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COINCIDENCE POINTS UNDER WEAK CONTRACTIONS ON SYMMETRIC SPACES

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Abstract. In this paper we prove some results on the existence of coincidence points for weak hybrid contractions on symmetric spaces. These results improve and generalize some known results. In particular, recent fixed point results due to Hicks [3] are generalized.

1. Introduction

The well-known Banach contraction principle has been extended in different directions by several authors. In [6], Jungck proved a common fixed point result for single-valued commuting self maps in metric spaces, extending the Banach contraction principle. Nadler [12] initiated a study of fixed points for multivalued maps. Using the concept of Hausdorff metric, he proved a multivalued version of the Banach contraction principle. The study of coincidence points for hybrid contractions (that is, contraction types involving single-valued and multivalued maps) was initiated by Singh and Kulshrestha [14]. Subsequently, a number of authors have further studied such maps and proved coincidence point results. In [7], Kaneko studied multivalued f-contraction maps (hybrid contractions) and proved coincidence and common fixed point results for such maps defined on metric spaces, extending the results of Jungck [6] and Nadler [12]. Later, many other results on coincidence and fixed points of hybrid contractions have appeared. For example, see [8, 10, 11, 13] and others.

It is a fact that the proofs of certain coincidence and fixed point results in the setting of metric spaces do not need the full force of a distance function. Motivated by this idea, Hicks and Rhoades [4](also see Hicks [3]) introduced a notion of symmetric space, proved the common fixed point result of Jungck [6] for such general spaces, and consequently, generalized the Banach contraction principle. Recently, several authors proved some more general

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common fixed point results for single-valued self maps of a symmetric space (see, e.g.,[1, 5]). On the other hand, in this new setting of symmetric spaces, Hicks [3] extended the Multivalued Contraction Principle of Nadler [12]. Recently, Latif and Hajar [9, Theorem 2.1] obtained a coincidence point result for f-contraction maps which extends some known coincidence point results and contains fixed point results due to Hicks [3, Theorem 3] and Nadler [12, Theorem 5] as special cases.

In this paper, we follow ideas in [3, 4] to prove some coincidence point results for weak hybrid contractions, that is, the weaker contractive conditions involving single-valued and multivalued maps on symmetric spaces. Our results generalize and extend a number of known fixed point and coincidence point results.

We recall the following definitions and notions [3, 4]:

Definition 1.1. *Let X be a nonempty set. Let d be a real-valued function defined on the product $X \times X$ such that*

(1) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$, *and*
(2) $d(x, y) = d(y, x)$.

Such a function d is said to be *symmetric* and X together with d written as (X, d) is called a *symmetric space*.

Let (X, d) be a symmetric space. For $x \in X$ and $\varepsilon > 0$, define $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. Note that in a symmetric space, $\lim_{n\to\infty} d(x_n, x) = 0$ if $x_n \to x$ in the topology $t(d)$. Similarly, a sequence in X is d-Cauchy if it satisfies the usual metric condition. A symmetric space (X, d) is complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that there exists x in X such that $\lim_{n\to\infty} d(x_n, x) = 0$. A map $f : X \to X$ is d-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies that $\lim_{n\to\infty} d(fx_n, fx) = 0$.

The following two axioms were given by Wilson [15] (also, see [3, 4]). Let (X, d) be a symmetric space.

(W.3) Given $\{x_n\}$, x and y in X, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply that $x = y$.
(W.4) Given $\{x_n\}$, $\{y_n\}$ and an x in X, $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply that $d(y_n, x) \to 0$.

Let (X, d) be a symmetric space with d bounded. We use $CL(X)$ to denote the collection of all nonempty closed subsets of X, $K(X)$ for the collection of nonempty compact subsets of X and ρ for the Hausdorff distance function on $CL(X)$ induced by d; i.e.,

$$\rho(A, B) = \max\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\},$$

for all A, B in $CL(X)$, where $d(x,A) = \inf\{d(x,y) : y \in A\}$. Clearly, ρ is symmetric on $CL(X)$. Let f be a singlevalued self map on X. A multivalued map $T : X \to CL(X)$ is said to be an f-contraction if and only if for a fixed constant $h \in [0,1)$ and for each $x, y \in X$,

$$\rho(Tx, Ty) \leq h\, d(fx, fy).$$

In particular, if f is the identity map on X then a multivalued map is an f-contraction if and only if it is a contraction. Note that each singlevalued map is an f-contraction if and only if it is a multivalued f-contraction. We say a sequence $\{x_n\}$ in X is an f-iterative sequence of T at $x_0 \in X$ if and only if $fx_n \in Tx_{n-1}$ for all $n \geq 1$. We say that f and T weakly commute if and only if $fTx \subset Tfx$ for all $x \in X$. Clearly, if f and T commute, then they also weakly commute. A point $x \in X$ is called (i) a fixed point of the multivalued map T if and only if $x \in Tx$; (ii) a coincidence point of f and T if and only if $fx \in Tx$. We denote by $C(f,T)$ the set of coincidence points of f and T, and $T(X) = \bigcup_{x \in X} Tx$.

We need the following result due to Hicks [3].

Lemma 1.1. *Suppose $T : (X,d) \to CL(X)$ where d is a bounded symmetric. Then $\lim_{n\to\infty} d(x_n, Tx) = 0$ if and only if there exists $y_n \in Tx$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

2. Main results

First we prove two results on the existence of f-iterative sequences in the setting of symmetric spaces. In the sequel, we need a non-decreasing function $k : [0,\infty) \to [0,\infty)$ with $k(0) = 0$.

Lemma 2.1. *Let (X,d) be a complete symmetric space with d bounded. Let f be a single valued self map on X and let $T : X \to K(X)$ be such that $T(X) \subset f(X)$ and for all $x, y \in X$,*

$$\rho(Tx, Ty) \leq k[d(fx, fy)]. \tag{2.1.1}$$

Let $y \to d(fx, y)$ be continuous for fixed x. If $\sum_{n=1}^{\infty} k^n[d(fx_0, Tx_0)] < \infty$ for some $x_0 \in X$, then there exists an f-iterative sequence $\{x_n\}$ of T at x_0 such that $\{fx_n\}$ converges to some $p \in X$.

Proof. Suppose that there exists $x_0 \in X$ such that $\sum_{n=1}^{\infty} k^n[d(fx_0, Tx_0)] < \infty$. Since $y \to d(fx_0, y)$ is continuous on a compact set Tx_0 and $T(x_0) \subset f(X)$, there exists $x_1 \in X$ and $fx_1 \in Tx_0$ such that

$$d(fx_0, fx_1) = d(fx_0, Tx_0).$$

Proceeding inductively, we obtain a sequence $\{x_n\}$ in X such that $fx_{n+1} \in Tx_n$, and

$$d(fx_n, fx_{n+1}) = d(fx_n, Tx_n).$$

Now, since for every $n \geq 1$, $fx_n \in Tx_{n-1}$ we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &= d(fx_n, Tx_n) \\ &\leq \sup\{d(y, Tx_n) : y \in Tx_{n-1}\} \\ &\leq \rho(Tx_{n-1}, Tx_n) \\ &\leq k[d(fx_{n-1}, fx_n]. \end{aligned}$$

Also, since k is nondecreasing, we get

$$\begin{aligned} d(fx_{n+1}, fx_n) &\leq k[d(fx_{n-1}, fx_n] \\ &\leq k^2[d(fx_{n-2}, fx_{n-1})] \quad \leq \cdots \\ &\leq k^n[d(fx_0, fx_1)] = k^n[d(fx_0, Tx_0)], \end{aligned}$$

and thus

$$\sum_{n=1}^{\infty} d(fx_{n+1}, fx_n) \leq \sum_{n=1}^{\infty} k^n[d(fx_0, Tx_0)] < \infty.$$

Now, by the completeness of X there exists some $p \in X$ such that

$$\lim_{n\to\infty} d(fx_n, p) = 0,$$

which completes the proof. □

Lemma 2.2. *Assume that all the hypothesis of Lemma* 2.1 *except the inequality* (2.1.1) *hold. If T fulfills the inequality*

$$\rho(Tx, Ty) \leq k[max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\}], \qquad (2.2.1)$$

for all $x, y \in X$ and k satisfies $k(t) < t$ for each $t > 0$. Then there exists an f-iterative sequence $\{x_n\}$ of T at x_0 such that $\{fx_n\}$ converges to some element of X.

Proof. Following the proof of the Lemma 2.1, we obtain a sequence $\{x_n\}$ in X such that $fx_n \in Tx_{n-1}$ and

$$d(fx_n, fx_{n+1}) = d(fx_n, Tx_n) \leq \rho(Tx_{n-1}, Tx_n).$$

Thus, using the definition of T, we get

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \rho(Tx_{n-1}, Tx_n) \\ &\leq k[max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\}] \\ &= k[max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}]. \end{aligned}$$

Now, if there exists an n for which $d(fx_n, fx_{n+1}) > d(fx_{n-1}, fx_n)$, then by using the above inequality and the fact that $k(t) < t$ for each $t > 0$, we obtain

$$d(fx_n, fx_{n+1}) \leq k[d(fx_n, fx_{n+1})] < d(fx_n, fx_{n+1}),$$

which is not possible. Thus, for each $n \geq 1$,

$$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n),$$

and hence

$$d(fx_n, fx_{n+1}) \leq k[d(fx_{n-1}, fx_n)].$$

Now, following the same argument as in the proof of Lemma 2.1, there exists an element $p \in X$ such that $\lim_{n\to\infty} d(fx_n, p) = 0$. □

Applying Lemma 2.1, we obtain the following result on the existence of coincidence points.

Theorem 2.1. *Assume that all the hypotheses of Lemma* 2.1 *hold. Let f be a d-continuous self map on X which commutes weakly with T and assume that* (W.4) *holds. Then $C(f, T) \neq \emptyset$ iff there exists $x_0 \in X$ with $\sum_{n=1}^{\infty} k^n[d(fx_0, Tx_0] < \infty$.*

Proof. If $C(f, T) \neq \emptyset$, then there exists $x \in X$ such that $d(fx, Tx) = 0 = k(0) = k^2(0)$ and thus $\sum_{n=1}^{\infty} k^n[d(fx, Tx] < \infty$. Now, suppose there is an element $x_0 \in X$ with $\sum_{n=1}^{\infty} k^n[d(fx_0, Tx_0] < \infty$. Then, by Lemma 2.1, there exists an f-iterative sequence $\{x_n\}$ of T at x_0 such that

$$\lim_{n\to\infty} d(fx_n, p) = 0,$$

for some $p \in X$. Then the d-continuity of f implies $\lim_{n\to\infty} d(f^2x_n, fp) = 0$. Note that

$$\rho(Tfx_n, Tp) \leq k[d(f^2x_n, fp)],$$

and thus $\lim_{n\to\infty} \rho(Tfx_n, Tp) = 0$. Since $fx_{n+1} \in Tx_n$, using the weak commutativity of f and T we have $ffx_{n+1} \in fTx_n \subset Tfx_n$, and thus

$$d(ffx_{n+1}, Tp) \leq \sup\{d(y, Tp) : y \in Tfx_n\} \leq \rho(Tfx_n, Tp).$$

Hence, $\lim_{n\to\infty} d(f^2x_n, Tp) = 0$. By Lemma 1.1, there exists $y_n \in Tp$ such that $\lim_{n\to\infty} d(f^2x_n, y_n) = 0$, and since $\lim_{n\to\infty} d(f^2x_n, fp) = 0$, (W.4) we have

$$\lim_{n\to\infty} d(fp, y_n) = 0.$$

As Tp is closed and $y_n \in Tp$, we get $fp \in Tp$, that is, $C(f, T) \neq \emptyset$. □

Now, applying Lemma 2.2, we have the following coincidence point result.

Theorem 2.2. *Assume that all the hypotheses of Lemma* 2.2 *hold. Let f be a d-continuous self map on X which commutes weakly with T and assume that* (W.4) *holds. Then $C(f, T) \neq \emptyset$ iff there exists $x_0 \in X$ with $\sum_{n=1}^{\infty} k^n[d(fx_0, Tx_0] < \infty$.*

Proof. Fix an $x_0 \in X$. By Lemma 2.2, there exists an f-iterative sequence $\{x_n\}$ of T at x_0 such that

$$\lim_{n\to\infty} d(fx_n, p) = 0,$$

for some $p \in X$. Note that $\lim_{n\to\infty} d(f^2x_n, fp) = 0$, for each n and

$$ffx_{n+1} \in fTx_n \subset Tfx_n.$$

Since Tfx_n is closed, $fp \in Tfx_n$. Therefore, applying the inequality (2.2.1) we have

$$\begin{aligned}\rho(Tfx_n, Tp) &\leq k[max\{d(f^2x_n, fp), d(f^2x_n, Tfx_n), d(fp, Tp)\}]\\ &\leq k[max\{d(f^2x_n, fp), d(fp, Tp)\}].\end{aligned}$$

Suppose that $max\{d(f^2x_n, fp), d(fp, Tp)\} = d(fp, Tp)$, then we obtain

$$\rho(Tfx_n, Tp) \leq k[d(fp, Tp)] < d(fp, Tp) \leq \rho(Tfx_n, Tp),$$

which is not possible. Thus,

$$\rho(Tfx_n, Tp) \leq k[d(f^2x_n, fp)].$$

Use the same arguments as in the proof of Theorem 2.1 to infer that $fp \in Tp$. □

Remark 2.1. Theorem 2.1 and Theorem 2.2 extend the fixed point theorems due to Hicks [3, Theorem 4] and Hicks [3, Theorem 5] respectively.

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