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ON THE FUNCTIONAL EQUATION $U_t + U_{-t} = V_t + V_{-t}$ IN A BANACH SPACE

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Abstract. In this paper we consider commuting one-parameter groups, $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ of unitary operators and the functional equation $U_t+U_{-t} = V_t+V_{-t}$ on a reflexive strictly convex Banach space with Gâteaux differentiable norm.

The operator equation $\alpha + \alpha_{-1} = \beta + \beta_{-1}$, where α and β are *-automorphisms on a Von Neumann algebra, has an important role in the geometric interpretation of Tomita-Takesaki modular theory and its generalization for Jordan algebras. Commuting one-parameter groups $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ of unitary operators on Hilbert space H such that $U_t+U_{-t} = V_t+V_{-t}$, for all $t \in R$, are considered in [4].

The following theorem is proved in [4].

Theorem 1. *Let $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ be two commuting one-parameter groups of unitary operators on a Hilbert space H such that $U_t + U_{-t} = V_t + V_{-t}$ for all $t \in R$. Then there is projection P on H such that $U_t = V_t$ on PH, $U_t = V_{-t}$ on $(I - P)H$ and P commutes with U_t and V_t for all $t \in R$.*

We can remark that it is not explicitly stated in Theorem 1 that U_t and V_t are strongly continuous groups. However, in the proof of this theorem in [4] it is taken into account that U_t and V_t are strongly continuous groups. In this note, we consider one-parameter groups $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ of unitary operators on reflexive strictly convex Banach space with Gâteaux differentiable norm.

Let X be real normed linear space and f a functional defined on X. Recall that by the first right-hand Gâteaux derivative of f at x in the direction h we mean

$$f'_+(x)(h) = \lim_{t\to+0} \frac{f(x+th) - f(x)}{t}. \tag{1}$$

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We have an analogous definition for first left-hand Gâteaux derivative of f at x in the direction h. If $f'_+(x)(h) = f'_-(x)(h)$ we say that f is Gâteaux differentiable at x in the direction h. Let $f(x) = \frac{1}{2}\|x\|^2$ and $\langle x, y\rangle = f'_+(x)(y)$. It is easy to prove that this derivative exists.

The proof of the next proposition is given in [3].

Proposition 2. *Every real normed linear space is a generalized inner product space in the sense that*

(a) $\langle x, y\rangle$ *is well defined;*
(b) $\|x\| = \langle x, x\rangle^{1/2}$*;*
(c) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ *(Cauchy - Schwarz - Buniakovsky inequality);*
(d) *If* X *is an inner product space with inner product* $[x, y]$ *then* $\langle x, y\rangle = [x, y]$.

Definition. *A normed linear space* X *is said to be strictly convex if* $\|x\| = \|y\| = 1$ *and* $x \neq y$ *imply* $\|\frac{x+y}{2}\| < 1$.

Some features of strictly convex space are given in [2].

The next theorem is proved in [3].

Theorem 3. *Suppose* X *is real Banach space. Then the Riesz representation theorem holds: Given* $\delta \in X^*$, *there exist* $x_\delta \in X$ *such that*

$$\delta(y) = < x_\delta, y > \quad (\forall y \in X) \ \text{ and } \ \|x\| = \|x_\delta\|$$

if and only if X *is reflexive with the Gâteaux differentiable norm.*

Furthermore x_δ *is unique (and mapping* $\delta \to x_\delta$ *is continuous from the norm topology on* X^* *to the weak topology on* X*) if and only if* X *is also strictly convex.*

In addition the mapping $\delta \to x_\delta$ *is also continuous from the norm topology on* X^* *to the norm topology on* X *if and only if* X *is also weakly uniformly convex.*

From now on, let X be complex strictly convex Banach space X with Gâteaux differentiable norm. Let

$$(x, y) \stackrel{def}{=} \langle x, y\rangle - i\langle x, iy\rangle.$$

It is easy to prove that

$$(x, iy) = i(x, y), \tag{2}$$

$$(ix, y) = -i(x, y). \tag{3}$$

Function (x, y) is linear relative to y, however (x, y) is not linear relative to x. It can be shown that $(\lambda x, y) = \overline{\lambda}(x, y)$. Using known methods it can be proved that

$$|(x, y)| \leq \|x\| \cdot \|y\|.$$

Similar to Theorem 3, in the case of a complex Banach space the following holds.

Theorem 3'. *Given $\delta \in X^*$, there exist unique $x_\delta \in X$ such that*

$$\delta(y) = (x_\delta, y) \ \ (\forall y \in X) \ \ \text{and} \ \ \|x\| = \|x_\delta\|$$

if and only if X is reflexive strictly convex Banach space with Gâteaux differentiable norm.

As $\delta \to x_\delta$ is a bijective mapping from X^* to X we can define its inverse mapping $\varphi : X \to X^*$

$$\varphi(x)(y) = (x, y) \quad y \in X.$$

Now, for all $x, y \in X$ we can introduce a new operation $\overset{*}{+}$ in X by

$$x \overset{*}{+} y = \varphi^{-1}(\varphi(x) + \varphi(y)) \quad x, y \in X.$$

The space X provided with the operation $+$ will be denoted by $(X, +)$, and by $(X, \overset{*}{+})$ we mean the space X provided with the operation $\overset{*}{+}$. Previously, we noted that (x, y) is linear relative to y in the $(X, +)$.

Next note that

$$(x \overset{*}{+} y, z) = (x, z) + (y, z)$$

i.e. (x, y) is linear relative to x in the space $(X, \overset{*}{+})$.

Definition. *A linear operator $U : X \to X$ is said to be isometric, i.e. unitary if*

1) $\|Ux\| = \|x\| \quad \forall x \in X$,
2) $UX = X$, *i.e. U is mapping from X on X.*

Let us show that U preserves inner product, i.e.

$$(Ux, Uy) = (x, y).$$

We have

$$\begin{aligned}(Ux, Uy) &= \langle Ux, Uy\rangle - i\langle Ux, Uiy\rangle \\ &= \|Ux\| \lim_{t\to 0} \frac{\|Ux + tUy\| - \|Ux\|}{t} - i\|Ux\| \lim_{t\to 0} \frac{\|Ux + itUy\| - \|Ux\|}{t} \\ &= \|x\| \lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} - i\|x\| \lim_{t\to 0} \frac{\|x + ity\| - \|x\|}{t} \\ &= \|x\|T(y) - i\|x\|T(iy) = \langle x, y\rangle - i\langle x, iy\rangle = (x, y).\end{aligned}$$

Remark*. Taking into account that U preserves inner product and that (x,y) is linear relative to x in the space $(X,\overset{*}{+})$ we have that every unitary operator $U : X \to X$ is linear in the space $(X,\overset{*}{+})$, i.e.

$$U(x \overset{*}{+} y) = Ux \overset{*}{+} Uy,$$
$$U(\lambda x) = \lambda Ux.$$

The next theorem is proved in [5].

Theorem 4. *Let X be complex reflexive and strictly convex Banach space with Gâteaux differentiable norm. For every closed linear subspace L of the space $(X,+)$ there exists a subspace L^* of the space $(X,\overset{*}{+})$ such that $X = L \oplus L^*$ (i.e. every $x \in X$ can be written in unique way in the form $x = l + l^*,\ l \in L,\ l^* \in L^*,\ \langle l^*, l\rangle = 0$).*

Note that a theorem similar to Theorem 4 is valid for the space $(X,\overset{*}{+})$.

Theorem 5. *Let X be Banach space with the same properties as in Theorem 4. Let L and L^* be subspaces of $(X,+)$ and $(X,\overset{*}{+})$ respectively, such that $X = L \oplus L^*$ and let U_t be group of unitary operators on X. If L is invariant under U_t then L^* is also invariant under U_t.*

Proof. Suppose L is invariant under U_t. This means, for $l \in L$ we have $U_t l \in L,\ t \in R$. Take $l \in L$ and $l^* \in L^*$. We have

$$0 = \langle l^*, U_t l\rangle = \|l^*\| \lim_{t\to 0} \frac{\|l^* + hU_t l\| - \|l^*\|}{h}$$
$$= \|U_{-t} l^*\| \lim_{t\to 0} \frac{\|U_{-t} l^* + hl\| - \|U_{-t} l^*\|}{h} = \langle U_{-t} l^*, l\rangle.$$

Thus, $U_{-t} l^* \in L^*$.

Since this holds for every $t \in R$, the theorem is proved. □

Before we give a generalization of Theorem 1, let us prove the following lemma.

Lemma 6. *Let $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ be two commuting, strongly continuous one-parameter groups of unitary operators on reflexive strictly convex Banach space X with Gâteaux differentiable norm. Suppose that*

a) *$U_t + U_{-t} = V_t + V_{-t}$, for all $t \in R$.*
b) *There is no $x \in X, x \neq 0$, such that $U_t x = V_t x$ for all $t \in R$.*

Then

$$U_t x = V_{-t} x, \quad \forall x \in X,\ \forall t \in R.$$

Proof. Let $C_t = \frac{U_t+U_{-t}}{2}$, $\forall t \in R$, i.e. $C_t = \frac{V_t+V_{-t}}{2}$, $\forall t \in R$.

Then, C_t is strongly continuous cosine operator function on X. If C is infinitesimal generator of C_t, and A and B are infinitesimal generators of U_t and V_t respectively, then $-C = A^2 = B^2$.

Using the fact that U_t and V_t commute, we get

$$(A - B)(A + B)x = 0, \quad x \in \mathcal{D}_{c^2}.$$

Let $(A + B)x = y$. Then $(A - B)y = 0$. Furthermore, for all $t \in R$ we have

$$\frac{d(U_t V_{-t})}{dt} y = AU_t V_{-t}\, y - BU_t V_{-t}\, y = U_t V_{-t}(A - B)\, y = 0.$$

Thus, for all $t \in R$ it is $U_t V_{-t}\, y = \bar{y}$, where $\bar{y}$ is a constant vector.

Hence, for $t = 0$, we get $\bar{y} = y$. So, $U_t y = V_t y$ for all $t \in R$. Taking into account b), this implies $y = 0$, i.e. $(A + B)x = 0$.

Thus, taking $\frac{d(U_t V_t x)}{dt}$ we get $U_t V_t x = x$, i.e. $U_t x = V_{-t} x$, $x \in \mathcal{D}_{c^2}$.

As the set $\mathcal{D}_{c^2}$ is dense in X and the groups U_t and V_t are strongly continuous, we have

$$U_t x = V_{-t} x, \quad \forall x \in X,\ \forall t \in R.$$

□

Now, we can prove a generalization of the Theorem 1.

Theorem 7. *Let $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ be two commuting, strongly continuous one-parameter groups of unitary operators on reflexive strictly convex Banach space X with Gâteaux differentiable norm such that*

$$\frac{U_t + U_{-t}}{2} = \frac{V_t + V_{-t}}{2}, \quad \forall t \in R. \tag{4}$$

Then there exist subspaces $L \subseteq (X, +)$ and $L^* \subseteq (X, \overset{*}{+})$ such that $X = L \oplus L^*$ and $U_t = V_t$ on L and $U_t = V_{-t}$ on L^*.

Proof. Let $L = \{l | U_t l = V_t l,\ \forall t \in R\}$. It can be easily shown that L is linear closed subspace of the space $(X, +)$. L is invariant under $\{U_t\}$.

Let $l \in L$. Then $U_t l = V_t l$. Performing U_s on $U_t l$ we get

$$U_s(U_t l) = U_{s+t} l,$$

and on the other hand

$$U_s(U_t l) = U_s(V_t l) = V_t U_s l = V_t V_s l = V_s U_t l = V_s(U_t l),$$

i.e. $U_t l \in L$. It can be easily proved that L is invariant under $\{V_t\}$.

According to Theorem 4, there is subspace L^* of the space $(X, \overset{*}{+})$ such that $X = L \oplus L^*$.

By Theorem 5, the subspace L^* is also invariant under U_t and under V_t.

Moreover, taking in the account Remark*, it can be easily proved that operators U_t and V_t are linear in $(X, \overset{*}{+})$, and thus, in L^*. As X is reflexive space, U_t and V_t are strongly continuous semigroups in L^*. From the definition of the subspace L^* it follows that there is no $x \in L^*$, $x \neq 0$ such that $U_t x = V_t x$, $\forall t \in R$.

Now, we can apply Lemma 6 to obtain the statement of the theorem. □

The notion of the Hilbert transform on local convex space X is given in paper [1]. In our note, we will consider the limit $\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} H_{\varepsilon,N}$ on reflexive strictly convex Banach space X with Gâteaux differentiable norm, where $H_{\varepsilon,N}\, x = \int_{\varepsilon\leq t<N} \frac{U_t x}{t}\, dt, \quad (x \in X,\ U_t$ unitary group of operators).

If $\bar{x} = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} H_{\varepsilon,N}\, x$ exists then $\bar{x}$ is called the Hilbert transform of the element x generated by group U_t and it is denoted by $\bar{x} = H\, x$.

In the following part of paper we will need some results obtained in [5]. From the Theorem 7 in [5] it follows that there are subspaces L and M of $(X, +)$ such that

a) Space $(X, +)$ is the direct sum of L and M.
b) $C_t x = x$, $\forall x \in L$, $\forall t \in R$.
c) M is invariant under all operators $C_t = \frac{U_t + U_{-t}}{2} = \frac{V_t + V_{-t}}{2}$.
d) If we consider C_t on M, then point 0 does not belong to punctual spectrum of infinitesimal generator C of C_t.

From d) it follows, as seen in [5], that in the subspace M operators A and B do not have eigenvectors that correspond to the eigenvalue 0. Also, taking into account that U_t and V_t are unitary operators and X is strictly convex, a) implies that

$$U_t x = V_t x = x, \quad \forall t \in R \text{ if and only if } x \in L.$$

Finally, from the proof of Theorem 7 in [5], it follows that $< l, m >= 0$, $\forall l \in L$, $\forall m \in M$.

Using the previously mentioned facts we can now easily prove that M is invariant under unitary operators U_t and V_t.

For $m \in M$, let $U_t m = l_t + m_t$, $l_t \in L$, $m_t \in M$. Then for any $x \in L$ and $m \in M$, we have

$$\begin{aligned} 0 =< x, m >=< U_{-t}x, m >=< x, U_t m >=< x, l_t + m_t > \\ =< x, l_t > + < x, m_t >=< x, l_t >, \quad \text{i.e. } l_t = 0. \end{aligned}$$

Thus, $U_t m \in M$. We can show in a similar way that M is invariant under V_t.

Also, it can be easily seen that there is no vector $m \in M$, $m \neq 0$ such that $U_t m = V_t m = m$.

Now, by applying Theorem 7 on space M we can obtain subspaces M_1 of $M, +)$ and M_2 of $(M, \overset{*}{+})$ such that $M = M_1 \oplus M_2$, and

$$U_t x = V_t x, \quad \forall x \in M_1, \forall t \in R$$
$$U_t x = V_{-t} x, \quad \forall x \in M_2, \forall t \in R$$

and $< m_2, m_1 >= 0$ for all $m_1 \in M_1$ and $m_2 \in M_2$. Moreover, we know that M_1 and M_2 are invariant under U_t and V_t.

Let us show that M_2 is linear subspace of $(M, +)$. Take $m', m'' \in M_2$. Obviously we can write $m' + m'' = m_1 + m_2$, $(m_1 \in M_1, m_2 \in M_2)$. Also, we have $U_t m' = V_{-t} m'$,$U_t m'' = V_{-t} m''$. Applying U_t on $m' + m'' = m_1 + m_2$ and using previous equalities we get:

$$V_{-t} m' + V_{-t} m'' = V_t m_1 + V_{-t} m_2. \tag{5}$$

On the other hand, if we apply V_{-t} on $m' + m'' = m_1 + m_2$ we obtain

$$V_{-t} m' + V_{-t} m'' = V_{-t} m_1 + V_{-t} m_2. \tag{6}$$

Comparing (5) and (6) we have

$$V_t m_1 + V_{-t} m_2 = V_{-t} m_1 + V_{-t} m_2, \text{ for all } t \in R,$$

where $m_1 \in M_1$, $m_2 \in M_2$. Thus, $V_t m_1 = V_{-t} m_1$, for all $t \in R$. This implies $V_t m_1 = m_1$, for all $t \in R$. So, $m_1 = 0$.

Therefore we have proved following theorem:

Theorem 8. *Under the previous assumptions on the space X and the operators U_t and V_t, X can be written as direct sum of its three linear subspaces L, M_1 and M_2 that are invariant under all operators U_t and V_t, $t \in R$ and such that*

1) $U_t x = V_t x = x, \ \forall x \in L, \forall t \in R.$
2) $U_t x = V_t x, \ \forall x \in M_1, \forall t \in R.$
3) $U_t x = V_{-t} x, \ \forall x \in M_2, \forall t \in R.$
4) $< l, m_1 + m_2 >= 0$ *and* $< m_2, m_1 >= 0$ *for all* $l \in L, m_1 M_1, m_2 \in M_2.$

Note. If we use the notation from [5] where A_+ represents positive second root of $-C(= A^2 = B^2)$, then it is easy to see that L, M_1 and M_2 are invariant under A_+.

From Theorem 8 we have the next theorem:

Theorem 9. *If $\{U_t : t \in R\}$ and $\{V_t : t \in R\}$ are two commuting, strongly continuous one-parameter groups of unitary operators on reflexive strictly convex Banach space X with Gâteaux differentiable norm and if*

$$U_t + U_{-t} = V_t + V_{-t}, \quad \forall t \in R.$$

and if one of those groups generates the bounded Hilbert transform on the whole space X than the same holds for other group.

Proof. According to Theorem 8, space X can be written as direct sum of its subspaces L, M_1 and M_2. Obviously, those subspaces are invariant under infinitesimal generators A and B of groups U_t and V_t, respectively, and as well, under the infinitesimal generator C of cosine operator C_t. According to Theorem 8, $A = B = 0$ on L, $A = B$ on L_1, and $A = -B$ on L_2.

Now, using Theorem 8 from [5] and previously given Note, we can easily obtain the statement given in our theorem. □

References

[1] S. Ishikawa, *Hilbert transforms on one-parametar groups of operators I, II*, Tokyo J. Math., 9 (2) (1986), 383–393, 395–414.

[2] S. Kurepa, *Funkcionalna analiza,* Školska knjiga, Zagreb, 1990.

[3] R. A. Tapia, *A characterization of inner product spaces,* Proc. Am. Math. Soc., 41 (1973), 569–574.

[4] A. B. Thaheem i Noor Mohammad, *On functional equation over Hilbert spaces*, Funkc. Ekvacioj, 32 (1989), 479–481.

[5] F. Vajzović and A. Šahović, *Cosine operator functions and Hilbert transforms*, Novi Sad J. Math., 35 (2) (2005), 41–55.

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