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ON k-WEAKLY PRIMARY IDEALS OVER SEMIRINGS

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ABSTRACT. Since ideals in rings and semirings are closely related, two-sided k-ideals occur frequently in semiring theory. Let R be a commutative semiring. For an ideal of R, the notion of k-weakly primary ideals is defined. It is shown that this notion inherits most of the essential properties of the weakly primary ideals of a commutative ring (see [1], [4]). For example, it is proved that a k-weakly primary ideal A of R, that is not primary, satisfies $A^2 = 0$ and $\mathrm{rad}(A) = \mathrm{rad}(0)$. Also, it is shown that an intersection of a family of k-weakly primary ideals, that are not primary, is k-weakly primary.

1. INTRODUCTION

This paper is concerned with generalizing some results of ring theory to semiring theory. There are many different definitions of a semiring appearing in the literature. Throughout this paper, a semiring will be defined as follows:

A set R together with two associative binary operations called addition and multiplication (denoted by $+$ and ., respectively) will be called a semiring provided: (i) addition is a commutative operation and that the multiplication is distributive with respect to the addition both from the left and from the right; (ii) there exists $0 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ for each $r \in R$. A semiring R is commutative if $(R, .)$ is a commutative semigroup.

Convention. In this paper all semirings considered will be assumed to be commutative.

A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k–ideal) K is an ideal such that if $x, x + y \in K$ then $y \in K$ (so $\{0\}$ is a k-ideal of R). A prime ideal of R is a proper ideal P of R for which $x \in P$ or $y \in P$ whenever $xy \in P$. Therefore P is prime if and only if A and B are ideals in R such

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that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ where $AB = \{ab : a \in A \text{ and } b \in B\}$ (see [3, theorem 5]). A primary ideal of P of R is a proper ideal of R such that, if $xy \in P$ and $x \notin P$, then $y^n \in P$ for some positive integer n. If I is an ideal of R, then the radical of I, denoted by rad(I), is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R, that contains I, and if $1 \in R$ then it is the intersection of all the prime ideals of R that contain I (see [2]).

Let R be a semiring. We define a proper ideal A of R to be weakly primary (resp. weakly prime) if $0 \neq ab \in A$ implies $a \in A$ or $b^m \in A$ for some positive integer m (resp. $a \in A$ or $b \in A$). So a primary ideal (resp. prime ideal) is a weakly primary (resp. weakly prime). However, since 0 is always weakly primary (resp. weakly prime) by definition, a weakly primary ideal (a weakly prime ideal) need not be primary (resp. prime). Clearly, every weakly prime is weakly primary (see [1] and [4]).

We shortly summarize the content of the paper. In Theorem 2.6, it is shown that if A is a k–weakly primary ideal (resp. k–weakly prime) of a semiring R that is not primary (resp. is not prime) then $A^2 = 0$. In Theorem 2.8, three other characterizations of k–weakly prime ideals of a semiring with identity are given. Finally, in Theorem 2.13, it is proved that if A and B are k–weakly prime ideals of a semiring that are not prime, then $AB = 0$.

2. Weakly prime ideals

Our starting point is the following lemma:

Lemma 2.1. *Let I, J be ideals of a semiring R with I a k–ideal, and let $x \in R$. Then the following hold:*

(i) $(I : J) = \{r \in R : rJ \subseteq I\}$ *is a k–ideal of R where* $rJ = \{rc : c \in J\}$.

(ii) $(0 : x) = \{r \in R : rx = 0\}$ *and* $(I : x) = \{r \in R : rx \in I\}$ *are k–ideals of R.*

Proof. (i) Clearly, $(I : J)$ is an ideal of R. Let $a \in (I : J)$, $b \in R$ and $a + b \in (I : J$. It suffices to show that $bc \in I$ for every $c \in J$. By assumption, $ac + bc, ac \in I$, so $bc \in I$ since I is a k–ideal of R, as required.

(ii) This follows from (i). □

Lemma 2.2. *Let R be a semiring. If an ideal of R is the union of two k–ideals, then it is equal to one of them.*

Proof. The proof is completely straightforward. □

Proposition 2.3. *For a proper k–ideal A of a semiring R, the following statements are equivalent.*

(i) *A is a weakly prime ideal of R.*

(ii) *For $x \in R - A$, $(A : x) = A \cup (0 : x)$.*

(iii) *For* $x \in R - A$, $(A : x) = A$ *or* $(A : x) = (0 : x)$.

Proof. $(i) \to (ii)$ Since $A \cup (0 : x) \subseteq (A : x)$ is trivial, we will only prove the reverse inclusion. Let $y \in (A : x)$ where $x \in R - A$. Then $xy \in A$. If $xy \neq 0$, then $y \in A$ since A is a weakly prime. If $xy = 0$, then $y \in (0 : x)$. So $(A : x) \subseteq A \cup (0 : x)$, hence we have equality. $(ii) \to (iii)$ follows from Lemma 2.1 and Lemma 2.2. $(iii) \to (i)$ is clear. □

Proposition 2.4. *For a proper k–ideal A of a semiring R, the following statements are equivalent.*

(i) *A is a weakly primary ideal of R.*
(ii) *For* $x \in R - \mathrm{rad}(A)$, $(A : x) = A \cup (0 : x)$.
(iii) *For* $x \in R - \mathrm{rad}(A)$, $(A : x) = A$ *or* $(A : x) = (0 : x)$.

Proof. $(i) \to (ii)$ Let $x \in R - \mathrm{rad}(A)$. Clearly, $A \cup (0 : x) \subseteq (A : x)$. For the other inclusion, suppose that $y \in (A : x)$, so $xy \in A$. If $xy \neq 0$, then A weakly primary gives $y \in A$. If $xy = 0$, then $y \in (0 : x)$, so we have equality. $(ii) \to (iii)$ follows from Lemma 2.1 and Lemma 2.2. $(iii) \to (i)$ is clear. □

Lemma 2.5. *Let A be a k–primary ideal of a semiring R. If $a \in A$ and $a + b \in \mathrm{rad}(A)$, then $b \in \mathrm{rad}(A)$.*

Proof. By assumption, there exists a positive integer m such that $(a+b)^m = c+b^m \in A$ where $c \in A$, so $b^m \in A$ since A is a k-ideal; hence $b \in \mathrm{rad}(A)$. □

Theorem 2.6.

(i) *Let A be a k-weakly primary ideal of a semiring R. If A is not primary, then $A^2 = \{ab : a, b \in A\} = 0$.*
(ii) *Let A be a k-weakly prime ideal of a semiring R. If A is not prime, then $A^2 = 0$.*

Proof. (i) Suppose that $A^2 \neq 0$; we show that A is a primary ideal of R. Let $xy \in A$ where $x, y \in R$. If $xy \neq 0$, then A weakly primary gives $x \in A$ or $y^m \in A$ for some m. So assume that $xy = 0$. If $0 \neq xA \subseteq A$, then there is a non-zero element d of A such that $xd \neq 0$; hence $0 \neq xd = x(d + y) \in A$. Then either $x \in A$ or $y \in \mathrm{rad}(A)$ by Lemma 2.5. So we can assume that $xA = 0$. Likewise, we can assume that $yA = 0$. Since $A^2 \neq 0$, there are elements $e, f \in A$ such that $ef \neq 0$. Then $0 \neq ef = (x + e)(y + f) \in A$, so either $x \in A$ or $y \in \mathrm{rad}(A)$ by Lemma 2.5. Thus A is a primary ideal. (ii) follows from (i). □

Theorem 2.7. *Let A be a k–weakly prime ideal of a semiring R. Then for ideals I and J of R with $0 \neq IJ \subseteq A$, either $I \subseteq A$ or $J \subseteq A$.*

Proof. Assume that A is a weakly prime ideal of R and let I and J be ideals of R with $IJ \subseteq A$, but $I \nsubseteq A$ and $J \nsubseteq A$. We show that $IJ = 0$ which is a contradiction. By [3, Theorem 5], A is a k-weakly prime ideal that is not prime, so $A^2 = 0$ by Theorem 2.6. Let $ab \in IJ$ where $a \in I$ and $b \in J$. First, suppose that $a \in I - A$. Now $aJ \subseteq A$, so $J \subseteq (A : a)$. Since $J \nsubseteq A$, by Proposition 2.3 ($(i) \to (iii)$), $aJ = 0$; hence $ab = 0$. Next suppose that $a \in A \cap I$. If $b \in A$, then $ab \in A^2 = 0$. If $b \in J - A$, then as previously noted, $bJ = 0$, and hence $ab = 0$. So $IJ = 0$. □

Theorem 2.8. *For a proper k–ideal A of a semiring R with identity, the following statements are equivalent.*

(i) *A is a weakly prime ideal of R.*
(ii) *For $x \in R - A$, $(A : x) = A \cup (0 : x)$.*
(iii) *For $x \in R - A$, $(A : x) = A$ or $(A : x) = (0 : x)$.*
(iv) *For ideals I and J with $0 \neq IJ \subseteq A$, either $I \subseteq A$ or $J \subseteq A$.*

Proof. By Proposition 2.3 and Theorem 2.7, it suffices to show that $(iv) \to (i)$. Suppose that $0 \neq xy \in A$. Then $0 \neq (xR)(yR) \subseteq A$, so $xR \subseteq A$ or $yR \subseteq A$; hence $x \in A$ or $y \in A$ since $1 \in R$, as required. □

Theorem 2.9. *Let R be a semiring. Then the following hold:*

(i) *Let A be a k–weakly primary ideal that is not primary. then* $\text{rad}(A) = \text{rad}(0)$.
(ii) *Let A be a k–weakly prime ideal that is not prime. then* $\text{rad}(A) = \text{rad}(0)$.

Proof. Clearly, $\text{rad}(0) \subseteq \text{rad}(A)$. By Theorem 2.6, $A^2 = 0$ gives $A \subseteq \text{rad}(0)$; hence $\text{rad}(A) \subseteq \text{rad}(0)$ by [3, Corollary 2.4], so we have the equality. (ii) follows from (i). □

Theorem 2.10. *Let R be a semiring, and let $\{A_i\}_{i \in I}$ be a family of k–weakly primary ideals of R that are not primary. Then $A = \bigcap_{i \in I} A_i$ is a weakly primary ideal of R.*

Proof. By Theorem 2.9, it is easy to check that $\text{rad}(A) = \text{rad}(0) \neq R$, so A is a proper ideal of R. Suppose that $a, b \in R$ are such that $0 \neq ab \in A$ but $b \notin A$. Then there exists $s \in I$ such that $b \notin A_s$ and $0 \neq ab \in A_s$. Then A_s weakly primary gives $a \in \text{rad}(A_s) = \text{rad}(0) = \text{rad}(A)$, as required. □

Theorem 2.11. *Let R be a semiring, and let $\{A_i\}_{i \in I}$ be a family of k-weakly prime ideals of R that are not prime. Then $A = \bigcap_{i \in I} A_i$ is a weakly prime ideal of R.*

Proof. By Theorem 2.9, it is easy to check that $\text{rad}(A) = \text{rad}(0) \neq R$, so A is a proper ideal of R. Suppose that $a, b \in R$ are such that $0 \neq ab \in A$ but $b \notin A$. Then there exists $s \in I$ such that $b \notin A_s$ and $0 \neq ab \in A_s$. Then A_s weakly prime gives $a \in A_s \subseteq \text{rad}(A_s) = \text{rad}(0)$, so $a^n = 0$ for some n. It follows that $a^n \in A_i$ for every $i \in I$; hence $a \in A$ since A_i is a weakly prime ideal for every $i \in N$. Thus A is a weakly prime ideal of R. □

Theorem 2.12. *Let R be a semiring, and let A be a k–weakly prime ideal that is not prime. then $A\text{rad}(0) = 0$.*

Proof. Let $ab \in A\text{rad}(0)$ where $a \in A$ and $b \in \text{rad}(0)$. If $b \in A$, then $ab \in A^2 = 0$ by Theorem 2.6. So suppose that $b \notin A$. By Proposition 2.3, either $(A : b) = A$ or $(A : b) = (0 : b)$. As $ab \in A \subseteq (A : b)$, the second case gives $ab = 0$. So suppose that $(A : b) = A$. Assume that $b^m = 0$, but $b^{m-1} \neq 0$. Then $0 \neq b^{m-1} \in (A : b) = A$, so $b \in A$ which is a contradiction. Thus $A\text{rad}(0) = 0$. □

Theorem 2.13. *Let R be a semiring, and let A and B be k-weakly prime ideals that are not prime. Then $AB = 0$.*

Proof. Let $ab \in AB$ where $a \in A$ and $b \in B$. By Theorem 2.6, $B \subseteq \text{rad}(0)$, so $ab \in A\text{rad}(0) = 0$ by Theorem 2.12, as required. □

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