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ON TRIANGLES WITH FIBONACCI AND LUCAS NUMBERS AS COORDINATES

ZVONKO ČERIN

ABSTRACT. We consider triangles in the plane with coordinates of points from the Fibonacci and Lucas sequences.

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \text{ for } n \geqslant 3,$$

and

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \text{ for } n \geqslant 3.$$

Let k be a positive integer. Let Δ_k and Γ_k denote the triangles with vertices $A_k = (F_k, F_{k+1})$, $B_k = (F_{k+1}, F_{k+2})$, $C_k = (F_{k+2}, F_{k+3})$ and $P_k = (L_k, L_{k+1})$, $Q_k = (L_{k+1}, L_{k+2})$, $R_k = (L_{k+2}, L_{k+3})$, respectively.

Our goal in this paper is to explore some common properties of the triangles Δ_k and Γ_k. We begin with the following theorem which shows that these triangles share the property of orthology.

Recall that the triangles ABC and XYZ are *orthologic* when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. The point of concurrence is $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto the corresponding sides of ABC are concurrent at the point $[XYZ, ABC]$.

By replacing in the above definition perpendiculars with parallels we get the analogous notion of *paralogic* triangles and of two points $\langle ABC, XYZ\rangle$ and $\langle XYZ, ABC\rangle$.

The triangle ABC is paralogic to its first Brocard triangle $A_bB_bC_b$ which has the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides as vertices (see [4] and [5]).

Theorem 1. *For all positive integers m and n, the following are pairs of orthologic triangles: (Δ_m, Δ_n), (Δ_m, Γ_n), and (Γ_m, Γ_n).*

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Proof. It is well-known (see [1]) that the triangles ABC and XYZ with coordinates of points $(a_1,\, a_2)$, $(b_1,\, b_2)$, $(c_1,\, c_2)$, $(x_1,\, x_2)$, $(y_1,\, y_2)$, and $(z_1,\, z_2)$ are orthologic if and only if

$$a_1(y_1 - z_1) + b_1(z_1 - x_1) + c_1(x_1 - y_1) + \\ a_2(y_2 - z_2) + b_2(z_2 - x_2) + c_2(x_2 - y_2) = 0.$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By the Binet formula F_k is equal to $\frac{\alpha^k - \beta^k}{\alpha - \beta}$ and L_k is equal to $\alpha^k + \beta^k$ (see [3] and [6]). When we substitute the coordinates of the vertices of Δ_m and Γ_n into the left hand side of the above criterion we get $(1-\alpha)(\beta-1)(\alpha\,\beta+1)(\alpha^m\,\beta^n + \alpha^n\,\beta^m)$. For the pairs Δ_m, Δ_n and Γ_m, Γ_n we get $\frac{(1-\alpha)(\beta-1)(\alpha\,\beta+1)(\alpha^m\,\beta^n+\alpha^n\,\beta^m)}{\beta-\alpha}$ and $(\beta-\alpha)(1-\alpha)(\beta-1)(\alpha\,\beta+1)$ $(\alpha^m\,\beta^n - \alpha^n\,\beta^m)$. From this the conclusion of the theorem is obvious because $\alpha\,\beta + 1 = 0$. □

Theorem 2. *For all positive integers m the orthocenters $H(\Delta_m)$ and $H(\Gamma_m)$ of the triangles Δ_m and Γ_m and the orthology centers $[\Delta_m,\, \Gamma_m]$ and $[\Gamma_m,\, \Delta_m]$ satisfy*

$$\frac{|H(\Delta_m)[\Delta_m,\, \Gamma_m]|}{|H(\Gamma_m)[\Gamma_m,\, \Delta_m]|} = \frac{\sqrt{5}}{5}.$$

Proof. Let us use θ_b^a as a short notation for the expression $a + b\sqrt{5}$. Let $A = \alpha^m$ and $B = \beta^m$.

Using the Binet formula for Fibonacci and Lucas numbers it is easy to check that $H(\Delta_m)$ has the coordinates $\frac{\theta_{-1}^3\left(\theta_3^7\,A^3+\theta_4^8\,A^2\,B-\theta_2^2\,A\,B^2+2\,B^3\right)}{20\,A\,B}$ and $\frac{\theta_{-1}^1\left(\theta_1^3\,A^3-\theta_4^8\,A^2\,B-\theta_{-2}^2\,A\,B^2-2\,B^3\right)}{20\,A\,B}$. The coordinates of the orthocenter $H(\Gamma_m)$ are $\frac{\theta_{-3}^5\left(\theta_3^7\,A^3-\theta_4^8\,A^2\,B-\theta_2^2\,A\,B^2-2\,B^3\right)}{20\,A\,B}$ and $\frac{\theta_{-1}^1\left(\theta_1^3\,A^3+\theta_4^8\,A^2\,B+\theta_{-2}^2\,A\,B^2+2\,B^3\right)}{20\,A\,B}$.

The same method for $[\Delta_m,\, \Gamma_m]$ gives $\frac{\theta_1^{-3}\left(\theta_3^7\,A^3-\theta_4^8\,A^2\,B+\theta_2^2\,A\,B^2+2\,B^3\right)}{20\,A\,B}$ and $\frac{\theta_1^{-1}\left(\theta_1^3\,A^3+\theta_4^8\,A^2\,B-\theta_{-2}^2 AB^2-2\,B^3\right)}{20\,A\,B}$. Finally, the second orthology center $[\Gamma_m,\, \Delta_m]$ has $\frac{\theta_3^{-5}\left(\theta_3^7\,A^3+\theta_4^8\,A^2\,B+\theta_2^2\,A\,B^2-2\,B^3\right)}{20\,A\,B}$ and $\frac{\theta_1^{-5}\left(\theta_1^3\,A^3-\theta_4^8\,A^2\,B-\theta_{-2}^2\,A\,B^2+2\,B^3\right)}{20\,A\,B}$ as coordinates. The square of the distance between $H(\Gamma_m)$ and $[\Gamma_m,\, \Delta_m]$ is

$$\frac{\theta_{-2}^5\left(\theta_3^7\,A^4 - \theta_1^3\,A^2\,B^2 + 2\,B^4\right)\left(\theta_1^3\,A^2 + 2\,B^2\right)}{5\,A^2\,B^2}$$

while the square of the distance between the points $H(\Delta_m)$ and $[\Delta_m,\, \Gamma_m]$ is exactly one fifth of this value. □

Theorem 3. *For all positive integers m the oriented areas $|\Delta_m|$ and $|\Gamma_m|$ of the triangles Δ_m and Γ_m are as follows:*

$$|\Delta_m| = \frac{(-1)^m}{2} \quad \textit{and} \quad |\Gamma_m| = 5\,|\Delta_{m+1}| = \frac{5\,(-1)^{m+1}}{2}.$$

Proof. Let us again assume that $\alpha^m = A$ and $B = \beta^m$. Note that $\alpha\,\beta = -1$ so that $A\,B = (-1)^m$. Recall that the triangle with the vertices whose coordinates are $(x_1,\,x_2)$, $(y_1,\,y_2)$, and $(z_1,\,z_2)$ has the oriented area equal to $\frac{(z_1-y_1)x_2+(x_1-z_1)y_2+(y_1-x_1)z_2}{2}$. By direct substitution and simplification we get that $|\Delta_m| = \frac{AB}{2} = \frac{(-1)^m}{2}$. On the other hand, for Γ_m we get $|\Gamma_m| = -\frac{5\,A\,B}{2} = 5\,|\Delta_{m+1}| = \frac{5\,(-1)^{m+1}}{2}$. □

At this point we can go back and keep coordinates of vertices according to their original definition and discover that the first claim in the above theorem is equivalent to the identity

$$F_{m+1}\,(F_{m+4} - F_m) = F_{m+1}^2 + F_{m+2}^2 + (-1)^m,$$

while the second claim in the above theorem is equivalent to the identity

$$L_{m+1}\,(L_{m+4} - L_m) = L_{m+1}^2 + L_{m+2}^2 + 5\,(-1)^{m+1}.$$

Theorem 4. *For all positive integers m the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ of the triangles Δ_m and Γ_m are at the distance $\frac{4}{3}\sqrt{F_{2m+3}}$.*

Proof. With the notation from the proof of Theorem 2 we get that the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ have as coordinates $\left(\frac{\theta_{-3}^{5}(2\,B-\theta_3^7\,A)}{30}, \frac{\theta_{-4}^{10}(B+\theta_4^9\,A)}{15}\right)$ and $\left(\frac{\theta_{-1}^{3}(B+\theta_3^7\,A)}{6}, \frac{\theta_{-2}^{4}(B-\theta_4^9\,A)}{3}\right)$. The square of their distance is $\frac{\theta_{-32}^{80}(\theta_4^9\,A^2+B^2)}{45}$ which in turn is precisely $\frac{16}{9}F_{2m+3}$. □

Theorem 5. *For all positive integers m the de Longchamps points $L(\Delta_m)$ and $L(\Gamma_m)$ of the triangles Δ_m and Γ_m are at the distance $4\,F_{2m+2}\sqrt{F_{2m+1}}$.*

Proof. With the notation from the proof of Theorem 2 we get that the de Longchamps points $L(\Delta_m)$ and $L(\Gamma_m)$ have

$$\left(\frac{\theta_{-1}^{3}\,(B-A)\left(\theta_6^{14}\,A^2-4\,B^2\right)}{20\,A\,B}, \frac{\theta_{-1}^{1}\left(\theta_{-1}^{3}\,A+2\,B\right)\left(\theta_6^{14}\,A^2-4\,B^2\right)}{40\,A\,B}\right)$$

and

$$\left(\frac{\theta_{3}^{-5}\,(B+A)\left(\theta_6^{14}\,A^2-4\,B^2\right)}{20\,A\,B}, \frac{\theta_{1}^{-5}\left(\theta_{-1}^{3}\,A-2\,B\right)\left(\theta_6^{14}\,A^2-4\,B^2\right)}{40\,A\,B}\right)$$

as coordinates. The square of their distance is

$$\frac{\theta_{352}^{800}\,A^6 + \theta_{64}^{0}\,A^2\,B^2\,(A^2-B^2) + \theta_{-352}^{800}\,B^6}{100}$$

which is equal to $16\,F_{2m+2}^2\,F_{2m+1}$. □

In an analogous fashion one can show also the following.

Theorem 6. *For all positive integers m the de Longchamps point $L(\Delta_m)$ of the triangle Δ_m and the centroid $G(\Gamma_m)$ of the triangle Γ_m are at the distance $\frac{2}{3}\sqrt{F_{2m+1}(9\,F_{2m+1}^2+1)}$.*

Theorem 7. *For every positive integer m, the triangles Γ_m and Δ_m are reversely similar and the sides of Γ_m are $\sqrt{5}$ times longer than the corresponding sides of Δ_m.*

Proof. It is well-known that two triangles are reversely similar if and only if they are ortologic and paralogic (see [2]). Since, by Theorem 1, we know that triangles Γ_m and Δ_m are orthologic, it remains to see that they are paralogic.

Recall that triangles ABC and XYZ with coordinates of points $(a_1,\, a_2)$, $(b_1,\, b_2)$, $(c_1,\, c_2)$, $(x_1,\, x_2)$, $(y_1,\, y_2)$ and $(z_1,\, z_2)$ are paralogic if and only if the expression $U+V$ is equal to zero where

$$U = (z_2 - y_2)\, a_1 + (x_2 - z_2)\, b_1 + (y_2 - x_2)\, c_1$$

and $V = (y_1 - z_1)\, a_2 + (z_1 - x_1)\, b_2 + (x_1 - y_1)\, c_2$. In our situation when we represent coordinates of vertices of triangles Δ_m and Γ_m by Binet formula in terms of α and β by substitution and easy simplification we get that $U+V=0$ so that these triangles are indeed paralogic. In a similar way one can easily show that $\frac{|P_m Q_m|^2}{|A_m B_m|^2} = 5$. □

Theorem 8. *For every positive integer m, the triangles Γ_m and Δ_m are both orthologic and paralogic. The centers $[\Delta_m,\, \Gamma_m]$ and $\langle\Delta_m,\, \Gamma_m\rangle$ are antipodal points on the circumcircle of Δ_m. The centers $[\Gamma_m,\, \Delta_m]$ and $\langle\Gamma_m,\, \Delta_m\rangle$ are antipodal points on the circumcircle of Γ_m.*

Proof. The first claim has been established in the previous theorem. In order to prove the second claim we shall prove that the orthology center $[\Delta_m,\, \Gamma_m]$ lies on the circumcircle of Δ_m by showing that it has the same distance from its circumcenter $O(\Delta_m)$ as the vertex A_m and that the reflection of the point $\langle\Delta_m,\, \Gamma_m\rangle$ in the circumcenter $O(\Delta_m)$ agrees with the point $[\Delta_m,\, \Gamma_m]$ (because their distance is equal to zero!).

The point $O(\Delta_m)$ has coordinates $\frac{\theta_1^{-3}\left(\theta_3^7\, A^3 - \theta_{10}^{22}\, A^2\, B - \theta_{-2}^2\, A\, B^2 + 2\, B^3\right)}{40\, A\, B}$ and $\frac{\theta_1^{-1}\left(\theta_1^3\, A^3 + \theta_{10}^{22}\, A^2\, B - \theta_{-4}^8\, A\, B^2 - 2\, B^3\right)}{40\, A\, B}$. The coordinates of the center $[\Delta_m,\, \Gamma_m]$ are $\frac{\theta_{-1}^3\left(\theta_3^7\, A + 2\, B\right)}{10}$ and $\frac{2\,\theta_1^{-2}\left(\theta_4^9\, A - B\right)}{5}$ while $\frac{\theta_1^{-3}\left(\theta_3^7\, A^3 - \theta_4^8\, A^2\, B + \theta_2^2\, A\, B^2 + 2\, B^3\right)}{20\, A\, B}$ and $\frac{\theta_1^{-1}\left(\theta_1^3\, A^3 + \theta_4^8\, A^2\, B - \theta_{-2}^2\, A\, B^2 - 2\, B^3\right)}{20\, A\, B}$ are coordinates of the center $\langle\Delta_m,\, \Gamma_m\rangle$. Now it is easy to establish that $|[\Delta_m,\, \Gamma_m] O(\Delta_m)|^2 - |O(\Delta_m)\, A_m|^2 = 0$. On the other hand, if W denotes the reflection of the point $\langle\Delta_m,\, \Gamma_m\rangle$ in the circumcenter $O(\Delta_m)$ (i. e., W divides the segment $\langle\Delta_m,\, \Gamma_m\rangle O(\Delta_m)$ with the ratio -2), then $|W[\Delta_m,\, \Gamma_m]|^2 = 0$.

The third claim has a similar proof. □

Theorem 9. *The square of the diameter of the circumcircle of the triangle* Δ_m *is equal to* $F_{2m+3}\,F_{2m+1}\,F_{2m-1}$.

Proof. In the proof of the previous theorem we found the coordinates of the circumcenter $O(\Delta_m)$. Hence, the square of its distance from the vertex A_m is

$$\frac{\theta_{-2}^{5}\left(\theta_4^9\,A^2+B^2\right)\left(\theta_1^3\,A^2+2\,B^2\right)\left(\theta_{-1}^3\,A^2+2\,B^2\right)}{400}.$$

However, this expression is in fact $\frac{F_{2m+3}\,F_{2m+1}\,F_{2m-1}}{4}$. □

In a similar way one can show the following result.

Theorem 10. *The cotangent of the Brocard angle of the triangle* Δ_m *is equal to* $\frac{(-1)^m}{2\,F_{2m+1}}$.

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Kopernikova 7
10010 Zagreb
Croatia
E–mail: cerin@math.hr