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# ON WEAKLY SEMI-$\mathcal{I}$-OPEN SETS

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Abstract. A decomposition of continuity via ideals is given. Characterizations of completely codense ideals are given in terms of weakly semi-$\mathcal{I}$-open sets. Also, properties of weakly semi-$\mathcal{I}$-open sets and their relation with other sets are discussed.

## 1. Introduction and Preliminaries.

Hatir and Jafari [7] have introduced the notions of weakly semi-$\mathcal{I}$-open sets and weakly semi-$\mathcal{I}$-continuous functions and obtained a decomposition of continuity. In this paper, we further study the properties of weakly semi-$\mathcal{I}$-open sets. We define weakly semi-$\mathcal{I}$-interior and weakly semi-$\mathcal{I}$-closure for subsets of ideal spaces, discuss their properties, give a decomposition of continuity and characterize completely codense ideals.

By a space, we always mean a topological space $(X,\tau)$ with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of $A$ in $(X,\tau)$. A subset $A$ of a space $(X,\tau)$ is said to be *regularclosed* if $cl(int(A)) = A$. $A$ is said to be *semiopen* [13] (resp. $\beta$-open [1]) if $A \subset cl(int(A))$ (resp. $A \subset cl(int(cl(A))))$. The complement of a semiopen set is said to be *semiclosed*. Also, $A$ is semiclosed if and only if $int(A) = int(cl(A))$ [6, Proposition 1]. An *ideal* $\mathcal{I}$ on a topological space $(X,\tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space $(X,\tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\wp(X)$ is the set of all subsets of $X$, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a *local function* [12] of $A$ with respect to $\tau$ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I},\tau)=\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local function [11, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$

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for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star-$ *topology*, finer than $\tau$ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I},\tau)$ [16, 17]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on $X$, then $(X,\tau,\mathcal{I})$ is called an ideal space. Given an ideal space $(X,\tau,\mathcal{I}),\mathcal{I}$ is said to be *codense* [5] if $\tau\cap\mathcal{I}=\{\emptyset\}$. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be $\star$*–dense in itself* [10] (resp. $\star$*–perfect* [10]) if $A \subset A^\star$ (resp. $A = A^\star$). The following lemmas will be useful in the sequel.

**Lemma 1.1.** *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A$ be a $\star$–dense in itself subset of $X$. Then $A^\star = cl(A) = cl^\star(A)$* [15*, Theorem* 5].

**Lemma 1.2.** *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent* [11*, Theorem* 6.1].
(a) *$\mathcal{I}$ is codense.*
(b) *$G \subset G^\star$ for every open set $G$.*

**Lemma 1.3.** *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense. Then the following hold* [15*, Corollary* 2].
(a) *$cl(G) = cl^\star(G)$ for every semiopen set $G$.*
(b) *$int(F) = int^\star(F)$ for every semiclosed set $F$.*

**Lemma 1.4.** *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. Then $cl^\star(int(cl^\star(int(A)))) = cl^\star(int(A))$ (The proof follows from Lemma* 1.13 *of* [3] *if we take $\iota = int$ and $\kappa = cl^\star$).*

## 2. Weakly semi-$\mathcal{I}$-open sets

A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be *almost strong $\mathcal{I}$-open* [8] (resp. *almost $\mathcal{I}$-open* [2]) if $A \subset cl^\star(int(A^\star))$ (resp. $A \subset cl(int(A^\star))$ ). Every almost strong $\mathcal{I}$-open set is almost $\mathcal{I}$-open but not the converse [8]. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be *strong $\beta$-$\mathcal{I}$-open* [8] if $A \subset cl^\star(int(cl^\star(A)))$. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be *$\beta$-$\mathcal{I}$-open* [9] if $A \subset cl(int(cl^\star(A)))$. Every almost strong $\mathcal{I}$-open set is a strong $\beta$-$\mathcal{I}$-open set, every strong $\beta$-$\mathcal{I}$-open set is a $\beta$-$\mathcal{I}$-open set and every $\beta$-$\mathcal{I}$-open set is a $\beta-$open set [8, Propositions 1,2]. The reverse implications are not true [8]. Every almost $\mathcal{I}$-open set is $\beta$-$\mathcal{I}$-open but not the converse [8]. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be *weakly semi-$\mathcal{I}$-open* [7] if $A \subset cl^\star(int(cl(A)))$. Every weakly semi-$\mathcal{I}$-open set is a $\beta-$open set but not the converse [7, Remark 2.2]. Since $\tau \subset \tau^\star$, it follows that every strong $\beta$-$\mathcal{I}$-open set is a weakly semi-$\mathcal{I}$-open set. Example 2.1 below shows that the converse is not true and Theorem 2.2 gives a decomposition for almost strong $\mathcal{I}$-open sets. Example 2.3 shows that weakly semi-$\mathcal{I}$-open sets and $\star$–dense in itself sets are independent. Theorem 2.4 shows that if the ideal is codense, then the concepts $\beta$-openness and weakly semi-$\mathcal{I}$-openness coincide.

**Example 2.1.** Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. If $A = \{a, b\}$, then $A^\star = \{b\}$ and so $cl^\star(A) = \{a, b\}$. Now $cl^\star(int(cl^\star(A))) = cl^\star(int(\{a, b\})) = cl^\star(\{a\}) = \{a\} \not\supset A$ and so $A$ is not strong $\beta$-$\mathcal{I}$-open. But $cl^\star(int(cl(A))) = cl^\star(int(X)) = X \supset A$ and so A is weakly semi-$\mathcal{I}$-open.

**Theorem 2.2.** *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then the following are equivalent.*
(a) *$A$ is almost strong $\mathcal{I}$-open.*
(b) *$A$ is both strong $\beta$-$\mathcal{I}$-open and almost $\mathcal{I}$-open.*
(c) *$A$ is both weakly semi-$\mathcal{I}$-open and almost $\mathcal{I}$-open.*
(d) *$A$ is both weakly semi-$\mathcal{I}$-open and $\star$–dense in itself.*

*Proof.* It is enough to prove that (d)⇒(a). If A is weakly semi-$\mathcal{I}$-open, then $A \subset cl^\star(int(cl(A)))$. If $A \subset A^\star$, by Lemma 1.1, $cl(A) = A^\star$ and so $A \subset cl^\star(int(A^\star))$ which implies that $A$ is almost strong $\mathcal{I}$-open. □

**Example 2.3.** Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{I}$={$\emptyset$, {a}}. If $A = \{b\}$, then $A^\star = \{b, d\}$ and so $A$ is $\star$–dense in itself. Now $cl^\star(int(cl(A))) = cl^\star(int(\{b, d\})) = cl^\star(int(\emptyset)) = \emptyset \not\supset A$ and so $A$ is not weakly semi-$\mathcal{I}$-open. Also if $B = \{a, c\}$, then $B^\star = \{b, c, d\}$ which does not contain $B$ and hence $B$ is not $\star$–dense in itself. But $cl^\star(int(cl(B))) = cl^\star(int(X)) = X$ and so $B$ is weakly semi-$\mathcal{I}$-open.

**Theorem 2.4.** *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense. If $A$ is $\beta$-open, then $A$ is weakly semi-$\mathcal{I}$-open.*

*Proof.* If $A$ is $\beta$-open, then $A \subset cl(int(cl(A)))$. By Lemmas 1.1 and 1.2, $cl(int(cl(A))) = cl^\star(int(cl(A)))$ and so $A \subset cl^\star(int(cl(A)))$ which implies that $A$ is weakly semi-$\mathcal{I}$-open. □

**Corollary 2.5.** *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense. If $A$ is $\star$–dense in itself, then the following are equivalent.*
(a) *$A$ is almost strong $\mathcal{I}$-open.*
(b) *$A$ is strong $\beta$-$\mathcal{I}$-open.*
(c) *$A$ is $\beta$-$\mathcal{I}$-open.*
(d) *$A$ is $\beta$-open.*
(e) *$A$ is weakly semi-$\mathcal{I}$-open.*

The following Examples 2.6 and 2.7 show that weakly semi-$\mathcal{I}$-openness and $\beta$-$\mathcal{I}$-openness are independent concepts.

**Example 2.6.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$ [7, Example 2.2]. In [7], it is shown that $A = \{a, c\}$, is not weakly semi-$\mathcal{I}$-open. Since $cl(int(cl^\star(A))) = cl(int(\{a, c\})) = cl(\{a\}) = \{a, c\} = A$, $A$ is $\beta$-$\mathcal{I}$-open.

**Example 2.7.** Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. If $A = \{b, c\}$, then $cl^\star(A) = \{b, c\}$ and $cl(int(cl^\star(A))) = cl(int(\{b, c\})) = cl(\emptyset) = \emptyset \not\supset A$. Hence $A$ is not $\beta$-$\mathcal{I}$-open. So $A$ is a weakly semi-$\mathcal{I}$-open set since $cl^\star(int(cl(A))) = cl^\star(int(\{a, b, c\})) = cl^\star(\{a, c\}) = \{a, b, c\} \supset A$.

A subset $A$ of an ideal space $(X, \tau)$ is said to be *preopen* [14] if $A \subset int(cl(A))$. The family of all preopen sets is denoted by $PO(X)$. The following Theorem 2.8 discuss the relation between preopen and weakly semi-$\mathcal{I}$-open sets. Example 2.9 shows that a weakly semi-$\mathcal{I}$-open set need not be preopen.

**Theorem 2.8.** *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following hold.*
(a) *If $A$ is preopen, then $A$ is weakly semi-$\mathcal{I}$-open.*
(b) *If open sets are $\star$–closed, then every weakly semi-$\mathcal{I}$-open set is preopen.*

*Proof.* (a) If $A$ is preopen, then $A \subset int(cl(A))$ and so $A \subset cl^\star(int(cl(A)))$ which implies that $A$ is weakly semi-$\mathcal{I}$-open.
(b) If $A$ is weakly semi-$\mathcal{I}$-open, then $A \subset cl^\star(int(cl(A)))$. Since $int(cl(A))$ is open, by hypothesis, $int(cl(A)) = cl^\star(int(cl(A)))$ and so $A \subset int(cl(A))$ which implies that $A$ is preopen. □

**Example 2.9.** Consider the ideal space $(X, \tau, \mathcal{I})$ of Example 2.7. If $A = \{b, c\}$, then $A$ is weakly semi-$\mathcal{I}$-open. Also, $int(cl(A)) = int(\{a, b, c\}) = \{a, c\} \not\supset A$. Therefore, $A$ is not preopen.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be *$\alpha$-$\mathcal{I}$-open* [9] if $A \subset int(cl^\star(int(A)))$. Every open set is an $\alpha$-$\mathcal{I}$-open set but not the converse. In Theorem 2.1(2) of [7], it is established that the intersection of an open set and a weakly semi-$\mathcal{I}$-open set is weakly semi-$\mathcal{I}$-open. The following Theorem 2.10 is a generalization of this result. Theorem 2.11 below gives a property of weakly semi-$\mathcal{I}$-open sets.

**Theorem 2.10.** *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A$ is $\alpha$-$\mathcal{I}$-open and $B$ is weakly semi-$\mathcal{I}$-open, then $A \cap B$ is weakly semi-$\mathcal{I}$-open.*

*Proof.* Since $A$ is $\alpha$-$\mathcal{I}$-open, $A \subset int(cl^\star(int(A)))$ and $B$ is weakly semi-$\mathcal{I}$-open, $B \subset cl^\star(int(cl(B)))$. Now $A \cap B \subset int(cl^\star(int(A))) \cap cl^\star(int(cl(B))) \subset cl^\star(\ int(cl^\star(int(A))) \cap int(cl(B))\ ) = cl^\star(int(\ cl^\star(int(A)) \cap int(cl(B))\ )) \subset cl^\star(int(cl^\star(\ int(A) \cap int(cl(B))\ ))) = cl^\star(int(cl^\star(int(\ int(A) \cap cl(B)\ )))) \subset cl^\star(int(cl^\star(int(cl(\ int(A) \cap B\ ))))) \subset cl^\star(int(cl^\star(int(cl(A \cap B))))) = cl^\star(int(cl(A \cap B)))$ by Lemma 1.4, which implies that $A \cap B$ is weakly semi-$\mathcal{I}$-open. □

**Theorem 2.11.** *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A \subset B \subset cl^\star(A)$ and $A$ is weakly semi-$\mathcal{I}$-open, then $B$ is weakly semi-$\mathcal{I}$-open. In particular, if $A$ is weakly semi-$\mathcal{I}$-open, then $cl^\star(A)$ is weakly semi-$\mathcal{I}$-open.*

*Proof.* If $A$ is weakly semi-$\mathcal{I}$-open, then $A \subset cl^\star(int(cl(A)))$. Since $B \subset cl^\star(A) \subset cl^\star(cl^\star(int(cl(A)))) = cl^\star(int(cl(A))) \subset cl^\star(int(cl(B)))$. Therefore, $B$ is weakly semi-$\mathcal{I}$-open. □

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-locally closed* [4] if $A = U \cap V$ where $U$ is open and $V$ is $\star$–perfect or equivalently, $A = U \cap A^\star$ for some open set $U$. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be semi-$\mathcal{I}$-open [9] if $A \subset cl^\star(int(A))$. Every semi-$\mathcal{I}$-open set is weakly semi-$\mathcal{I}$-open [7, Remark 2.1] and but not the converse [7, Example 2.1]. The following Theorem 2.12 deals with the reverse direction. Example 2.13 below shows that the condition semiclosed or $\mathcal{I}$-locally closed on the subset in Theorem 2.12 cannot be dropped.

**Theorem 2.12.** *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$ be a weakly semi-$\mathcal{I}$-open. If $A$ is either semiclosed or $\mathcal{I}$-locally closed, then $A$ is semi-$\mathcal{I}$-open.*

*Proof.* Suppose $A$ is $\mathcal{I}$-locally closed. $A$ is $\mathcal{I}$-locally closed implies that $A = U \cap A^\star$ for some open set $U$. $A$ is weakly semi-$\mathcal{I}$-open implies that $A \subset cl^\star(int(cl(A)))$. Now $A = U \cap A^\star \subset U \cap (cl^\star(int(cl(A))))^\star \subset U \cap cl^\star(cl^\star(int(cl(U \cap A^\star)))) = U \cap cl^\star(int(cl(U \cap A^\star))) \subset cl^\star(U \cap int(cl(U \cap A^\star))) = cl^\star(int(U \cap cl(U \cap A^\star))) \subset cl^\star(int(U \cap cl(U) \cap cl(A^\star))) = cl^\star(int(U \cap A^\star)) = cl^\star(int(A))$. Hence $A$ is semi-$\mathcal{I}$-open.

Suppose $A$ is semiclosed. Then $int(cl(A)) = int(A)$. Since $A$ is weakly semi-$\mathcal{I}$-open, $A \subset cl^\star(int(cl(A))) = cl^\star(int(A))$. Hence $A$ is semi-$\mathcal{I}$-open. □

**Example 2.13.** Consider the ideal space $(X, \tau, \mathcal{I})$ with $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. If A ={a}, then $A^\star = X$ and $cl^\star(int(cl(A))) = cl^\star(int(X)) = X \supset A$ and so $A$ is weakly semi-$\mathcal{I}$-open. Also $cl^\star(int(A)) = cl^\star(\emptyset) = \emptyset$. Hence A is not semi-$\mathcal{I}$-open. Moreover, $A$ is neither $\mathcal{I}$-locally closed nor semiclosed.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be *weakly semi-$\mathcal{I}$-closed* [7] if $X - A$ is weakly semi-$\mathcal{I}$-open or equivalently, $int^\star(cl(int(A))) \subset A$ [7, Theorem 2.2]. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be *$\alpha^\star$-$\mathcal{I}$-set* [9] if $int(cl^\star(int(A))) = int(A)$. The following Theorem 2.14 examines the relation between weakly semi-$\mathcal{I}$-closed set and $\alpha^\star$-$\mathcal{I}$-set. Example 2.15 below shows that an $\alpha^\star$-$\mathcal{I}$-set need not be a weakly semi-$\mathcal{I}$-closed set.

**Theorem 2.14.** *The following hold in any ideal space $(X, \tau, \mathcal{I})$.*
(a) *If $A$ is a weakly semi-$\mathcal{I}$-closed subset of $X$, then $A$ is an $\alpha^\star$-$\mathcal{I}$-set.*
(b) *If $\mathcal{I}$ is codense, then $A$ is weakly semi-$\mathcal{I}$-closed if and only if $int(cl^\star(int(A))) \subset A$.*

*Proof.* (a) If $A$ is weakly semi-$\mathcal{I}$-closed, then by Theorem 2.3 of [7], $int(cl^\star(int(A))) \subset A$ and so $int(cl^\star(int(A))) \subset int(A)$. Hence it follows that $int(cl^\star(int(A))) = int(A)$ which implies that $A$ is an $\alpha^\star$-$\mathcal{I}$-set.
(b) If $A$ is any subset of $X$, then $int^\star(cl(X-A)) = int(cl(X-A))$ by Lemma 1.3(b). Therefore, $cl(int^\star(cl(X-A))) = cl(int(cl(X-A)))$. By Lemma 1.3(a), $cl(int(cl(X-A))) = cl^\star(int(cl(X-A)))$ and so $cl(int^\star(cl(X-A))) = cl^\star(int(cl(X-A)))$ which implies that $X - int(cl^\star(int(A))) = cl^\star(int(cl(X-A)))$. By Corollary 2.1 of [7], (b) follows. □

**Example 2.15.** Let $X = \{a,b,c,d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a,d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. If $A = \{a,b\}$, then $cl^\star(int(A)) = cl^\star(\{a\}) = \{a,b,c\}$ and so $int(cl^\star(int(A))) = int(\{a,b,c\}) = \{a\} = int(A)$. Hence $A$ is an $\alpha^\star$-$\mathcal{I}$-set. Since $int^\star(cl(int(A))) = int^\star(cl(\{a\})) = int^\star(\{a,b,c\}) = \{a,b,c\} \not\subset A$, $A$ is not weakly semi-$\mathcal{I}$-closed.

A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be *weakly $S_\mathcal{I}$-set* (resp. *$C_\mathcal{I}$-set* [9]) if $A = G \cap V$ where $G$ is open and $V$ is weakly semi-$\mathcal{I}$-closed (resp. $\alpha^\star$-$\mathcal{I}$-set). Clearly, every open set is a weakly $S_\mathcal{I}$-set and every weakly $S_\mathcal{I}$-set is a $C_\mathcal{I}$-set. Hence we have the following decomposition of open sets. Example 2.17 shows that the concepts $\alpha$-$\mathcal{I}$-open sets and weakly $S_\mathcal{I}$-sets are independent.

**Theorem 2.16.** *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) *$A$ is open.*
(b) *$A$ is $\alpha$-$\mathcal{I}$-open and a weakly $S_\mathcal{I}$-set.*
(c) *$A$ is $\alpha$-$\mathcal{I}$-open and a $C_\mathcal{I}$-set.*

*Proof.* (a)⇒(b) and (b)⇒(c) are clear. (c)⇒(a) follows from Proposition 3.3 of [9]. □

**Example 2.17.** Consider the ideal space $(X,\tau,\mathcal{I})$ with $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. If $A = \{b\}$, then $int^\star(cl(int(A))) = int^\star(cl(\emptyset)) = \emptyset \subset A$ and so $A$ is weakly semi-$\mathcal{I}$-closed. Hence $A$ is a weakly $S_\mathcal{I}$-set. Also, $int(cl^\star(int(A))) = \emptyset$ and so $A$ is not $\alpha$-$\mathcal{I}$-open. If $B = \{c\}$, then $int^\star(cl(int(B))) = int^\star(cl(\{c\})) = int^\star(\{b,c,d\}) = \{b,c,d\} \not\subset B$ and so $B$ is not a weakly $S_\mathcal{I}$-set. But $int(cl^\star(int(B))) = int(\{b,c,d\}) = \{c\} = B$. Therefore, $B$ is $\alpha$-$\mathcal{I}$-open.

A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be *$\alpha$-$\mathcal{I}$- continuous* [9] (resp. *$C_\mathcal{I}$-continuous* [9], *weakly $S_\mathcal{I}-$continuous*) if for every $V \in \sigma, f^{-1}(V)$ is an $\alpha$-$\mathcal{I}$-open set (resp. $C_\mathcal{I}$-set, weakly $S_\mathcal{I}-$set). The following Theorem 2.18 is a decomposition continuity which follows from Theorem 2.16.

**Theorem 2.18.** *If $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is a function, then the following are equivalent.*

(a) *$f$ is continuous.*
(b) *$f$ is $\alpha$-$\mathcal{I}$-continuous and weakly $S_{\mathcal{I}}-$continuous.*
(c) *$f$ is $\alpha$-$\mathcal{I}$-continuous and $C_{\mathcal{I}}$ continuous.*

## 3. Completely codense ideals

An ideal $\mathcal{I}$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be *completely codense* [5] if $PO(X)\cap\mathcal{I}=\{\emptyset\}$. In Theorem 4.13 of [5], it is established that $\mathcal{I}$ is completely codense if and only if $\mathcal{I}\subset\mathcal{N}$, where $\mathcal{N}$ is the ideal of all nowhere dense sets in $X$. Also, every completely codense ideal is codense but not the converse[5]. In this section, we define weakly semi-$\mathcal{I}$-interior and weakly semi-$\mathcal{I}$-closure of a subset of an ideal space, discuss its properties and characterize completely codense ideals in terms of weakly semi-$\mathcal{I}$-open sets. The *weakly semi-$\mathcal{I}$-interior* of a subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is the largest weakly semi-$\mathcal{I}$-open set contained in $A$ and is denoted by $ws\mathcal{I}int(A)$. By Theorem 2.1(a) of [7], $ws\mathcal{I}int(A)$ is a weakly semi-$\mathcal{I}$-open set and it is clear that $A$ is a weakly semi-$\mathcal{I}$-open set if and only if $A=ws\mathcal{I}int(A)$. The *weakly semi-$\mathcal{I}$-closure* of a subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is the smallest weakly semi-$\mathcal{I}$-closed set containing $A$ and is denoted by $ws\mathcal{I}cl(A)$. It is clear that $ws\mathcal{I}cl(A)$ is a weakly semi-$\mathcal{I}$-closed set and $A$ is a weakly semi-$\mathcal{I}$-closed set if and only if $A=ws\mathcal{I}cl(A)$. The following results are essential to characterize completely codense ideals in terms of weakly semi-$\mathcal{I}$-open sets.

**Theorem 3.1.** *If $(X,\tau,\mathcal{I})$ is an ideal space and $A\subset X$, then the following holds.*
(a) $ws\mathcal{I}int(A)=A\cap cl^\star(int(cl(A)))$.
(b) $ws\mathcal{I}cl(A)=A\cup int^\star(cl(int(A)))$.

*Proof.* (a) If $A$ is any subset of $X$, then $A\cap cl^\star(int(cl(A)))\subset cl^\star(int(cl(A)))$ $=cl^\star(int(\ cl(A)\cap int(cl(A))\ ))\subset cl^\star(int(cl(A\cap int(cl(A))\ )))\subset cl^\star(int(cl(A\cap cl^\star(int(cl(A)))\ )))$ and so $A\cap cl^\star(int(cl(A)))$ is a weakly semi-$\mathcal{I}$-open set contained in $A$. Therefore, $A\cap cl^\star(int(cl(A)))\subset ws\mathcal{I}int(A)$. Since $ws\mathcal{I}int(A)$ is weakly semi-$\mathcal{I}$-open, $ws\mathcal{I}int(A)\subset cl^\star(int(cl(ws\mathcal{I}int(A))))\subset cl^\star(int(cl(A)))$ and so $ws\mathcal{I}int(A)\subset A\cap cl^\star(int(cl(A)))$. Therefore $ws\mathcal{I}int(A)=A\cap cl^\star(int(cl(A)))$.
(b) Now $int^\star(cl(int(\ A\cup int^\star(cl(int(A)))\ )))\subset int^\star(cl(int(A\cup cl(int(A)))))$ $\subset int^\star(cl(\ int(A)\cup cl(int(A))\ ))=int^\star(cl(cl(int(A))))=int^\star(cl(int(A)))\subset A\cup int^\star(cl(int(A)))$. Hence $A\cup int^\star(cl(int(A)))$ is a weakly semi$-\mathcal{I}$-closed set containing $A$ and so $ws\mathcal{I}cl(A)\subset A\cup int^\star(cl(int(A)))$. Since $ws\mathcal{I}cl(A)$ is weakly semi-$\mathcal{I}$-closed, we have $int^\star(cl(int(A)))\subset int^\star(cl(int(ws\mathcal{I}cl(A))))\subset ws\mathcal{I}cl(A)$. Therefore, $A\cup int^\star(cl(int(A)))\subset A\cup ws\mathcal{I}cl(A)=ws\mathcal{I}cl(A)$. Hence $ws\mathcal{I}cl(A)=A\cup int^\star(cl(int(A)))$. □

The following Theorem 3.2 characterizes completely codense ideals. We will denote the family of all weakly semi-$\mathcal{I}$-open sets in any ideal space $(X, \tau, \mathcal{I})$ by $WS\mathcal{I}O(X)$.

**Theorem 3.2.** *If $(X, \tau, \mathcal{I})$ is an ideal space, then the following are equivalent.*
(a) *$\mathcal{I}$ is completely codense.*
(b) $WS\mathcal{I}O(X) \cap \mathcal{I} = \{\emptyset\}$.
(c) $A \subset A^\star$ *for every* $A \in WS\mathcal{I}O(X)$.
(d) $ws\mathcal{I}int(A) \subset ws\mathcal{I}int(A^\star)$ *for every subset $A$ of $X$.*
(e) $ws\mathcal{I}int(A) = \emptyset$ *for every* $A \in \mathcal{I}$.

*Proof.* (a)⇒(b). Suppose $A \in WS\mathcal{I}O(X) \cap I$. $A \in \mathcal{I}$ implies that $A \in \mathcal{N}$ by Lemma 1.3 and so $int(cl(A)) = \emptyset$. Since $A \in WS\mathcal{I}O(X)$, $A \subset cl^\star(int(cl(A))) = cl^\star(\emptyset) = \emptyset$ and so $A = \emptyset$. Therefore, $WS\mathcal{I}O(X) \cap \mathcal{I} = \{\emptyset\}$.
(b)⇒(c). Let $A \in WS\mathcal{I}O(X)$. Suppose that $x \notin A^\star$. Then there exists an open set $G$ containing $x$ such that $G \cap A \in \mathcal{I}$. Since $A \in WS\mathcal{I}O(X)$, by Theorem 2.1(2) of [7], $G \cap A \in WS\mathcal{I}O(X)$ and so by hypothesis, $G \cap A = \emptyset$ which implies that $x \notin A$. Hence $A \subset A^\star$.
(c)⇒(d). For any subset $A$ of X, $ws\mathcal{I}int(A) \in WS\mathcal{I}O(X)$ and so $ws\mathcal{I}int(A) \subset (ws\mathcal{I}int(A))^\star \subset A^\star$. Therefore, $ws\mathcal{I}int(A) \subset ws\mathcal{I}int(A^\star)$.
(d)⇒(e). If $A \in \mathcal{I}$, then $A^\star = \emptyset$ and so by (d), $ws\mathcal{I}int(A) \subset ws\mathcal{I}int(\emptyset) = \emptyset$. Therefore, $ws\mathcal{I}int(A) = \emptyset$.
(e)⇒(a). Suppose $A \in PO(X) \cap \mathcal{I}$. $A \in PO(X)$ implies that $A \subset int(cl(A))$. $A \in \mathcal{I}$ implies that $ws\mathcal{I}int(A) = \emptyset$. By Theorem 3.1(a), $A \cap cl^\star(int(cl(A))) = \emptyset$ which implies that $A \cap int(cl(A)) = \emptyset$. Since $A \subset int(cl(A))$, we have $A = \emptyset$. Therefore, $PO(X) \cap \mathcal{I} = \{\emptyset\}$ which implies that $\mathcal{I}$ is completely codense. □

**Corollary 3.3.** *If $(X, \tau, \mathcal{I})$ is an ideal space and $A \in WS\mathcal{I}O(X)$, then the following holds.*
(a) *$cl(A)$ is regularclosed and* $cl(A) = cl(int(cl(A))) = A^\star(\mathcal{N})$.
(b) *If $\mathcal{I}$ is completely codense, then* $A^\star(\mathcal{I}) = A^\star(\mathcal{N})$.

*Proof.* (a) If $A \in WS\mathcal{I}O(X)$, then $A \subset cl^\star(int(cl(A))) \subset cl(int(cl(A))) \subset cl(A)$ and so, it follows that $cl(A) = cl(int(cl(A)))$. Since $A^\star(\mathcal{N}) = cl(int(cl(A)))$, [17] (a) follows.
(b) If $\mathcal{I}$ is completely codense, by Theorem 3.2(c), $A \subset A^\star$. By Lemma 1.1, $cl(A) = A^\star$. Therefore, the proof follows from (a). □

The following Theorem 3.4 gives another characterization of completely codense ideals where $A\mathcal{I}O(X)$ is the family of all almost $\mathcal{I}$-open sets in $(X, \tau, \mathcal{I})$.

**Theorem 3.4.** *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is completely codense if and only if* $WS\mathcal{I}O(X) = A\mathcal{I}O(X)$.

*Proof.* Suppose $\mathcal{I}$ is completely codense. If $A \in WS\mathcal{I}O(X)$, then $A \subset cl^{\star}(int(cl(A)))$ and by Theorem 3.2(c), $A \subset A^{\star}$. Since every completely codense ideal is codense, by Lemma 1.3, $cl^{\star}(int(cl(A))) = cl(int(cl(A))) \subset cl(int(cl(A^{\star}))) = cl(int(A^{\star}))$, since $A^{\star}$ is closed. Therefore, $A \subset cl(int(A^{\star}))$ which implies that $A \in A\mathcal{I}O(X)$. If $A \in A\mathcal{I}O(X)$, then $A \subset cl(int(A^{\star}))$ and so $A \subset cl(int(cl(A))) = cl^{\star}(int(cl(A)))$ which implies that $A \in WS\mathcal{I}O(X)$. Conversely, suppose $WS\mathcal{I}O(X) = A\mathcal{I}O(X)$. If $A \in WS\mathcal{I}O(X)$, then $A \in A\mathcal{I}O(X)$ and so $A \subset cl(int(A^{\star})) \subset cl(A^{\star}) = A^{\star}$. By Theorem 3.2, $\mathcal{I}$ is completely codense. □

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