ON TRIANGLES WITH FIBONacci AND LUCAS NUMBERS AS COORDINATES

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ABSTRACT. We consider triangles in the plane with coordinates of points from the Fibonacci and Lucas sequences.

The Fibonacci and Lucas sequences $F_n$ and $L_n$ are defined by the recurrence relations

\[ F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 3, \]

and

\[ L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 3. \]

Let $k$ be a positive integer. Let $\Delta_k$ and $\Gamma_k$ denote the triangles with vertices $A_k = (F_k, F_{k+1})$, $B_k = (F_{k+1}, F_{k+2})$, $C_k = (F_{k+2}, F_{k+3})$ and $P_k = (L_k, L_{k+1})$, $Q_k = (L_{k+1}, L_{k+2})$, $R_k = (L_{k+2}, L_{k+3})$, respectively.

Our goal in this paper is to explore some common properties of the triangles $\Delta_k$ and $\Gamma_k$. We begin with the following theorem which shows that these triangles share the property of orthology.

Recall that the triangles $ABC$ and $XYZ$ are orthologic when the perpendiculars at vertices of $ABC$ onto the corresponding sides of $XYZ$ are concurrent. The point of concurrence is $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of $XYZ$ onto the corresponding sides of $ABC$ are concurrent at the point $[XYZ, ABC]$.

By replacing in the above definition perpendiculars with parallels we get the analogous notion of paralogic triangles and of two points $\langle ABC, XYZ \rangle$ and $\langle XYZ, ABC \rangle$.

The triangle $ABC$ is paralogic to its first Brocard triangle $A_bB_bC_b$ which has the orthogonal projections of the symmedian point $K$ onto the perpendicular bisectors of sides as vertices (see [4] and [5]).

Theorem 1. For all positive integers $m$ and $n$, the following are pairs of orthologic triangles: $(\Delta_m, \Delta_n)$, $(\Delta_m, \Gamma_n)$, and $(\Gamma_m, \Gamma_n)$.

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Proof. It is well-known (see [1]) that the triangles $ABC$ and $XYZ$ with coordinates of points $(a_1, a_2), (b_1, b_2), (c_1, c_2), (x_1, x_2), (y_1, y_2),$ and $(z_1, z_2)$ are orthologic if and only if

$$a_1(y_1 - z_1) + b_1(z_1 - x_1) + c_1(x_1 - y_1) + a_2(y_2 - z_2) + b_2(z_2 - x_2) + c_2(x_2 - y_2) = 0.$$ 

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By the Binet formula $F_k$ is equal to $\frac{\alpha^k - \beta^k}{\alpha - \beta}$ and $L_k$ is equal to $\alpha^k + \beta^k$ (see [3] and [6]). When we substitute the coordinates of the vertices of $\Delta_m$ and $\Gamma_n$ into the left hand side of the above criterion we get $(1 - \alpha)(\beta - 1)(\alpha + 1)(\alpha^m \beta^n + \alpha^n \beta^m)$. For the pairs $\Delta_m, \Delta_n$ and $\Gamma_m, \Gamma_n$ we get $\frac{(1-\alpha)(\beta-1)(\alpha+1)(\alpha^m \beta^n + \alpha^n \beta^m)}{\beta - \alpha}$ and $(\beta - 1)(\alpha + 1)\frac{(\alpha^m \beta^n - \alpha^n \beta^m)}{\beta - \alpha}$. From this the conclusion of the theorem is obvious because $\alpha + 1 = 0$. \hfill \Box

**Theorem 2.** For all positive integers $m$ in the orthocenters $H(\Delta_m)$ and $H(\Gamma_m)$ of the triangles $\Delta_m$ and $\Gamma_m$ and the orthology centers $[\Delta_m, \Gamma_m]$ and $[\Gamma_m, \Delta_m]$ satisfy

$$\frac{|H(\Delta_m)|}{|\Delta_m, \Gamma_m|} = \frac{\sqrt{5}}{5}.$$ 

Proof. Let us use $\theta_b^a$ as a short notation for the expression $a + b \sqrt{5}$. Let $A = \alpha^m$ and $B = \beta^m$. 

Using the Binet formula for Fibonacci and Lucas numbers it is easy to check that $H(\Delta_m)$ has the coordinates $\frac{\theta_1^3 (\theta_3^2 A^3 + \theta_4^2 A^2 B - \theta_2^2 A B^2 + 2 B^3)}{20 A B}$ and $\frac{\theta_1^1 (\theta_3^1 A^3 - \theta_4^1 A^2 B - \theta_2^1 A B^2 - 2 B^3)}{20 A B}$. The coordinates of the orthocenter $H(\Gamma_m)$ are $\frac{\theta_3^5 (\theta_4^2 A^3 - \theta_2^2 A B^2 + 2 B^3)}{20 A B}$ and $\frac{\theta_1^1 (\theta_3^1 A^3 + \theta_4^1 A^2 B + \theta_2^1 A B^2 + 2 B^3)}{20 A B}$. 

The same method for $[\Delta_m, \Gamma_m]$ gives $\frac{\theta_1^3 (\theta_3^2 A^3 + \theta_4^2 A^2 B + \theta_2^2 A B^2 + 2 B^3)}{20 A B}$ and $\frac{\theta_1^1 (\theta_3^1 A^3 - \theta_4^1 A^2 B - \theta_2^1 A B^2 - 2 B^3)}{20 A B}$. Finally, the second orthology center $[\Gamma_m, \Delta_m]$ has $\frac{\theta_3^5 (\theta_4^2 A^3 - \theta_2^2 A B^2 + 2 B^3)}{20 A B}$ and $\frac{\theta_1^1 (\theta_3^1 A^3 - \theta_4^1 A^2 B + \theta_2^1 A B^2 + 2 B^3)}{20 A B}$ as coordinates. The square of the distance between $H(\Gamma_m)$ and $[\Gamma_m, \Delta_m]$ is

$$\frac{\theta_3^{10} - 2 \theta_2^8 A^4 - \theta_4^8 A^2 B^2 + 2 B^4 \theta_1^8 A^2 + 2 B^4}{5 A^2 B^2}$$

while the square of the distance between the points $H(\Delta_m)$ and $[\Delta_m, \Gamma_m]$ is exactly one fifth of this value. \hfill \Box

**Theorem 3.** For all positive integers $m$ in the oriented areas $|\Delta_m|$ and $|\Gamma_m|$ of the triangles $\Delta_m$ and $\Gamma_m$ are as follows:

$$|\Delta_m| = \frac{(-1)^m}{2}, \quad \text{and} \quad |\Gamma_m| = 5 |\Delta_{m+1}| = \frac{5 (-1)^{m+1}}{2}.$$
Proof. Let us again assume that $\alpha^m = A$ and $B = \beta^m$. Note that $\alpha \beta = -1$ so that $AB = (-1)^m$. Recall that the triangle with the vertices whose coordinates are $(x_1, x_2)$, $(y_1, y_2)$, and $(z_1, z_2)$ has the oriented area equal to $\frac{1}{2} (z_1-y_1)x_2 + (x_1-z_1)y_2 + (y_1-x_1)z_2$. By direct substitution and simplification we get that $|\Delta_m| = \frac{AB}{2} = (-1)^m \frac{m}{2}$. On the other hand, for $\Gamma_m$ we get $|\Gamma_m| = -\frac{5AB}{2} = 5 |\Delta_{m+1}| = \frac{5(-1)^{m+1}}{2}$.

At this point we can go back and keep coordinates of vertices according to their original definition and discover that the first claim in the above theorem is equivalent to the identity

$$F_{m+1} (F_{m+4} - F_m) = F_{m+1}^2 + F_{m+2}^2 + (-1)^m,$$

while the second claim in the above theorem is equivalent to the identity

$$L_{m+1} (L_{m+4} - L_m) = L_{m+1}^2 + L_{m+2}^2 + 5 (-1)^{m+1}.$$

**Theorem 4.** For all positive integers $m$ the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ of the triangles $\Delta_m$ and $\Gamma_m$ are at the distance $\frac{1}{2} \sqrt{F_{2m+3}}$.

Proof. With the notation from the proof of Theorem 2 we get that the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ have as coordinates $\left(\frac{\theta_{\Delta}^m (2B - \theta_A^m A)}{30}, \frac{\theta_{\Gamma}^m (B + \theta_A^m A)}{15}\right)$ and $\left(\frac{\theta_{\Delta}^m (B + \theta_A^m A)}{6}, \frac{\theta_{\Gamma}^m (B - \theta_A^m A)}{3}\right)$. The square of their distance is $\frac{\theta_{\Delta}^{m+2} (\theta_A^m A^2 + B^2)}{45}$ which in turn is precisely $\frac{15}{2} F_{2m+3}$.

**Theorem 5.** For all positive integers $m$ the de Longchamps points $L(\Delta_m)$ and $L(\Gamma_m)$ of the triangles $\Delta_m$ and $\Gamma_m$ are at the distance $4 F_{2m+2} \sqrt{F_{2m+1}}$.

Proof. With the notation from the proof of Theorem 2 we get that the de Longchamps points $L(\Delta_m)$ and $L(\Gamma_m)$ have

$$\left(\frac{\theta_{\Delta}^m (B - A)}{20 AB} \left(\frac{\theta_A^m A^2 - 4B^2}{14 A B}\right), \frac{\theta_{\Gamma}^m (\theta_A^m A + 2B)}{40 A B} \left(\frac{\theta_A^m A^2 - 4B^2}{14 A B}\right)\right)$$

and

$$\left(\frac{\theta_{\Delta}^{m-5} (B + A)}{20 AB} \left(\frac{\theta_A^{m-5} A^2 - 4B^2}{14 A B}\right), \frac{\theta_{\Gamma}^{m-5} (\theta_A^{m-5} A - 2B)}{40 A B} \left(\frac{\theta_A^{m-5} A^2 - 4B^2}{14 A B}\right)\right)$$

as coordinates. The square of their distance is

$$\frac{\theta_{\Delta}^{m-352} A^6 + \theta_{\Delta}^{m-64} A^2 B^2 (A^2 - B^2) + \theta_{\Delta}^{m-352} B^6}{100}$$

which is equal to $16 F_{2m+2}^2 F_{2m+1}$.

In an analogous fashion one can show also the following.
Theorem 6. For all positive integers $m$ the de Longchamps point $L(\Delta_m)$ of the triangle $\Delta_m$ and the centroid $G(\Gamma_m)$ of the triangle $\Gamma_m$ are at the distance $\frac{2}{3}\sqrt{F_{2m+1}(9F_{2m+1}^2+1)}$.

Theorem 7. For every positive integer $m$, the triangles $\Gamma_m$ and $\Delta_m$ are reversely similar and the sides of $\Gamma_m$ are $\sqrt{5}$ times longer than the corresponding sides of $\Delta_m$.

Proof. It is well-known that two triangles are reversely similar if and only if they are ortologic and paralogic (see [2]). Since, by Theorem 1, we know that triangles $\Gamma_m$ and $\Delta_m$ are orthologic, it remains to see that they are paralogic.

Recall that triangles $ABC$ and $XYZ$ with coordinates of points $(a_1, a_2)$, $(b_1, b_2)$, $(c_1, c_2)$, $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2)$ are paralogic if and only if the expression $U + V$ is equal to zero where

$$U = (z_2 - y_2)a_1 + (x_2 - z_2)b_1 + (y_2 - x_2)c_1$$

and $V = (y_1 - z_1)a_2 + (z_1 - x_1)b_2 + (x_1 - y_1)c_2$. In our situation when we represent coordinates of vertices of triangles $\Delta_m$ and $\Gamma_m$ by Binet formula in terms of $\alpha$ and $\beta$ by substitution and easy simplification we get that $U + V = 0$ so that these triangles are indeed paralogic. In a similar way one can easily show that $|P_mQ_m|^2 = 5$. \hfill \Box

Theorem 8. For every positive integer $m$, the triangles $\Gamma_m$ and $\Delta_m$ are both orthologic and paralogic. The centers $[\Delta_m, \Gamma_m]$ and $[\Delta_m, \Gamma_m]$ are antipodal points on the circumcircle of $\Delta_m$. The centers $[\Gamma_m, \Delta_m]$ and $[\Gamma_m, \Delta_m]$ are antipodal points on the circumcircle of $\Gamma_m$.

Proof. The first claim has been established in the previous theorem. In order to prove the second claim we shall prove that the orthology center $[\Delta_m, \Gamma_m]$ lies on the circumcircle of $\Delta_m$ by showing that it has the same distance from its circumcenter $O(\Delta_m)$ as the vertex $A_m$ and that the reflection of the point $[\Delta_m, \Gamma_m]$ in the circumcenter $O(\Delta_m)$ agrees with the point $[\Delta_m, \Gamma_m]$ (because their distance is equal to zero).

The point $O(\Delta_m)$ has coordinates $\frac{\theta_1^{-3}(\theta_2^2 A^3 - \theta_2^2 A^3 B - \theta_2^2 A B^2 + 2 B^3)}{40 A B}$ and $\frac{\theta_1^{-1}(\theta_2^4 A^3 + \theta_2^4 A^3 B - \theta_2^4 A B^2 - 2 B^3)}{40 A B}$.

The coordinates of the center $[\Delta_m, \Gamma_m]$ are $\frac{\theta_1^3 (\theta_2^4 A + 2 B)}{40 A B}$ and $\frac{2 \theta_1^{-2} (\theta_2^4 A - B)}{50 A B}$ while $\frac{\theta_1^{-3}(\theta_2^2 A^3 - \theta_2^2 A^3 B + \theta_2^2 A B^2 + 2 B^3)}{20 A B}$ and $\frac{\theta_1^{-1}(\theta_2^4 A^3 + \theta_2^4 A^3 B - \theta_2^4 A B^2 - 2 B^3)}{20 A B}$ are coordinates of the center $[\Delta_m, \Gamma_m]$. Now it is easy to establish that $||[\Delta_m, \Gamma_m]O(\Delta_m)||^2 - |O(\Delta_m) A_m|^2 = 0$. On the other hand, if $W$ denotes the reflection of the point $[\Delta_m, \Gamma_m]$ in the circumcenter $O(\Delta_m)$ (i. e., $W$ divides the segment $[\Delta_m, \Gamma_m]O(\Delta_m)$ with the ratio $-2$), then $|W[\Delta_m, \Gamma_m]|^2 = 0$. 

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The third claim has a similar proof. □

**Theorem 9.** The square of the diameter of the circumcircle of the triangle $\Delta_m$ is equal to $F_{2m+3} F_{2m+1} F_{2m-1}$.

**Proof.** In the proof of the previous theorem we found the coordinates of the circumcenter $O(\Delta_m)$. Hence, the square of its distance from the vertex $A_m$ is

$$\frac{\theta_5^2 (\theta_4^3 A^2 + B^2) (\theta_4^3 A^2 + 2 B^2) (\theta_4^3 A^2 + 2 B^2)}{400}.$$  

However, this expression is in fact $\frac{F_{2m+3} F_{2m+1} F_{2m-1}}{4}$. □

In a similar way one can show the following result.

**Theorem 10.** The cotangent of the Brocard angle of the triangle $\Delta_m$ is equal to $\frac{(-1)^m}{2F_{2m+1}}$.

**References**


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