ON WEAKLY SEMI-$\mathcal{I}$-OPEN SETS

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Abstract. A decomposition of continuity via ideals is given. Characterizations of completely codense ideals are given in terms of weakly semi-$\mathcal{I}$-open sets. Also, properties of weakly semi-$\mathcal{I}$-open sets and their relation with other sets are discussed.

1. Introduction and Preliminaries.

Hatir and Jafari [7] have introduced the notions of weakly semi-$\mathcal{I}$-open sets and weakly semi-$\mathcal{I}$-continuous functions and obtained a decomposition of continuity. In this paper, we further study the properties of weakly semi-$\mathcal{I}$-open sets. We define weakly semi-$\mathcal{I}$-interior and weakly semi-$\mathcal{I}$-closure for subsets of ideal spaces, discuss their properties, give a decomposition of continuity and characterize completely codense ideals.

By a space, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subset X$, $\text{cl}(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau)$. A subset $A$ of a space $(X, \tau)$ is said to be regularclosed if $\text{cl}(\text{int}(A)) = A$. $A$ is said to be semiopen [13] (resp. $\beta$-open [1]) if $A \subset \text{cl}(\text{int}(A))$ (resp. $A \subset \text{cl}(\text{int}(\text{cl}(A)))$). The complement of a semiopen set is said to be semiclosed. Also, $A$ is semiclosed if and only if $\text{int}(A) = \text{int}(\text{cl}(A))$ [6, Proposition 1]. An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\varphi(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^* : \varphi(X) \to \varphi(X)$, called a local function [12] of $A$ with respect to $\tau$ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{ x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \}$ where $\tau(x) = \{ U \in \tau \mid x \in U \}$. We will make use of the basic facts concerning the local function [11, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(\cdot)$

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for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finer than $\tau$ is defined by $\text{cl}^*(A) = A \cup A^*(\mathcal{I}, \tau)$ \cite{16, 17}. When there is no chance for confusion, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on $X$, then $(X, \tau, \mathcal{I})$ is called an ideal space. Given an ideal space $(X, \tau, \mathcal{I})$, $\mathcal{I}$ is said to be codense \cite{5} if $\tau \cap \mathcal{I} = \{\emptyset\}$. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be $*$-dense in itself \cite{10} (resp. $*$-perfect \cite{10}) if $A \subseteq A^*$ (resp. $A = A^*$). The following lemmas will be useful in the sequel.

**Lemma 1.1.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A$ be a $*$-dense in itself subset of $X$. Then $A^* = \text{cl}(A) = \text{cl}^*(A)$ \cite{15, Theorem 5}.

**Lemma 1.2.** Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent \cite{11, Theorem 6.1}.
(a) $\mathcal{I}$ is codense.
(b) $G \subset G^*$ for every open set $G$.

**Lemma 1.3.** Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense. Then the following hold \cite{15, Corollary 2}.
(a) $\text{cl}(G) = \text{cl}^*(G)$ for every semiclosed set $G$.
(b) $\text{int}(F) = \text{int}^*(F)$ for every semiopen set $F$.

**Lemma 1.4.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. Then $\text{cl}^*({\text{int}(\text{cl}^*({\text{int}(A)}))}) = \text{cl}^*({\text{int}(A)})$ (The proof follows from Lemma 1.13 of \cite{3} if we take $i = \text{int}$ and $\kappa = \text{cl}^*$).

### 2. Weakly semi-$\mathcal{I}$-open sets

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be almost strong $\mathcal{I}$-open \cite{8} (resp. almost $\mathcal{I}$-open \cite{2}) if $A \subseteq \text{cl}^*({\text{int}(A^*)})$ (resp. $A \subseteq \text{cl}({\text{int}(A^*)})$).

Every almost strong $\mathcal{I}$-open set is almost $\mathcal{I}$-open but not the converse \cite{8}. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be strong $\beta$-$\mathcal{I}$-open \cite{8} if $A \subseteq \text{cl}^*({\text{int}(\text{cl}^*(A))})$.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be strong $\beta$-$\mathcal{I}$-open \cite{9} if $A \subseteq \text{cl}({\text{int}(\text{cl}^*(A))})$. Every almost strong $\mathcal{I}$-open set is a strong $\beta$-$\mathcal{I}$-open set, every strong $\beta$-$\mathcal{I}$-open set is a $\beta$-$\mathcal{I}$-open set and every $\beta$-$\mathcal{I}$-open set is a $\beta$-open set \cite{8, Propositions 1,2}. The reverse implications are not true \cite{8}. Every almost $\mathcal{I}$-open set is $\beta$-$\mathcal{I}$-open but not the converse \cite{8}. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be weakly semi-$\mathcal{I}$-open \cite{7} if $A \subseteq \text{cl}^*({\text{int}(\text{cl}(A))})$.

Every weakly semi-$\mathcal{I}$-open set is a $\beta$-open set but not the converse \cite{7, Remark 2.2}. Since $\tau \subset \tau^*$, it follows that every strong $\beta$-$\mathcal{I}$-open set is a weakly semi-$\mathcal{I}$-open set. Example 2.1 below shows that the converse is not true and Theorem 2.2 gives a decomposition for almost strong $\mathcal{I}$-open sets. Example 2.3 shows that weakly semi-$\mathcal{I}$-open sets and $*$-dense in itself sets are independent. Theorem 2.4 shows that if the ideal is codense, then the concepts $\beta$-openness and weakly semi-$\mathcal{I}$-openness coincide.
Example 2.1. Consider the ideal space \((X, \tau, I)\) where \(X = \{a, b, c\}\), \(\tau = \{\emptyset, \{a\}, \{a, c\}, X\}\) and \(I = \{\emptyset, \{a\}\}\). If \(A = \{a, b\}\), then \(A^* = \{b\}\) and so \(\text{cl}^*(A) = \{a, b\}\). Now \(\text{cl}^*(\text{int}(\text{cl}^*(A))) = \text{cl}^*(\text{int}(\{a, b\})) = \text{cl}^*(\{a\}) = \{a\} \not\subset A\) and so \(A\) is not strong \(\beta-I\)-open. But \(\text{cl}^*(\text{int}(\text{cl}(A))) = \text{cl}^*(\text{int}(X)) = X \supset A\) and so \(A\) is weakly semi-\(I\)-open.

Theorem 2.2. Let \((X, \tau, I)\) be an ideal space and \(A \subset X\). Then the following are equivalent.

(a) \(A\) is almost strong \(I\)-open.
(b) \(A\) is both strong \(\beta-I\)-open and almost \(I\)-open.
(c) \(A\) is both weakly semi-\(I\)-open and almost \(I\)-open.
(d) \(A\) is both weakly semi-\(I\)-open and \(\ast\)-dense in itself.

Proof. It is enough to prove that \((d) \Rightarrow (a)\). If \(A\) is weakly semi-\(I\)-open, then \(A \subset \text{cl}^*(\text{int}(\text{cl}(A)))\). If \(A \subset A^*\), by Lemma 1.1, \(\text{cl}(A) = A^*\) and so \(A \subset \text{cl}^*(\text{int}(A^*))\) which implies that \(A\) is almost strong \(I\)-open. \(\square\)

Example 2.3. Consider the ideal space \((X, \tau, I)\) where \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}\) and \(I = \{\emptyset, \{a\}\}\). If \(A = \{b\}\), then \(A^* = \{b, d\}\) and so \(A\) is \(\ast\)-dense in itself. Now \(\text{cl}^*(\text{int}(\text{cl}(A))) = \text{cl}^*(\text{int}(\{b, d\})) = \text{cl}^*(\text{int}(\emptyset)) = \emptyset \not\subset A\) and so \(A\) is not weakly semi-\(I\)-open. Also if \(B = \{a, c\}\), then \(B^* = \{b, c, d\}\) which does not contain \(B\) and hence \(B\) is not \(\ast\)-dense in itself. But \(\text{cl}^*(\text{int}(\text{cl}(B))) = \text{cl}^*(\text{int}(X)) = X\) and so \(B\) is weakly semi-\(I\)-open.

Theorem 2.4. Let \((X, \tau, I)\) be an ideal space where \(I\) is codense. If \(A\) is \(\beta\)-open, then \(A\) is weakly semi-\(I\)-open.

Proof. If \(A\) is \(\beta\)-open, then \(A \subset \text{cl}(\text{int}(\text{cl}(A)))\). By Lemmas 1.1 and 1.2, \(\text{cl}(\text{int}(\text{cl}(A))) = \text{cl}^*(\text{int}(\text{cl}(A)))\) and so \(A \subset \text{cl}^*(\text{int}(\text{cl}(A)))\) which implies that \(A\) is weakly semi-\(I\)-open. \(\square\)

Corollary 2.5. Let \((X, \tau, I)\) be an ideal space where \(I\) is codense. If \(A\) is \(\ast\)-dense in itself, then the following are equivalent.

(a) \(A\) is almost strong \(I\)-open.
(b) \(A\) is strong \(\beta-I\)-open.
(c) \(A\) is \(\beta-I\)-open.
(d) \(A\) is \(\beta\)-open.
(e) \(A\) is weakly semi-\(I\)-open.

The following Examples 2.6 and 2.7 show that weakly semi-\(I\)-openness and \(\beta-I\)-openness are independent concepts.

Example 2.6. Let \(X = \{a, b, c\}\), \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\) and \(I = \{\emptyset, \{a\}\}\) [7, Example 2.2]. In [7], it is shown that \(A = \{a, c\}\), is not weakly semi-\(I\)-open. Since \(\text{cl}(\text{int}(\text{cl}^*(A))) = \text{cl}(\text{int}(\{a, c\})) = \text{cl}(\{a\}) = \{a, c\} = A\), \(A\) is \(\beta-I\)-open.
Example 2.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. If $A = \{b, c\}$, then $\text{cl}^*(A) = \{b, c\}$ and $\text{cl}(\text{int}(\text{cl}^*(A))) = \text{cl}(\text{int}(\{b, c\})) = \text{cl}(\emptyset) = \emptyset \not\supset A$. Hence $A$ is not $\beta$-$\mathcal{I}$-open. So $A$ is a weakly semi-$\mathcal{I}$-open set since $\text{cl}^*(\text{int}(\text{cl}(A))) = \text{cl}^*(\text{int}(\{a, b, c\})) = \text{cl}^*(\{a, c\}) = \{a, b, c\} \supset A$.

A subset $A$ of an ideal space $(X, \tau)$ is said to be preopen [14] if $A \subset \text{int}(\text{cl}(A))$. The family of all preopen sets is denoted by $PO(X)$. The following Theorem 2.8 discuss the relation between preopen and weakly semi-$\mathcal{I}$-open sets. Example 2.9 shows that a weakly semi-$\mathcal{I}$-open set need not be preopen.

Theorem 2.8. Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following hold.
(a) If $A$ is preopen, then $A$ is weakly semi-$\mathcal{I}$-open.
(b) If open sets are $\ast$–closed, then every weakly semi-$\mathcal{I}$-open set is preopen.

Proof. (a) If $A$ is preopen, then $A \subset \text{int}(\text{cl}(A))$ and so $A \subset \text{cl}^*(\text{int}(\text{cl}(A)))$ which implies that $A$ is weakly semi-$\mathcal{I}$-open.
(b) If $A$ is weakly semi-$\mathcal{I}$-open, then $A \subset \text{cl}^*(\text{int}(\text{cl}(A)))$. Since $\text{int}(\text{cl}(A))$ is open, by hypothesis, $\text{int}(\text{cl}(A)) = \text{cl}^*(\text{int}(\text{cl}(A)))$ and so $A \subset \text{int}(\text{cl}(A))$ which implies that $A$ is preopen.

Example 2.9. Consider the ideal space $(X, \tau, \mathcal{I})$ of Example 2.7. If $A = \{b, c\}$, then $A$ is weakly semi-$\mathcal{I}$-open. Also, $\text{int}(\text{cl}(A)) = \text{int}(\{a, b, c\}) = \{a, b, c\} \not\supset A$. Therefore, $A$ is not preopen.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\alpha$-$\mathcal{I}$-open [9] if $A \subset \text{int}(\text{cl}^*(\text{int}(A)))$. Every open set is an $\alpha$-$\mathcal{I}$-open set but not the converse. In Theorem 2.1(2) of [7], it is established that the intersection of an open set and a weakly $\mathcal{I}$-open set is weakly $\mathcal{I}$-open. The following Theorem 2.10 is a generalization of this result. Theorem 2.11 below gives a property of weakly semi-$\mathcal{I}$-open sets.

Theorem 2.10. Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A$ is $\alpha$-$\mathcal{I}$-open and $B$ is weakly semi-$\mathcal{I}$-open, then $A \cap B$ is weakly semi-$\mathcal{I}$-open.

Proof. Since $A$ is $\alpha$-$\mathcal{I}$-open, $A \subset \text{int}(\text{cl}^*(\text{int}(A)))$ and $B$ is weakly semi-$\mathcal{I}$-open, $B \subset \text{cl}^*(\text{int}(\text{cl}(B)))$. Now $A \cap B \subset \text{int}(\text{cl}^*(\text{int}(A))) \cap \text{cl}^*(\text{int}(\text{cl}(B))) \subset \text{cl}^*(\text{int}(\text{cl}^*(\text{int}(A))) \cap \text{int}(\text{cl}(B))) = \text{cl}^*(\text{int}(\text{cl}^*(\text{int}(A))) \cap \text{int}(\text{cl}(B))) \subset \text{cl}^*(\text{int}(\text{cl}^*(\text{int}(A)) \cap \text{cl}(B))) \subset \text{cl}^*(\text{int}(\text{cl}^*(\text{int}(A) \cap \text{cl}(B)))) \subset \text{cl}^*(\text{int}(\text{cl}^*(\text{int}(A \cap B)))) = \text{cl}^*(\text{int}(\text{cl}(A \cap B)))$ by Lemma 1.4, which implies that $A \cap B$ is weakly semi-$\mathcal{I}$-open.

Theorem 2.11. Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A \subset B \subset \text{cl}^*(A)$ and $A$ is weakly semi-$\mathcal{I}$-open, then $B$ is weakly semi-$\mathcal{I}$-open. In particular, if $A$ is weakly semi-$\mathcal{I}$-open, then $\text{cl}^*(A)$ is weakly semi-$\mathcal{I}$-open.
Proof. If $A$ is weakly semi-$\mathcal{I}$-open, then $A \subseteq cl^*(int(cl(A)))$. Since $B \subseteq cl^*(A) \subseteq cl^*(cl^*(int(cl(A)))) = cl^*(int(cl(A))) \subseteq cl^*(int(cl(B)))$. Therefore, $B$ is weakly semi-$\mathcal{I}$-open.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-locally closed [4] if $A = U \cap V$ where $U$ is open and $V$ is $\mathcal{I}$-perfect or equivalently, $A = U \cap A^*$ for some open set $U$. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be semi-$\mathcal{I}$-open [9] if $A \subseteq cl^*(int(A))$. Every semi-$\mathcal{I}$-open set is weakly semi-$\mathcal{I}$-open [7, Remark 2.1] and but not the converse [7, Example 2.1]. The following Theorem 2.12 deals with the reverse direction. Example 2.13 below shows that the condition semiclosed or $\mathcal{I}$-locally closed on the subset in Theorem 2.12 cannot be dropped.

**Theorem 2.12.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$ be a weakly semi-$\mathcal{I}$-open. If $A$ is either semiclosed or $\mathcal{I}$-locally closed, then $A$ is semi-$\mathcal{I}$-open.

**Proof.** Suppose $A$ is $\mathcal{I}$-locally closed. $A$ is $\mathcal{I}$-locally closed implies that $A = U \cap A^*$ for some open set $U$. $A$ is weakly semi-$\mathcal{I}$-open implies that $A \subseteq cl^*(int(cl(A)))$. Now $A = U \cap A^* \subseteq U \cap (cl^*(int(cl(A))))^* \subseteq U \cap cl^*(cl^*(int(cl(U \cap A^*)))) = U \cap cl^*(int(cl(U \cap A^*))) \subseteq cl^*(U \cap int(cl(U \cap A^*))) = cl^*(int(U \cap cl(U \cap A^*))) = cl^*(int(cl(U \cap A^*))) = cl^*(int(U \cap A^*))$. Hence $A$ is semi-$\mathcal{I}$-open.

Suppose $A$ is semiclosed. Then $int(cl(A)) = int(A)$. Since $A$ is weakly semi-$\mathcal{I}$-open, $A \subseteq cl^*(int(cl(A))) = cl^*(int(A))$. Hence $A$ is semi-$\mathcal{I}$-open. $\square$

**Example 2.13.** Consider the ideal space $(X, \tau, \mathcal{I})$ with $X = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\mathcal{I} = \emptyset, \{c\}$. If $A = \{a\}$, then $A^* = X$ and $cl^*(int(cl(A))) = cl^*(int(X)) = X$ $\supseteq A$ and so $A$ is weakly semi-$\mathcal{I}$-open. Also $cl^*(int(A)) = cl^*(\emptyset) = \emptyset$. Hence $A$ is not semi-$\mathcal{I}$-open. Moreover, $A$ is neither $\mathcal{I}$-locally closed nor semiclosed.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be weakly semi-$\mathcal{I}$-closed [7] if $X - A$ is weakly semi-$\mathcal{I}$-open or equivalently, $int^*(cl(int(A))) \subset A$ [7, Theorem 2.2]. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\alpha^*-\mathcal{I}$-set [9] if $int(cl^*(int(A))) = int(A)$. The following Theorem 2.14 examines the relation between weakly semi-$\mathcal{I}$-closed set and $\alpha^*-\mathcal{I}$-set. Example 2.15 below shows that an $\alpha^*-\mathcal{I}$-set need not be a weakly semi-$\mathcal{I}$-closed set.

**Theorem 2.14.** The following hold in any ideal space $(X, \tau, \mathcal{I})$.

(a) If $A$ is a weakly semi-$\mathcal{I}$-closed subset of $X$, then $A$ is an $\alpha^*-\mathcal{I}$-set.

(b) If $\mathcal{I}$ is codense, then $A$ is weakly semi-$\mathcal{I}$-closed if and only if $int(cl^*(int(A))) \subset A$. 


Proof. (a) If $A$ is weakly semi-$\mathcal{I}$-closed, then by Theorem 2.3 of [7], $\text{int}(\text{cl}^{\ast}(\text{int}(A))) \subset A$ and so $\text{int}(\text{cl}^{\ast}(\text{int}(A))) \subset \text{int}(A)$. Hence it follows that $\text{int}(\text{cl}^{\ast}(\text{int}(A))) = \text{int}(A)$ which implies that $A$ is an $\alpha^{\ast}\mathcal{I}$-set.

(b) If $A$ is any subset of $X$, then $\text{int}^{\ast}(\text{cl}(X - A)) = \text{int}(\text{cl}(X - A))$ by Lemma 1.3(b). Therefore, $\text{cl}(\text{int}^{\ast}(\text{cl}(X - A))) = \text{cl}(\text{int}(\text{cl}(X - A)))$. By Lemma 1.3(a), $\text{cl}(\text{int}(\text{cl}(X - A))) = \text{cl}^{\ast}(\text{int}(\text{cl}(X - A)))$ and so $\text{cl}(\text{int}^{\ast}(\text{cl}(X - A))) = \text{cl}^{\ast}(\text{int}(\text{cl}(X - A)))$ which implies that $X - \text{int}(\text{cl}^{\ast}(\text{int}(A))) = \text{cl}^{\ast}(\text{int}(\text{cl}(X - A)))$. By Corollary 2.1 of [7], (b) follows.

Example 2.15. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. If $A = \{a, b\}$, then $\text{cl}(\text{int}(A)) = \text{cl}(\{a\}) = \{a, b, c\}$ and so $\text{int}(\text{cl}^{\ast}(\text{int}(A))) = \text{int}(\{a, b, c\}) = \{a\} = \text{int}(A)$. Hence $A$ is an $\alpha^{\ast}\mathcal{I}$-set. Since $\text{int}^{\ast}(\text{cl}(\text{int}(A))) = \text{int}^{\ast}(\text{cl}(\{a\})) = \text{int}^{\ast}(\{a, b, c\}) = \{a, b, c\} \not\subset A$, $A$ is not weakly semi-$\mathcal{I}$-closed.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be weakly $S_{\mathcal{I}}$-set (resp. $C_{\mathcal{I}}$-set [9]) if $A = G \cap V$ where $G$ is open and $V$ is weakly semi-$\mathcal{I}$-closed (resp. $\alpha^{\ast}\mathcal{I}$-set). Clearly, every open set is a weakly $S_{\mathcal{I}}$-set and every weakly $S_{\mathcal{I}}$-set is a $C_{\mathcal{I}}$-set. Hence we have the following decomposition of open sets. Example 2.17 shows that the concepts $\alpha\mathcal{I}$-open sets and weakly $S_{\mathcal{I}}$-sets are independent.

Theorem 2.16. Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.

(a) $A$ is open.

(b) $A$ is $\alpha\mathcal{I}$-open and a weakly $S_{\mathcal{I}}$-set.

(c) $A$ is $\alpha\mathcal{I}$-open and a $C_{\mathcal{I}}$-set.

Proof. (a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) are clear. (c)$\Rightarrow$(a) follows from Proposition 3.3 of [9].

Example 2.17. Consider the ideal space $(X, \tau, \mathcal{I})$ with $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. If $A = \{b\}$, then $\text{int}^{\ast}(\text{cl}(\text{int}(A))) = \text{int}^{\ast}(\text{cl}(\emptyset)) = \emptyset \subset A$ and so $A$ is weakly semi-$\mathcal{I}$-closed. Hence $A$ is a weakly $S_{\mathcal{I}}$-set. Also, $\text{int}(\text{cl}(\text{int}(A))) = \emptyset$ and so $A$ is not $\alpha\mathcal{I}$-open. If $B = \{c\}$, then $\text{int}^{\ast}(\text{cl}(\text{int}(B))) = \text{int}^{\ast}(\text{cl}(\{c\})) = \text{int}^{\ast}(\{b, c, d\}) = \{b, c, d\} \not\subset B$ and so $B$ is not a weakly $S_{\mathcal{I}}$-set. But $\text{int}(\text{cl}(\text{int}(B))) = \text{int}(\{b, c, d\}) = \{c\} = B$. Therefore, $B$ is $\alpha\mathcal{I}$-open.

A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $\alpha\mathcal{I}$-continuous [9] (resp. $C_{\mathcal{I}}$-continuous [9], weakly $S_{\mathcal{I}}$-continuous) if for every $V \in \sigma$, $f^{-1}(V)$ is an $\alpha\mathcal{I}$-open set (resp. $C_{\mathcal{I}}$-set, weakly $S_{\mathcal{I}}$-set). The following Theorem 2.18 is a decomposition continuity which follows from Theorem 2.16.

Theorem 2.18. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a function, then the following are equivalent.
(a) \( f \) is continuous.
(b) \( f \) is \( \alpha\mathcal{I} \)-continuous and weakly \( S_\mathcal{I} \)-continuous.
(c) \( f \) is \( \alpha\mathcal{I} \)-continuous and \( C_\mathcal{I} \) continuous.

3. Completely codense ideals

An ideal \( \mathcal{I} \) of an ideal space \((X, \tau, \mathcal{I})\) is said to be completely codense [5] if \( PO(X) \cap \mathcal{I} = \{\emptyset\} \). In Theorem 4.13 of [5], it is established that \( \mathcal{I} \) is completely codense if and only if \( \mathcal{I} \subset \mathcal{N} \), where \( \mathcal{N} \) is the ideal of all nowhere dense sets in \( X \). Also, every completely codense ideal is codense but not the converse[5]. In this section, we define weakly \( \mathcal{I} \)-interior and weakly \( \mathcal{I} \)-closure of a subset of an ideal space, discuss its properties and characterize completely codense ideals in terms of weakly \( \mathcal{I} \)-open sets.

The weakly \( \mathcal{I} \)-interior of a subset \( A \) of an ideal space \((X, \tau, \mathcal{I})\) is the largest weakly \( \mathcal{I} \)-open set contained in \( A \) and is denoted by \( ws\mathcal{I} \text{Int}(A) \).

By Theorem 2.1(a) of [7], \( ws\mathcal{I} \text{Int}(A) \) is a weakly \( \mathcal{I} \)-open set and it is clear that \( A \) is a weakly \( \mathcal{I} \)-open set if and only if \( A = ws\mathcal{I} \text{Int}(A) \).

The weakly \( \mathcal{I} \)-closure of a subset \( A \) of an ideal space \((X, \tau, \mathcal{I})\) is the smallest weakly \( \mathcal{I} \)-closed set containing \( A \) and is denoted by \( ws\mathcal{I} \text{Cl}(A) \).

It is clear that \( ws\mathcal{I} \text{Cl}(A) \) is a weakly \( \mathcal{I} \)-closed set and \( A \) is a weakly \( \mathcal{I} \)-closed set if and only if \( A = ws\mathcal{I} \text{Cl}(A) \). The following results are essential to characterize completely codense ideals in terms of weakly \( \mathcal{I} \)-open sets.

**Theorem 3.1.** If \((X, \tau, \mathcal{I})\) is an ideal space and \( A \subset X \), then the following holds.

(a) \( ws\mathcal{I} \text{Int}(A) = A \cap \text{cl}^*(\text{int}(\text{cl}(A))) \).
(b) \( ws\mathcal{I} \text{Cl}(A) = A \cup \text{int}^*(\text{cl}(\text{int}(A))) \).

**Proof.** (a) If \( A \) is any subset of \( X \), then \( A \cap \text{cl}^*(\text{int}(\text{cl}(A))) \subset \text{cl}^*(\text{int}(\text{cl}(A))) = \text{cl}^*(\text{int}(\text{cl}(A) \cap \text{int}(\text{cl}(A)))) \subset \text{cl}^*(\text{int}(\text{cl}(A) \cap \text{cl}(\text{int}(A)))) \subset \text{cl}^*(\text{int}(\text{cl}(A) \cap \text{cl}(\text{int}(A)))) \cap \text{cl}^*(\text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A))) \subset \text{cl}^*(\text{int}(\text{cl}(A) \cap \text{cl}(\text{int}(A)))) \)

Since \( ws\mathcal{I} \text{Int}(A) \) is weakly \( \mathcal{I} \)-open, \( ws\mathcal{I} \text{Int}(A) \subset \text{cl}^*(\text{int}(\text{cl}(ws\mathcal{I} \text{Int}(A)))) \subset \text{cl}^*(\text{int}(\text{cl}(A))) \)

and so \( ws\mathcal{I} \text{Int}(A) \subset A \cap \text{cl}^*(\text{int}(\text{cl}(A))) \). Therefore \( ws\mathcal{I} \text{Int}(A) = A \cap \text{cl}^*(\text{int}(\text{cl}(A))) \).

(b) Now \( \text{int}^*(\text{cl}(\text{int}(A) \cap \text{int}(\text{cl}(A)))) \subset \text{int}^*(\text{cl}(\text{int}(A) \cup \text{cl}(\text{int}(A)))) = \text{int}^*(\text{cl}(\text{int}(A))) \subset A \cup \text{int}^*(\text{cl}(\text{int}(A))) \). Hence \( A \cup \text{int}^*(\text{cl}(\text{int}(A))) \) is a weakly \( \mathcal{I} \)-closed set containing \( A \) and so \( ws\mathcal{I} \text{Cl}(A) \subset A \cup \text{int}^*(\text{cl}(\text{int}(A))) \).

Since \( ws\mathcal{I} \text{Cl}(A) \) is weakly \( \mathcal{I} \)-closed, we have \( \text{int}^*(\text{cl}(\text{int}(A))) \subset \text{int}^*(\text{cl}(ws\mathcal{I} \text{Cl}(A))) \subset \text{wsI} \text{Cl}(A) \).

Therefore, \( A \cup \text{int}^*(\text{cl}(\text{int}(A))) \subset A \cup ws\mathcal{I} \text{Cl}(A) = ws\mathcal{I} \text{Cl}(A) \).

Hence \( ws\mathcal{I} \text{Cl}(A) = A \cup \text{int}^*(\text{cl}(\text{int}(A))) \). \( \square \)
The following Theorem 3.2 characterizes completely codense ideals. We will denote the family of all weakly semi-$\mathcal{I}$-open sets in any ideal space $(X, \tau, \mathcal{I})$ by $WSIO(X)$.

**Theorem 3.2.** If $(X, \tau, \mathcal{I})$ is an ideal space, then the following are equivalent.  
(a) $\mathcal{I}$ is completely codense.  
(b) $WSIO(X) \cap \mathcal{I} = \{\emptyset\}$.  
(c) $A \subset A^*$ for every $A \in WSIO(X)$.  
(d) $ws\operatorname{Int}(A) \subset ws\operatorname{Int}(A^*)$ for every subset $A$ of $X$.  
(e) $ws\operatorname{Int}(A) = \emptyset$ for every $A \in \mathcal{I}$.

**Proof.** (a)$\Rightarrow$(b). Suppose $A \in WSIO(X) \cap \mathcal{I}$. $A \in \mathcal{I}$ implies that $A \in \mathcal{N}$ by Lemma 1.3 and so $\operatorname{int}(\operatorname{cl}(A)) = \emptyset$. Since $A \in WSIO(X)$, $A \subset \operatorname{cl}^*(\operatorname{int}(\operatorname{cl}(A))) = \operatorname{cl}^*(\emptyset) = \emptyset$ and so $A = \emptyset$. Therefore, $WSIO(X) \cap \mathcal{I} = \{\emptyset\}$.  
(b)$\Rightarrow$(c). Let $A \in WSIO(X)$. Suppose that $x \notin A^*$. Then there exists an open set $G$ containing $x$ such that $G \cap A \in \mathcal{I}$. Since $A \in WSIO(X)$, by Theorem 2.1(2) of [7], $G \cap A \in WSIO(X)$ and so by hypothesis, $G \cap A = \emptyset$ which implies that $x \notin A$. Hence $A \subset A^*$.  
(c)$\Rightarrow$(d). For any subset $A$ of $X$, $ws\operatorname{Int}(A) \in WSIO(X)$ and so $ws\operatorname{Int}(A) \subset (ws\operatorname{Int}(A))^* \subset A^*$. Therefore, $ws\operatorname{Int}(A) \subset ws\operatorname{Int}(A^*)$.  
(d)$\Rightarrow$(e). If $A \in \mathcal{I}$, then $A^* = \emptyset$ and so by (d), $ws\operatorname{Int}(A) \subset ws\operatorname{Int}(\emptyset) = \emptyset$. Therefore, $ws\operatorname{Int}(A) = \emptyset$.  
(e)$\Rightarrow$(a). Suppose $A \in PO(X) \cap \mathcal{I}$. $A \in PO(X)$ implies that $A \subset \operatorname{int}(\operatorname{cl}(A))$. $A \in \mathcal{I}$ implies that $ws\operatorname{Int}(A) = \emptyset$. By Theorem 3.1(a), $A \cap \operatorname{cl}^*(\operatorname{int}(\operatorname{cl}(A))) = \emptyset$ which implies that $A \cap \operatorname{int}(\operatorname{cl}(A)) = \emptyset$. Since $A \subset \operatorname{int}(\operatorname{cl}(A))$, we have $A = \emptyset$. Therefore, $PO(X) \cap \mathcal{I} = \{\emptyset\}$ which implies that $\mathcal{I}$ is completely codense. □

**Corollary 3.3.** If $(X, \tau, \mathcal{I})$ is an ideal space and $A \in WSIO(X)$, then the following holds.  
(a) $\operatorname{cl}(A)$ is regular closed and $\operatorname{cl}(A) = \operatorname{cl}^*(\operatorname{int}(\operatorname{cl}(A))) = A^*(\mathcal{N})$.  
(b) If $\mathcal{I}$ is completely codense, then $A^*(\mathcal{I}) = A^*(\mathcal{N})$.

**Proof.** (a) If $A \in WSIO(X)$, then $A \subset \operatorname{cl}^*(\operatorname{int}(\operatorname{cl}(A))) \subset \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A))) \subset \operatorname{cl}(A)$ and so, it follows that $\operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))$. Since $A^*(\mathcal{N}) = \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))$, [17] (a) follows.  
(b) If $\mathcal{I}$ is completely codense, by Theorem 3.2(c), $A \subset A^*$. By Lemma 1.1, $\operatorname{cl}(A) = A^*$. Therefore, the proof follows from (a). □

The following Theorem 3.4 gives another characterization of completely codense ideals where $AIO(X)$ is the family of all almost $\mathcal{I}$-open sets in $(X, \tau, \mathcal{I})$.

**Theorem 3.4.** Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is completely codense if and only if $WSIO(X) = AIO(X)$. 

V. RENUKA DEVI AND D. SIVARAJ
Proof. Suppose $I$ is completely codense. If $A \in WSIO(X)$, then $A \subset cl^*(int(cl(A)))$ and by Theorem 3.2(c), $A \subset A^*$. Since every completely codense ideal is codense, by Lemma 1.3, $cl^*(int(cl(A))) = cl(int(cl(A))) \subset cl(int(cl(A^*))) = cl(int(A^*))$, since $A^*$ is closed. Therefore, $A \subset cl((int(A^*)))$ which implies that $A \in AZO(X)$. If $A \in AZO(X)$, then $A \subset cl((int(A^*)))$ and so $A \subset cl(int(cl(A))) = cl^*(int(cl(A)))$ which implies that $A \in WSIO(X)$. Conversely, suppose $WSIO(X) = AZO(X)$. If $A \in WSIO(X)$, then $A \in AZO(X)$ and so $A \subset cl(int(A^*)) \subset cl(A^*) = A^*$. By Theorem 3.2, $I$ is completely codense. □

References

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