.

Furthermore $x_\delta$ is unique (and mapping $\delta \to x_\delta$ is continuous from the norm topology on $X^*$ to the weak topology on $X$) if and only if $X$ is also strictly convex.

In addition the mapping $\delta \to x_\delta$ is also continuous from the norm topology on $X^*$ to the norm topology on $X$ if and only if $X$ is also weakly uniformly convex.

From now on, let $X$ be complex strictly convex Banach space $X$ with Gâteaux differentiable norm. Let

$$\langle x, y \rangle \overset{\text{def}}{=} \langle x, y \rangle - i\langle x, iy \rangle.$$

It is easy to prove that

$$\langle x, iy \rangle = i\langle x, y \rangle,$$

$$\langle ix, y \rangle = -i\langle x, y \rangle. \quad \text{(3)}$$

Function $(x, y)$ is linear relative to $y$, however $(x, y)$ is not linear relative to $x$. It can be shown that $(\lambda x, y) = \overline{\lambda} (x, y)$. Using known methods it can be proved that

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$
ON THE FUNCTIONAL EQUATION $U_{t} + U_{-t} = V_{t} + V_{-t}$ IN A BANACH SPACE 243

Similar to Theorem 3, in the case of a complex Banach space the following holds.

**Theorem 3’.** Given $\delta \in X^*$, there exist unique $x_\delta \in X$ such that

$$\delta(y) = (x_\delta, y) \quad (\forall y \in X) \quad \text{and} \quad \|x\| = \|x_\delta\|$$

if and only if $X$ is reflexive strictly convex Banach space with Gâteaux differentiable norm.

As $\delta \to x_\delta$ is a bijective mapping from $X^*$ to $X$ we can define its inverse mapping $\varphi : X \to X^*$

$$\varphi(x)(y) = (x, y) \quad y \in X.$$ 

Now, for all $x, y \in X$ we can introduce a new operation $^*$ in $X$ by

$$x^* + y = \varphi^{-1}(\varphi(x) + \varphi(y)) \quad x, y \in X.$$ 

The space $X$ provided with the operation $+$ will be denoted by $(X, +)$, and by $(X, ^*)$ we mean the space $X$ provided with the operation $^*$. Previously, we noted that $(x, y)$ is linear relative to $y$ in the $(X, +)$.

Next note that

$$(x^* + y, z) = (x, z) + (y, z)$$

i.e. $(x, y)$ is linear relative to $x$ in the space $(X, ^*)$.

**Definition.** A linear operator $U : X \to X$ is said to be isometric, i.e. unitary if

1) $\|Ux\| = \|x\| \quad \forall x \in X,$
2) $UX = X$, i.e. $U$ is mapping from $X$ on $X$.

Let us show that $U$ preserves inner product, i.e.

$$(Ux, Uy) = (x, y).$$

We have

$$(Ux, Uy) = \langle Ux, Uy \rangle - i \langle Ux, Uiy \rangle$$

$$= \|Ux\| \lim_{t \to 0} \frac{\|Ux + tUy\| - \|Ux\| - i\|Ux\| \lim_{t \to 0} \|Ux + itUy\| - \|Ux\|}{t}$$

$$= \|x\| \lim_{t \to 0} \frac{\|x + ty\| - \|x\| - i\|x\| \lim_{t \to 0} \|x + ity\| - \|x\|}{t}$$

$$= \|x\| \|T(y) - i\|x\|T(iy) = (x, y) - i(x, iy) = (x, y).$$
Remark*. Taking into account that $U$ preserves inner product and that $(x, y)$ is linear relative to $x$ in the space $(X, +)$ we have that every unitary operator $U : X \to X$ is linear in the space $(X, +)$, i.e.

$$U(x + y) = Ux + Uy,$$
$$U(\lambda x) = \lambda Ux.$$ 

The next theorem is proved in [5].

**Theorem 4.** Let $X$ be complex reflexive and strictly convex Banach space with Gâteaux differentiable norm. For every closed linear subspace $L$ of the space $(X, +)$ there exists a subspace $L^*$ of the space $(X, +)$ such that $X = L \oplus L^*$ (i.e. every $x \in X$ can be written in unique way in the form $x = l + l^*$, $l \in L$, $l^* \in L^*$, $\langle l^*, l \rangle = 0$).

Note that a theorem similar to Theorem 4 is valid for the space $(X, +)$.

**Theorem 5.** Let $X$ be Banach space with the same properties as in Theorem 4. Let $L$ and $L^*$ be subspaces of $(X, +)$ and $(X, +)$ respectively, such that $X = L \oplus L^*$ and let $U_t$ be group of unitary operators on $X$. If $L$ is invariant under $U_t$ then $L^*$ is also invariant under $U_t$.

**Proof.** Suppose $L$ is invariant under $U_t$. This means, for $l \in L$ we have $U_t l \in L$, $t \in \mathbb{R}$. Take $l \in L$ and $l^* \in L^*$. We have

$$0 = \langle l^*, U_t l \rangle = \|l^*\| \lim_{t \to 0} \|l^* + h U_t l\| - \|l^*\|$$
$$= \|U_{-t} l^*\| \lim_{t \to 0} \|U_{-t} l^* + h l\| - \|U_{-t} l^*\| = \langle U_{-t} l^*, l \rangle.$$ 

Thus, $U_{-t} l^* \in L^*$.

Since this holds for every $t \in \mathbb{R}$, the theorem is proved. \qed

Before we give a generalization of Theorem 1, let us prove the following lemma.

**Lemma 6.** Let $\{U_t : t \in \mathbb{R}\}$ and $\{V_t : t \in \mathbb{R}\}$ be two commuting, strongly continuous one-parameter groups of unitary operators on reflexive strictly convex Banach space $X$ with Gâteaux differentiable norm. Suppose that

a) $U_t + U_{-t} = V_t + V_{-t}$, for all $t \in \mathbb{R}$.

b) There is no $x \in X$, $x \neq 0$, such that $U_t x = V_t x$ for all $t \in \mathbb{R}$.

Then

$$U_t x = V_{-t} x,$$  \quad $\forall x \in X$, $\forall t \in \mathbb{R}$. 

ON THE FUNCTIONAL EQUATION $U_t + U_{-t} = V_t + V_{-t}$ IN A BANACH SPACE 245

Proof. Let $C_t = \frac{U_t + U_{-t}}{2}$, \forall $t \in R$, i.e. $C_t = \frac{V_t + V_{-t}}{2}$, \forall $t \in R$.

Then, $C_t$ is strongly continuous cosine operator function on $X$. If $C$ is infinitesimal generator of $C_t$, and $A$ and $B$ are infinitesimal generators of $U_t$ and $V_t$ respectively, then $-C = A^2 = B^2$.

Using the fact that $U_t$ and $V_t$ commute, we get

$$(A - B)(A + B)x = 0, \quad x \in D_c.$$  

Let $(A + B)x = y$. Then $(A - B)y = 0$. Furthermore, for all $t \in R$ we have

$$\frac{d(U_t V_{-t})}{dt} y = AU_t V_{-t} y - BU_t V_{-t} y = U_t V_{-t} (A - B)x = 0.$$  

Thus, for all $t \in R$ it is $U_t V_{-t} y = \tilde{y}$, where $\tilde{y}$ is a constant vector.

Hence, for $t = 0$, we get $\tilde{y} = y$. So, $U_0 y = V_0 y$ for all $t \in R$. Taking into account b), this implies $y = 0$, i.e. $(A + B)x = 0$.

Thus, taking $\frac{d(U_t V_{-t})}{dt}$ we get $U_t V_{-t} x = x$, i.e. $U_t x = V_{-t} x$, $x \in D_c$.

As the set $D_c \subset X$ is dense in $X$ and the groups $U_t$ and $V_t$ are strongly continuous, we have

$$U_t x = V_{-t} x, \quad \forall x \in X, \forall t \in R.$$  

\square

Now, we can prove a generalization of the Theorem 1.

**Theorem 7.** Let $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ be two commuting, strongly continuous one-parameter groups of unitary operators on reflexive strictly convex Banach space $X$ with Gâteaux differentiable norm such that

$$\frac{U_t + U_{-t}}{2} = \frac{V_t + V_{-t}}{2}, \quad \forall t \in R.$$  

Then there exist subspaces $L \subseteq (X, +)$ and $L^* \subseteq (X^*, +)$ such that $X = L \oplus L^*$ and $U_t = V_t$ on $L$ and $U_t = V_{-t}$ on $L^*$.

Proof. Let $L = \{l | U_t l = V_t l, \forall t \in R\}$. It can be easily shown that $L$ is linear closed subspace of the space $(X, +)$. $L$ is invariant under $\{U_t\}$.

Let $l \in L$. Then $U_t l = V_t l$. Performing $U_s$ on $U_t l$ we get

$$U_s(U_t l) = U_{s+t} l,$$

and on the other hand

$$U_s(U_t l) = U_s(V_t l) = V_t U_s l = V_t V_s l = V_s U_t l = V_s(U_t l),$$

i.e. $U_t l \in L$. It can be easily proved that $L$ is invariant under $\{V_t\}$.

According to Theorem 4, there is subspace $L^*$ of the space $(X, +)$ such that $X = L \oplus L^*$.

By Theorem 5, the subspace $L^*$ is also invariant under $U_t$ and under $V_t$. 

Moreover, taking in the account Remark\(^\ast\), it can be easily proved that operators \(U_t\) and \(V_t\) are linear in \((X, +^\ast)\), and thus, in \(L^\ast\). As \(X\) is reflexive space, \(U_t\) and \(V_t\) are strongly continuous semigroups in \(L^\ast\). From the definition of the subspace \(L^\ast\) it follows that there is no \(x \in L^\ast\), \(x \neq 0\) such that \(U_t x = V_t x\), \(\forall t \in R\).

Now, we can apply Lemma 6 to obtain the statement of the theorem. □

The notion of the Hilbert transform on local convex space \(X\) is given in paper [1]. In our note, we will consider the limit \(\lim_{\varepsilon \to 0} \lim_{N \to \infty} H_{\varepsilon,N} x\) on reflexive strictly convex Banach space \(X\) with Gâteaux differentiable norm, where \(H_{\varepsilon,N} x = \int_{\varepsilon \leq t < N} U_t x \frac{dt}{t^2} (x \in X, U_t\) unitary group of operators).

If \(\bar{x} = \lim_{N \to \infty} H_{\varepsilon,N} x\) exists then \(\bar{x}\) is called the Hilbert transform of the element \(x\) generated by group \(U_t\) and it is denoted by \(\bar{x} = H x\).

In the following part of paper we will need some results obtained in [5]. From the Theorem 7 in [5] it follows that there are subspaces \(L\) and \(M\) of \((X, +)\) such that

a) Space \((X, +)\) is the direct sum of \(L\) and \(M\).

b) \(C_t x = x, \forall x \in L, \forall t \in R\).

c) \(M\) is invariant under all operators \(C_t = \frac{U_t + U_{-t}}{2} = \frac{V_t + V_{-t}}{2}\).

d) If we consider \(C_t\) on \(M\), then point \(0\) does not belong to punctual spectrum of infinitesimal generator \(C\) of \(C_t\).

From d) it follows, as seen in [5], that in the subspace \(M\) operators \(A\) and \(B\) do not have eigenvectors that correspond to the eigenvalue \(0\). Also, taking into account that \(U_t\) and \(V_t\) are unitary operators and \(X\) is strictly convex, a) implies that

\[ U_t x = V_t x = x, \quad \forall t \in R \text{ if and only if } x \in L. \]

Finally, from the proof of Theorem 7 in [5], it follows that \(< l, m > = 0, \forall l \in L, \forall m \in M\).

Using the previously mentioned facts we can now easily prove that \(M\) is invariant under unitary operators \(U_t\) and \(V_t\).

For \(m \in M\), let \(U_t m = l_t + m_t, l_t \in L, m_t \in M\). Then for any \(x \in L\) and \(m \in M\), we have

\[ 0 = < x, m > = < U_{-t} x, m > = < x, U_t m > = < x, l_t + m_t > = < x, l_t > + < x, m_t > = < x, l_t >, \text{ i.e. } l_t = 0. \]

Thus, \(U_t m \in M\). We can show in a similar way that \(M\) is invariant under \(V_t\).

Also, it can be easily seen that there is no vector \(m \in M, m \neq 0\) such that \(U_t m = V_t m = m\).
Now, by applying Theorem 7 on space $M$ we can obtain subspaces $M_1$ of $M, +$) and $M_2$ of $(M, +)$ such that $M = M_1 \oplus M_2$, and
\[ U_1 x = V_1 x, \quad \forall x \in M_1, \forall t \in R \]
\[ U_2 x = V_2 x, \quad \forall x \in M_2, \forall t \in R \]
and $< m_2, m_1 > = 0$ for all $m_1 \in M_1$ and $m_2 \in M_2$. Moreover, we know that $M_1$ and $M_2$ are invariant under $U_t$ and $V_t$.

Let us show that $M_2$ is linear subspace of $(M, +)$. Take $m', m'' \in M_2$. Obviously we can write $m' + m'' = m_1 + m_2$, $(m_1 \in M_1, m_2 \in M_2)$. Also, we have $U_t m' = V_{-t} m', U_t m'' = V_{-t} m''$. Applying $U_t$ on $m' + m'' = m_1 + m_2$ and using previous equalities we get:
\[ V_{-t} m' + V_{-t} m'' = V_t m_1 + V_{-t} m_2. \tag{5} \]
On the other hand, if we apply $V_{-t}$ on $m' + m'' = m_1 + m_2$ we obtain
\[ V_{-t} m' + V_{-t} m'' = V_{-t} m_1 + V_{-t} m_2. \tag{6} \]
Comparing (5) and (6) we have
\[ V_t m_1 + V_{-t} m_2 = V_{-t} m_1 + V_{-t} m_2, \text{ for all } t \in R, \]
where $m_1 \in M_1$, $m_2 \in M_2$. Thus, $V_t m_1 = V_{-t} m_1$, for all $t \in R$. This implies $V_t m_1 = m_1$, for all $t \in R$. So, $m_1 = 0$.

Therefore we have proved following theorem:

**Theorem 8.** Under the previous assumptions on the space $X$ and the operators $U_t$ and $V_t$, $X$ can be written as direct sum of its three linear subspaces $L, M_1$ and $M_2$ that are invariant under all operators $U_t$ and $V_t$, $t \in R$ and such that
1) $U_t x = V_t x = x$, $\forall x \in L$, $\forall t \in R$.
2) $U_t x = V_t x$, $\forall x \in M_1$, $\forall t \in R$.
3) $U_t x = V_{-t} x$, $\forall x \in M_2$, $\forall t \in R$.
4) $< l, m_1 + m_2 >= 0$ and $< m_2, m_1 > = 0$ for all $l \in L, m_1 M_1, m_2 \in M_2$.

**Note.** If we use the notation from [5] where $A_+$ represents positive second root of $-C(= A^2 = B^2)$, then it is easy to see that $L, M_1$ and $M_2$ are invariant under $A_+$.

From Theorem 8 we have the next theorem:

**Theorem 9.** If $\{ U_t : t \in R \}$ and $\{ V_t : t \in R \}$ are two commuting, strongly continuous one-parameter groups of unitary operators on reflexive strictly convex Banach space $X$ with Gâteaux differentiable norm and if
\[ U_t + U_{-t} = V_t + V_{-t}, \quad \forall t \in R. \]
and if one of those groups generates the bounded Hilbert transform on the whole space $X$ than the same holds for other group.

Proof. According to Theorem 8, space $X$ can be written as direct sum of its subspaces $L, M_1$ and $M_2$. Obviously, those subspaces are invariant under infinitesimal generators $A$ and $B$ of groups $U_t$ and $V_t$, respectively, and as well, under the infinitesimal generator $C$ of cosine operator $C_t$. According to Theorem 8, $A = B = 0$ on $L$, $A = B$ on $L_1$, and $A = -B$ on $L_2$.

Now, using Theorem 8 from [5] and previously given Note, we can easily obtain the statement given in our theorem. □

References


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