

ON FOCI AND ASYMPTOTES OF CONICS IN THE ISOTROPIC PLANE

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ABSTRACT. The paper shows that every conic with foci in the isotropic plane can be represented by the equation of the form $y^2 = \epsilon x^2 + x$, where $\epsilon \in \{-1, 0, 1\}$ for an ellipse, a parabola and a hyperbola with foci respectively. Using this equation some important properties of the foci are proved. According to duality the properties of asymptotes of the hyperbola in the isotropic plane are valid as well.

A *conic section* or *conic* is the locus of all points $(x : y : z)$ in a real projective plane that are the solutions of an equation written in the form

$$lx^2 + my^2 + nz^2 + 2qxy + 2pxz + 2oyz = 0, \quad l, m, n, q, p, o \in \mathbb{R} \quad (1)$$

where

$$\delta = \det \begin{pmatrix} l & q & p \\ q & m & o \\ p & o & n \end{pmatrix} = lmn + 2opq - lo^2 - mp^2 - nq^2. \quad (2)$$

We are going to consider the proper conics, i. e. the conics with $\delta \neq 0$. To any point $T(x_0 : y_0 : z_0)$ its *polar* \mathcal{T} with respect to (w.r.t.) the conic \mathcal{C} is the straight line with the equation

$$lx_0x + my_0y + nz_0z + o(y_0z + z_0y) + p(z_0x + x_0z) + q(x_0y + y_0x) = 0$$

which can be also written in the form

$$(lx_0 + qy_0 + pz_0)x + (qx_0 + my_0 + oz_0)y + (px_0 + oy_0 + nz_0)z = 0. \quad (3)$$

Conversely, one says that the point T is the *pole* of the line \mathcal{T} with respect to the conic \mathcal{C} . The isotropic plane is a projective metric plane whose *absolute figure* is a pair consisting of a point Ω and a line ω incident to it, i.e. of the *absolute point* Ω and the *absolute line* ω respectively. As the absolute figure is dual to itself, the principle of duality from the projective plane is preserved in the isotropic plane. The points of the absolute line ω are called

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isotropic points and the lines incident to the absolute point Ω are *isotropic lines*. Two lines are *parallel* if they have the same isotropic point, and two points are *parallel* if they are incident to the same isotropic line.

All the notions related to the geometry of the isotropic plane can be found for example in Sachs [1] and Strubecker [2].

If the absolute figure is determined by the line ω with the equation $z = 0$ and by the point $\Omega = (0 : 1 : 0)$ of the line ω , then the non-isotropic points $(x : y : z)$ are characterized by $z \neq 0$. As a consequence, it can be assumed that $z = 1$, and the non-isotropic points can be denoted by (x, y) . Letting $z = 1$ the equation (1) turns into the form

$$lx^2 + 2qxy + my^2 + 2px + 2oy + n = 0, \quad (4)$$

being the equation of the conic \mathcal{C} in the isotropic plane.

The conic \mathcal{C} in the isotropic plane is a *circle* if the absolute line ω is tangent to it at the absolute point Ω , a *parabola* if the line ω is tangent to it at an isotropic point different from Ω , an *ellipse* if it does not intersect the line ω , and a *hyperbola* if they intersect at two different points.

Common points of the line ω and the conic \mathcal{C} given in (1) fulfill the equation

$$lx^2 + 2qxy + my^2 = 0. \quad (5)$$

The conic \mathcal{C} is a circle provided that the equation (5) has a double solution in y for $x = 0$. This is achieved if, and only if, $m = q = 0$. From (2) it follows that $\delta = -lo^2 \neq 0$, i.e. $l, o \neq 0$. Choosing $l = 1$, we get:

Theorem 1. *Every circle of the isotropic plane (with chosen affine coordinate system) has the equation $x^2 + 2px + 2oy + n = 0$ that can be written in the form $2py = x^2 + ux + v$. In the latter case ρ is the radius of this circle.*

Considering equation (5) it follows:

Theorem 2. *The conic \mathcal{C} with the equation (4) is an ellipse if $q^2 < lm$, a hyperbola if $q^2 > lm$ and a parabola if $q^2 = lm$ with $l, m, q \neq 0$.*

In the third case in Theorem 2 we do not allow $m = q = 0$. The condition $l, m \neq 0$ implies $q \neq 0$, the condition $q \neq 0$ implies $l, m \neq 0$, but it is possible that $l = q = 0, m \neq 0$.

The pole S of the absolute line ω w.r.t. the conic \mathcal{C} is called the *center* of the conic. Dually, the polar \mathcal{S} of the absolute point Ω w.r.t. the conic \mathcal{C} is called the *axis*. The axis of the conic passes through its center because the point Ω is incident to the line ω . Possible points of intersection of the conic \mathcal{C} with its axis are called *foci* of the conic (see [1]). Tangents to the conic \mathcal{C} at those points pass through the point Ω , i.e. they are isotropic lines. The isotropic tangents to \mathcal{C} at the foci are called *directrices* of the conic \mathcal{C} .

Considering duality, the tangents in the isotropic points of the hyperbola \mathcal{C} are called *asymptotes* of \mathcal{C} and they pass through its center.

In the case of the parabola \mathcal{C} the center S is an isotropic point. As the parabola is a conic that touches the line ω at the isotropic point S different from Ω , it is dual to the conic \mathcal{C} that passes through the point Ω , and at that point touches the isotropic line \mathcal{S} different from ω . This means that the line \mathcal{S} is an asymptote of the conic \mathcal{C} which is a hyperbola. Such a hyperbola with one isotropic line as one of its asymptotes is called a *special hyperbola*. At the same time the line \mathcal{S} is an axis as well, and besides the isotropic asymptote \mathcal{S} the special hyperbola has one more non-isotropic asymptote.

Lines through the center of the conic \mathcal{C} are its *diameters*. Possible intersection points of the conic with its isotropic diameter are called *vertices*. The line ω can be considered as the isotropic diameter of the parabola \mathcal{C} , while in the case of the special hyperbola \mathcal{C} the isotropic diameter coincides with its axis \mathcal{S} and at the same time it is an isotropic asymptote of \mathcal{C} .

For the absolute point $\Omega = (0 : 1 : 0)$ the polar equation (3) with $z = 1$ turns into the form

$$qx + my + o = 0. \tag{6}$$

Theorem 3. *The conic with the equation (4) has the axis given by equation (6).*

The line (6) is an isotropic line under $m = 0$ and $q \neq 0$, i.e. the next theorem is valid:

Theorem 4. *The conic with the equation (4) is a special hyperbola if and only if $m = 0$ and $q \neq 0$ is valid.*

Theorem 5. *Abscissae of the foci of the conic given in equation (4) are the solutions on x of the equation*

$$(q^2 - lm)x^2 + 2(oq - mp)x + o^2 - mn = 0. \tag{7}$$

With $m \neq 0$ this conic has two foci if $\delta m < 0$, and it has none if $\delta m > 0$, with δ given in (2).

Proof. The equation (4) can be written in the form

$$my^2 + 2(qx + o)y + lx^2 + 2px + n = 0.$$

We seek the values of x for which the equation has double solutions in y . A condition for obtaining this is

$$(qx + o)^2 - m(lx^2 + 2px + n) = 0$$

which satisfies (7). Because of (2) its discriminant is equal to

$$(oq - mp)^2 - (o^2 - mn)(q^2 - lm) = -m(lmn + 2opq - lo^2 - mp^2 - nq^2) = -\delta m,$$

from which the second claim of the theorem follows. □

Latter, we will be interested in the conic \mathcal{C} with two different foci. By an affine coordinate transformation, leaving ω and Ω invariant, every conic with two different foci can be represented as the conic having the x -axis as its axis and the origin as its focus.

Theorem 6. *Every conic with the origin as its focus and the x -axis as its axis has the equation of the form*

$$y^2 = \epsilon x^2 + x \quad (8)$$

where $\epsilon \in \{-1, 0, 1\}$. Depending on $\epsilon = -1, \epsilon = 0$ or $\epsilon = 1$ the conic given in (8) is an ellipse, a parabola or a hyperbola.

Proof. The axis given in the form of (6) coincides with the x -axis provided that $o = q = 0$ and $m \neq 0$. Under the same conditions the equation (7) divided by $-m$ has the form $lx^2 + 2px + n = 0$. With $n = 0$ this equation has one solution $x = 0$. The equation (4) therefore has the form

$$lx^2 + my^2 + 2px = 0, \quad (9)$$

by (2) it follows that $-mp^2 \neq 0$, i.e. $m, p \neq 0$. If $l \neq 0$, using the substitution

$$x \rightarrow \frac{2p\epsilon}{l}x, \quad y \rightarrow \frac{2p}{\sqrt{|lm|}}y,$$

where $\epsilon = -\text{sgn}(lm)$, the equation (9) turns into

$$\frac{4p^2}{l}x^2 - \frac{4p^2}{l\epsilon}y^2 + \frac{4p^2\epsilon}{l}x = 0$$

because of $|lm| = -lm\epsilon$. Because of $\epsilon^2 = 1$, multiplying the latter equation by $\frac{l\epsilon}{4p^2}$ it transforms into the form given in (8). On the other hand, if $l = 0$, using the substitution $x \rightarrow \frac{-m}{2p}x$ the equation $my^2 + 2px = 0$ turns into the form $my^2 - mx = 0$, i.e. $y^2 = x$ being the equation (8) with $\epsilon = 0$, representing a parabola. With $l = \epsilon$, $m = -1$, $p = \frac{1}{2}$ and $n = o = q = 0$ the equation (4) turns into (8) and so if $\epsilon \in \{-1, 1\}$, then $q = 0$ and $lm = -\epsilon$ are valid. If $\epsilon = -1$, then $lm = 1 > 0$ and by Theorem 2 the conic is an ellipse. By the same theorem, if $\epsilon = 1$, then $lm = -1 < 0$ and the conic is a hyperbola. \square

Under $y = 0$ from (8) it follows that $x = 0$ or $x = -\epsilon$. Thus we have:

Corollary 1. *The second focus of the ellipse or the hyperbola from Theorem 6 is the point $O' = (-\epsilon, 0)$.*

These observations do not lead to the conclusion that every ellipse or hyperbola has foci.

Theorem 7. *A line through the focus $(0, 0)$ with the equation $y = tx$ meets the conic (8) residually at the point*

$$T = \left(\frac{1}{t^2 - \epsilon}, \frac{t}{t^2 - \epsilon} \right). \tag{10}$$

Proof. The point T from (10) obviously lies on the line $y = tx$ but it lies on the conic (8) as well because of

$$\epsilon \frac{1}{(t^2 - \epsilon)^2} + \frac{1}{t^2 - \epsilon} = \frac{t^2}{(t^2 - \epsilon)^2}.$$

□

Corollary 2. *The conic (8) has parametric equations*

$$x = \frac{1}{t^2 - \epsilon}, \quad y = \frac{t}{t^2 - \epsilon}, \tag{11}$$

where $t \in \mathbb{R} \cup \{\infty\}$.

The parameter $t = \infty$ corresponds to the focus $(0, 0)$, and in the case of an ellipse or a hyperbola the parameter $t = 0$ corresponds to another focus $(-\epsilon, 0)$, while in the case of a parabola for $t = 0$ an isotropic point of the parabola is found. In the case of a hyperbola an isotropic point is reached for $t = 1$ or $t = -1$.

The point T from (10) will be denoted by $T = (t)$.

Theorem 8. *The conic with the equation (8), i.e. with the parametric equations (11) has a tangent whose equation is*

$$y = \frac{t^2 + \epsilon}{2t}x + \frac{1}{2t} \tag{12}$$

at the point $T = (t)$ given in (10).

Proof. Eliminating the variable y from (8) and (12) we get the equation

$$[(t^2 + \epsilon)x + 1]^2 = 4t^2(\epsilon x^2 + x),$$

where factors next to x^2 and x are

$$(t^2 + \epsilon)^2 - 4t^2\epsilon = (t^2 - \epsilon)^2$$

and

$$2(t^2 + \epsilon) - 4t^2 = -2(t^2 - \epsilon),$$

so the equation becomes of the form $[(t^2 - \epsilon)x - 1]^2 = 0$ with double solution $x = \frac{1}{t^2 - \epsilon}$. So the line (12) touches the considered conic at the point $T = (t)$ given in (10). □

Theorem 9. *The point of the intersection of the tangents \mathcal{T}_1 and \mathcal{T}_2 of the conic (8) at the points T_1 and T_2 with parameters $t = t_1$ and $t = t_2$ in (10) is*

$$T_{12} = \left(\frac{1}{t_1 t_2 - \epsilon}, \frac{t_1 + t_2}{2(t_1 t_2 - \epsilon)} \right). \quad (13)$$

Proof. It is sufficient to show that the point T_{12} from (13) is incident to the tangent \mathcal{T}_1 given in the equation (12) with $t = t_1$. As a matter of fact, the equality

$$\frac{t_1^2 + \epsilon}{2t_1} \cdot \frac{1}{t_1 t_2 - \epsilon} + \frac{1}{2t_1} = \frac{t_1 + t_2}{2(t_1 t_2 - \epsilon)}$$

is valid. \square

Let us prove now a few claims on points, tangents, and a focus of any conic. To begin with, we need the following definition: lines through the vertex of an angle and symmetric with respect to the bisector of the angle are called isogonal lines.

Theorem 10. *A line joining a focus of a conic to the point of intersection of two of its tangents is the bisector of the lines joining the focus to the points of contact of those tangents.*

Proof. The tangents \mathcal{T}_1 and \mathcal{T}_2 at the points $T_1 = (t_1)$ and $T_2 = (t_2)$ meet in the point T_{12} given in (13). Let O be the focus $(0, 0)$. The lines OT_1 and OT_2 have slopes t_1 and t_2 , and the line OT_{12} has the slope $\frac{t_1+t_2}{2}$, which proves the claim in the theorem. \square

Theorem 11. *The lines joining a focus of the conic to the point of intersection of its two tangents and to a point of contact of the third tangent are isogonal with respect to the lines joining the focus with the points of intersection of the first two tangents with the third one.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be tangents on the conic (8) at the points $T_1 = (t_1), T_2 = (t_2), T_3 = (t_3)$ and let $T_{12} = \mathcal{T}_1 \cap \mathcal{T}_2, T_{13} = \mathcal{T}_1 \cap \mathcal{T}_3, T_{23} = \mathcal{T}_2 \cap \mathcal{T}_3$. We need to show that OT_{13}, OT_{23} and OT_{12}, OT_3 are isogonal pairs of lines. Those lines have slopes $\frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}$ and $\frac{t_1+t_2}{2}, t_3$ respectively and we have

$$\frac{t_1 + t_3}{2} + \frac{t_2 + t_3}{2} = \frac{t_1 + t_2}{2} + t_3.$$

\square

Theorem 12. *The joint lines of the focus and the endpoints of the segment of any tangent of the conic cut by its two fixed tangents form the angle that is equal to half of an angle formed by the lines joining the focus to the points of contact of those fixed tangents.*

Proof. With the previously used notation let us prove for example the equality $\angle(OT_{13}, OT_{23}) = \frac{1}{2}\angle(OT_1, OT_2)$, where $\mathcal{T}_1, \mathcal{T}_2$ are fixed tangents of the conic and \mathcal{T}_3 its variable one. The lines OT_{13}, OT_{23} and OT_1, OT_2 have slopes $\frac{1}{2}(t_1 + t_3), \frac{1}{2}(t_2 + t_3)$ and t_1, t_2 respectively. Hence

$$\angle(OT_{13}, OT_{23}) = \frac{1}{2}(t_2 + t_3) - \frac{1}{2}(t_1 + t_3) = \frac{1}{2}(t_2 - t_1) = \frac{1}{2}\angle(OT_1, OT_2). \quad \square$$

Theorem 13. *Pairs of lines joining the focus of a conic to the pairs of opposite vertices of a complete quadrilateral made by four tangents of a conic have a common bisector.*

Proof. By $\mathcal{T}_i, i \in \{1, 2, 3, 4\}$ are denoted the tangents of the conic (8) at the points $T = (t_i)$. Let us put $\mathcal{T}_{ij} = \mathcal{T}_i \cap \mathcal{T}_j$ with $i, j \in \{1, 2, 3, 4\}$ and $i < j$. Then $T_{12}, T_{34}; T_{13}, T_{24}; T_{14}, T_{23}$ are pairs of opposite vertices of the quadrilateral made by the lines $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$. As lines OT_{12} and OT_{34} have slopes $\frac{1}{2}(t_1 + t_2)$ and $\frac{1}{2}(t_3 + t_4)$, their bisector has slope $\frac{1}{4}(t_1 + t_2 + t_3 + t_4)$. The symmetry in the all items of the latter sum proves the claim in the theorem. \square

Under $t_4 = t_3$ Theorem 11 follows from Theorem 13, and under $t_4 = t_1$ and $t_3 = t_2$ Theorem 10 follows from Theorem 13. Applying Theorem 10 twice on a variable tangent and on one fixed tangent, Theorem 12 can be proved as well.

Let us prove now a few more claims on points, tangents and two foci of the conic.

Theorem 14. *The tangent of an ellipse or a hyperbola is a bisector of the lines joining its point of contact to the foci of the conic.*

Proof. Let \mathcal{T} be a tangent of the conic (8) at the point $T = (t)$. The conic has foci O and $O' = (-\epsilon, 0)$. Slopes of the lines OT and $O'T$ are t and

$$\frac{\frac{t}{t^2 - \epsilon}}{\frac{1}{t^2 - \epsilon} + \epsilon} = \frac{\epsilon}{t}$$

respectively, and the line \mathcal{T} given in equation (12) has slope $\frac{1}{2t}(t^2 + \epsilon)$. The equality

$$\frac{1}{2}\left(t + \frac{\epsilon}{t}\right) = \frac{t^2 + \epsilon}{2t}$$

completes the proof. \square

Theorem 15. *Two tangents of an ellipse or a hyperbola are isogonal w.r.t. the lines joining their point of intersection to the foci of the conic.*

Proof. Using the notation from Theorem 10 and the foci O and O' from the previous proof, the lines \mathcal{T}_1 and \mathcal{T}_2 have slopes

$$\frac{t_1^2 + \epsilon}{2t_1} \quad \text{and} \quad \frac{t_2^2 + \epsilon}{2t_2},$$

while lines OT_{12} and $O'T_{12}$ have slopes $\frac{1}{2}(t_1 + t_2)$ and

$$\frac{\frac{t_1+t_2}{2(t_1t_2-\epsilon)}}{\frac{1}{t_1t_2-\epsilon} + \epsilon} = \frac{t_1 + t_2}{2t_1t_2\epsilon}$$

respectively. Adding the two pairs of slopes, we get

$$\begin{aligned} \frac{t_1^2 + \epsilon}{2t_1} + \frac{t_2^2 + \epsilon}{2t_2} &= \frac{1}{2t_1t_2}(t_1 + t_2)(t_1t_2 + \epsilon), \quad \text{and} \\ \frac{1}{2}(t_1 + t_2) + \frac{t_1 + t_2}{2t_1t_2\epsilon} &= \frac{1}{2t_1t_2}(t_1 + t_2)(t_1t_2 + \epsilon), \end{aligned}$$

i.e. equal sums. □

Theorems 14 and 15 have been proved in [1], p.74, p.75 in a different way and Theorems 10, 14, 15 have been stated in [2] without proofs. The claims of Theorem 14 and 15 are valid for a parabola provided that the other focus is considered as its isotropic point. In that case the line $O'T$ from the proof of Theorem 14 has slope 0 and the line \mathcal{T} has slope $\frac{t}{2}$. The lines $\mathcal{T}_1, \mathcal{T}_2$ and $OT_{12}, O'T_{12}$ in the proof of Theorem 15 have slopes $\frac{1}{2}t_1, \frac{1}{2}t_2$ and $\frac{1}{2}(t_1 + t_2), 0$ respectively.

Foci of a conic are the points of contact of its tangents drawn from the absolute point Ω as well as they are the points of intersection of a conic with its axis i.e. the polar of the absolute point Ω . Dual statement: asymptotes of a hyperbola are the tangents at its isotropic points. They are also the tangents drawn from the center of the hyperbola, the pole of the absolute line. Thus the notions of the center and the foci of a conic are dual to the notions of the axis and the asymptotes.

Theorems 10-15 are dual to the well known theorems in an affine plane. Before stating those theorems we need the following definition: two pairs of points incident to the same line are called *isotomic* if they have the same midpoint.

Theorem 16. *A point of intersection of a hyperbola's asymptote with a line joining two of its points is the midpoint of the points of intersection of the asymptote with the tangents of the hyperbola at these two points.*

Theorem 17. *Points of intersection of the hyperbola's asymptote with a line joining two of its points with the tangent at a third point are isotomic*

w.r.t. the points of intersection of the asymptote with the lines joining the first two points to the third one.

Theorem 18. *The lines joining any point of the hyperbola with two of its points cut its asymptote in the segment whose length is equal to half a length of the segment of the asymptote that is cut by the tangents of the hyperbola at these two chosen points.*

Theorem 19. *Pairs of the points of intersection of the hyperbola's asymptote with pairs of opposite sides of a complete quadrangle made by four points of the hyperbola have a common midpoint.*

Theorem 20. *Any point on a hyperbola is the midpoint of the points of intersection of its tangent at that point with the asymptotes of the hyperbola.*

Theorem 21. *Any two points of the hyperbola are isotomic w.r.t. the points of intersection of their joint line with the asymptotes of the hyperbola.*

Theorems 16, 20, 21 have been stated in [2]. The claim stated in Theorem 19 is a special case of the so called Butterfly theorem. Let us say a little bit more about that midpoint.

Theorem 22. *The common midpoint given in Theorem 19 is symmetrical to the center of the hyperbola w.r.t. the line parallel to another asymptote and passing through the centroid of the vertices of the considered complete quadrangle.*

Proof. The claim has affine character. Thus, it can be proved for any hyperbola in the affine plane. Let us take an affine coordinate system in such a way that the hyperbola has the equation $xy = 1$. With $i \in \{1, 2, 3, 4\}$ we have points of the form $T_i = (x_i, \frac{1}{x_i})$. The line joining the points T_i and T_j has the equation

$$x_i x_j y = -x + x_i + x_j.$$

This line meets the asymptote given with the equation $y = 0$ in the point having the abscissa $x_i + x_j$ and denoted by T_{ij} . Two points $T_{12} = (x_1 + x_2, 0)$ and $T_{34} = (x_3 + x_4, 0)$, for example, have the midpoint $(\frac{1}{2}(x_1 + x_2 + x_3 + x_4), 0)$. This midpoint is symmetrical to the center of the hyperbola, i.e. to the origin w.r.t. the line with the equation $x = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$ passing through the centroid of the points T_1, T_2, T_3, T_4 and parallel to the y -axis, i.e. another asymptote of the hyperbola. \square

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