ON SPLIT EXACT SEQUENCES AND PROJECTIVE SEMIMODULES

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Abstract. In this paper the notion of split exact sequences of semimodules is introduced. We also study some results on projective semimodules that are analogous to module theory.

1. Introduction and preliminaries

A semiring is a commutative monoid \((S, +)\) having additive identity zero \(0_S\) and a semigroup \((S, \cdot)\) which are connected by ring like distributivity. Let \(S\) be a semiring. A left \(S\)-semimodule \(M\) is a commutative monoid \((M, +)\) which has a zero element 0\(_M\), together with an operation \(S \times M \rightarrow M\); defined by \((a, x) \mapsto ax\) such that for all \(a, b \in S\) and \(x, y \in M\),

(i) \(a(x + y) = ax + ay\),
(ii) \((a + b)x = ax + bx\),
(iii) \((ab)x = a(bx)\),
(iv) \(0_Sx = 0_M = a0_M\).

A right \(S\)-semimodule is defined in an analogous manner. A non empty subset \(A\) of \(S\)-semimodule \(M\) is a subsemimodule of \(M\) if \(A\) is closed under addition and scalar multiplication. Let \(M\) and \(N\) be left \(S\)-semimodules. A homomorphism from \(M\) to \(N\) is a map \(f : M \rightarrow N\) such that,

(i) \(f(m_1 + m_2) = f(m_1) + f(m_2)\),
(ii) \(f(am) = af(m)\), for all \(m, m_1, m_2 \in M\) and for all \(a \in S\).

Definition 1.1 ([6]). Let \(A\) and \(B\) be \(S\)-semimodules and \(f : A \rightarrow B\) be \(S\)-semimodule homomorphism. Define

\[ K_f = \{(a, b) \in A \times A \mid f(a) + x = f(b) + x \text{ for some } x \in B\} \]
\[ I_f = \{(c, d) \in B \times B \mid c + f(a) = d + f(b) \text{ for some } a, b \in A\} \]
\[ \bar{I}_f = \{(c, d) \in B \times B \mid c + f(a) + x = d + f(b) + x \text{ for some } a, b \in A, \text{ some } x \in B\}. \]

2000 Mathematics Subject Classification. 16Y60.
Let $A$ be a monic if $K_f = \bar{\Delta}_A$, where $\Delta_A = \{(a, a) | a \in A\}$ and $\bar{\Delta}_A = \{(a, b) \in A \times A | a + x = b + x$ for some $x \in A\}$

and $f$ is said to be an epic if for any $b \in B$ there exist some $a_i \in A$, $i = 1, 2$ and $x \in B$ such that $b + f(a_1) + x = f(a_2) + x$.

**Definition 1.2** ([6]). Let $A, B$ be $S$-semimodules. Then an $S$-semimodules homomorphism $f : A \rightarrow B$ is said to be a $Z$-homomorphism if for each $a \in A$ there exists $x \in B$ such that $f(a) + x = x$.

**Definition 1.3** ([6]). A sequence of $S$-semimodules and $S$-semimodule homomorphism is a diagram of the form,

$$\ldots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_{i+1}} M_{i+1} \rightarrow \ldots$$

Such a sequence is said to be exact if $I_{f_{i-1}} = K_{f_i}$ for all $i$.

**Definition 1.4.** Let $A$ and $B$ be $S$-semimodules and $f : A \rightarrow B$ be $S$-semimodules homomorphism. Then $f$ is said to be $i$-regular if for each $b \in B$ there exist $a_1, a_2 \in A$ such that $b + f(a_1) = f(a_2)$ and $f$ is said to be $k$-regular if $f(a_1) + x = f(a_2) + x$ where $a_1, a_2 \in A$ and $x \in B$ implies $f(a_1) = f(a_2)$.

**Result 1.5** ([6]). Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be $S$-semimodule homomorphisms. Then $gf : A \rightarrow C$ is a $Z$-homomorphism if and only if $I_f \subseteq K_g$.

**Definition 1.6** (Projective Semimodule). A $S$-semimodule $P$ is called projective if it satisfies the following two properties.

(i) If a $S$-homomorphism $f : A \rightarrow B$ is an epic and $g : P \rightarrow B$ is $S$-semimodule homomorphism then there exists a $S$-homomorphism $\phi : P \rightarrow A$ such that $f \circ \phi = g$.

(ii) To every $k$-regular $S$-homomorphism $f : A \rightarrow B$ and to every $S$-homomorphisms $\psi_1, \psi_2 : P \rightarrow A$ with $f \circ \psi_1 = f \circ \psi_2$ there exist $S$-homomorphisms $k_1, k_2 : P \rightarrow A$ such that $f \circ k_1$ and $f \circ k_2$ are $Z$-homomorphisms and $\psi_1 + k_1 = \psi_2 + k_2$.

**Definition 1.7** ([7]). Let $\{M_i\}_{i \in I}$ be a family of $S$-semimodules. The cartesian product $\prod_{i \in I} M_i$ forms a $S$-semimodule under usual operations called the direct product of $\{M_i\}$.

In the direct product $\prod_{i} M_i$, the set of all elements whose components $x_i$ are equal to 0 except for a finite number of $i$ is denoted by $\bigoplus_{i \in I} M_i$ and is called the external direct sum of $\{M_i\}$. Then $\bigoplus_{i \in I} M_i$ is a subsemimodule of $\prod_{i} M_i$.

**Definition 1.8** ([8]). Let $M$ be a $S$-semimodule. Then $M$ is called cancellative if whenever $m + x = m' + x$ for $m, m', x \in M$, we have $m = m'$. 
2. Split exact sequences

In this section, we define split exact sequences of S-semimodules and prove some theorems that are analogous to module theory.

**Definition 2.1.** An exact sequence of S-semimodules of the form
\[ \ldots M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} M_0 \longrightarrow 0 \]
is said to split if there exist S-semimodule homomorphisms \( g_i : M_i \longrightarrow M_{i+1} \) such that
\begin{enumerate}
  \item \( f_1 \circ g_0 = I_{M_0} \)
  \item \( g_{i-1}f_i + f_{i+1}g_i = I_{M_i} \) for all \( i \geq 1 \)
\end{enumerate}
where addition of S-semimodule homomorphisms is defined in the usual manner.

**Theorem 2.2.** If the sequences of S-semimodules
\[ 0 \xrightarrow{u} M' \xrightarrow{q} M \xrightarrow{p} M'' \xrightarrow{v} 0 \]
are such that \( qu = I_{M''}, pv = I_{M''}, uq + vp = I_M \) then each of the above sequences are split exact.

**Proof.** We need only to show that the given sequences are exact. To show \( u \) is a monic let \( (a, b) \in K_u \). Then \( u(a) + m = u(b) + m \) for some \( m \in M \) implies \( qu(a) + q(m) = qu(b) + q(m) \) or, \( a + q(m) = b + q(m) \) implies \( (a, b) \in \Delta_{M''} \).

So, \( K_u \subseteq \Delta_{M''} \). Hence \( u \) is a monic.

To show \( K_p = I_u \), let \( (m_1, m_2) \in K_p \). Then \( p(m_1) + m'' = p(m_2) + m'' \) for some \( m'' \in M'' \) implies \( vp(m_1) + v(m'') = vp(m_2) + v(m'') \) and \( vp(m_1) + uq(m_1) + v(m'') = vp(m_2) + uq(m_1) + v(m'') \) or, \( m_1 + u(q(m_2)) + v(m'') = m_2 + u(q(m_1)) + v(m'') \) as \( vp + uq = I_M \) implies \( (m_1, m_2) \in I_u \). So, \( K_p \subseteq I_u \).

Again, let \( (m_1, m_2) \in I_u \). Then
\[ m_1 + u(m_1') + m = m_2 + u(m_2') + m \]
for some \( m_1', m_2' \in M' \), and \( m \in M \) implies
\[ p(m_1) + pu(m_1') + p(m) = p(m_2) + pu(m_2') + p(m). \]

Again, from (1) we have \( q(m_1) + qu(m_1') + q(m) = q(m_2) + qu(m_2') + q(m) \) or, \( m_1' + q(m) = m_2' + q(m) \) (as \( qu = I_{M''} \) which implies \( uq(m_1) + u(m_1') + uq(m) = uq(m_2) + u(m_2') + uq(m) \).

Adding \( vp(m_1) + vp(m_2) \) on both sides, we get \( m_1 + vp(m_2) + m_2 + vp(m_1) + uq(m) = m_2 + vp(m_1) + uq(m) + u(m_2') \) which implies \( p(m_1) + p(m_2) + \)
pu(m'\textsuperscript{1}) + puq(m) = p(m_2) + p(m_1) + puq(m) + pu(m'\textsuperscript{2}) \text{ or implies } p(m_2) + p(m_1) + pu(m'\textsuperscript{1}) + p(m) + puq(m) = p(m_2) + p(m_1) + puq(m) + pu(m'\textsuperscript{2}) + p(m).

Using (2), we get

\[ p(m_2) + p(m_1) + pu(m'\textsuperscript{1}) + p(m) + puq(m) = p(m_2) + p(m_1) + pu(m'\textsuperscript{2}) + puq(m) + p(m) \]

which implies \((m_1, m_2) \in K_p\). So, \(\bar{I}_u \subseteq K_p\).

Hence \(K_p = \bar{I}_u\).

Finally, \(p\) is an epic, because \(pv(m'') = m''\) implies

\[ m'' + p(0) + 0 = pv(m'') + 0. \]

Hence the given sequence is split exact. Similarly we can show that the other sequence is also split exact. □

Theorem 2.3. Consider a commutative diagram of \(S\)-semimodules

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & A' \\
\gamma & & \\
0 & \xrightarrow{\alpha} & M' \xrightarrow{\beta} A' & \xrightarrow{\beta'} & A'' & 0 \\
\theta & & \\
M & \xrightarrow{\delta} & M' & \xrightarrow{\delta'} & A'' & 0 \\
\phi & & \\
M'' & & \\
0 & & \\
\end{array}
\]

and suppose that all columns are exact and the row is split exact (i.e. there exists a \(S\)-homomorphism \(\beta' : A'' \rightarrow M'\) such that \(\beta' = I_{A''}\) and a \(S\)-homomorphism \(\alpha' : M' \rightarrow A'\) such that \(\alpha' + \beta' \beta = I_{M'}\)), then \(\bar{I}_{\eta} = K_{\phi\delta\delta'}\).

Proof. Let \((x, y) \in K_{\phi\delta\delta'}\). Then

\[ \phi\delta\delta'(x) + m'' = \phi\delta\delta'(y) + m'' \text{ for some } m'' \in M'' \]
implies \((\delta\delta'(x), \delta\delta'(y)) \in K_{\phi} = \bar{I}_{\theta}\) (by exactness).

Therefore \(\delta\delta'(x) + \theta(q_1) + m = \delta\delta'(y) + \theta(q_2) + m\) for some \(q_1, q_2 \in A'\) and \(m \in M\) or, \(\delta\delta'(x) + \delta\alpha(q_1) + m = \delta\delta'(y) + \delta\alpha(q_2) + m\) (as \(\theta = \delta\alpha\)) implies \((\beta'(x) + \alpha(q_1), \beta'(y) + \alpha(q_2)) \in K_{\delta} = \bar{I}_{\gamma}\) (by exactness).
Therefore $\beta(x) + \alpha(q_1) + \gamma(a_1) + r = \beta'(y) + \alpha(q_2) + \gamma(a_2) + r$ for some $a_1, a_2 \in A$ and $r \in M'$ implies $\beta\beta'(x) + \beta\alpha(q_1) + \beta\gamma(a_1) + \beta(r) = \beta\beta'(y) + \beta\alpha(q_2) + \beta\gamma(a_2) + \beta(r)$.

Since $\beta\alpha$ is a $Z$-homomorphism we have $\beta\alpha(q_1) + a''_1 = a'_1$ and $\beta\alpha(q_2) + a''_2 = a'_2$ for some $a''_1, a''_2 \in A''$.

Since $\beta\beta' = I$ and $\beta\gamma = \eta$ we obtain

$$x + \eta(a_1) + a''_1 + a''_2 + \beta(r) = y + \eta(a_2) + a''_1 + a''_2 + \beta(r)$$

which implies $(x, y) \in I_\eta$. Therefore $K_{\phi\delta\beta'} \subseteq I_\eta$.

Again, since the given row splits we have $\alpha\alpha' + \beta\beta' = I_{M'}$.

Let $a \in A$. Then $\gamma(a) \in M'$. Therefore

$$\alpha\alpha'\gamma(a) + \beta\beta'\gamma(a) = \gamma(a)$$

which implies $\phi\delta\alpha'\gamma(a) + \phi\delta\beta'\eta(a) = \phi\delta\gamma(a)$ (as $\beta\gamma = \eta$) or,

$$\phi\theta\alpha'\gamma(a) + \phi\delta\beta'\eta(a) = \phi\delta\gamma(a)$$ (as $\delta\alpha = \theta$). (3)

Since $\delta\gamma$ and $\phi\theta$ are $Z$-homomorphisms, we get $\phi\theta(\alpha'\gamma(a)) + u = u$ for some $u \in M''$ and $\delta\gamma(\alpha) + m = m$ for some $m \in M$, which implies $\phi\delta\gamma(a) + \phi(m) = \phi(m)$.

From (3), $\phi\delta\beta'\eta(a) + u + \phi(m) = u + \phi(m)$ which implies $\phi\delta\beta'\eta$ is a $Z$-homomorphism, therefore by Result 1.5, $I_\eta \subseteq K_{\phi\delta\beta'}$. Hence $K_{\phi\delta\beta'} = I_\eta$. □

3. Projective semimodule

In this section we study some results on projective semimodule which are analogous to module theory.

**Definition 3.1.** Let $A$ and $B$ be $S$-semimodules and $f : A \rightarrow B$ be an $S$-homomorphism. Define

$$\tilde{f} = \{b \in B | b + f(a_1) + x = f(a_2) + x \text{ for some } a_1, a_2 \in A, x \in B\}$$

Clearly, $\tilde{f}$ is a subsemimodule of $B$.

**Theorem 3.2.** Let $P$ be projective $S$-semimodule. If $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of $S$-semimodules, then for every $S$-homomorphism $\phi : P \rightarrow B$ such that $g \circ \phi$ is a $Z$-homomorphism there exists a $S$-homomorphism $\phi' : P \rightarrow A$ with $f \circ \phi' = \phi$.
Proof. Let \( b \in \bar{J}_\phi \). Then
\[
\phi(p_1) + u = \phi(p_2) + u,
\]
for some \( p_1, p_2 \in P \) and \( u \in B \). Since \( g \circ \phi \) is a \( Z \)-homomorphism, for some \( c_1, c_2 \in C \).

From (4), \( g(b) + c_1 = c_2 \) implies \( b \in \bar{J}_f \).

Hence, \( \bar{J}_\phi \subseteq \bar{J}_f \).

Consider the diagram
\[
\begin{array}{ccc}
P & \xrightarrow{\phi} & A \\
\downarrow{\phi'} & & \downarrow{f} \\
A & \rightarrow & \bar{J}_f & \rightarrow & 0
\end{array}
\]

Clearly \( f \) is an epic. Since \( P \) is projective there exists an \( S \)-homomorphism \( \phi' : P \rightarrow A \) such that \( f \circ \phi' = \phi \).

Theorem 3.3. Consider the diagram
\[
\begin{array}{ccc}
P & \xrightarrow{g} & B \\
\downarrow{f} & & \downarrow{A} \\
B & \rightarrow & A
\end{array}
\]
of \( S \)-semimodules and \( S \)-homomorphisms. If \( P \) is projective \( S \)-semimodule then the following are equivalent:

\( \text{(i)} \) there is an \( S \)-homomorphism \( h : P \rightarrow B \) such that \( f \circ h = g \),

\( \text{(ii)} \) \( \bar{J}_g \subseteq \bar{J}_f \).

Proof. (i)\( \Rightarrow \) (ii). Let \( y \in \bar{J}_g \). Then
\[
y + g(p_1) + b = g(p_2) + b,
\]
for some \( p_1, p_2 \in P \) and \( b \in A \).

which implies \( y + f \circ h(p_1) + b = f \circ h(p_2) + b \) (by (i)), hence \( y \in \bar{J}_f \).

Therefore \( \bar{J}_g \subseteq \bar{J}_f \).

Conversely, let \( \bar{J}_g \subseteq \bar{J}_f \). Consider
Since $P$ is projective, there exists an $S$-homomorphism $h : P \rightarrow B$ such that $f \circ h = g$. \hfill \Box

**Theorem 3.4.** Suppose that

\[ P \xrightarrow{f} L \xrightarrow{g} M \]

\[ A \xrightarrow{h} B \xrightarrow{k} C \]

is a commutative diagram of $S$-semimodules and $S$-homomorphisms, that $P$ is projective, $g \circ f$ is a $Z$-homomorphism and the lower row is exact then there exists an $S$-homomorphism $P \rightarrow A$ which makes the diagram commutative.

**Proof.** Let $p \in P$. Since $g \circ f$ is a $Z$-homomorphism then there exists an $m \in M$ such that $g \circ f(p) + m = m$ which implies $\gamma \circ (g \circ f(p)) + \gamma(m) = \gamma(m)$ or, $k \circ \beta(f(p)) + \gamma(m) = \gamma(m)$, hence $(\beta f(p), 0) \in K_k = J_h$ (as lower row is exact) which implies $\beta f(p) + h(a_1) + x = h(a_2) + x$ for some $a_1, a_2 \in A$ and $x \in B$. So, $\beta f(p) \in J_h$, for all $p \in P$.

Define $\theta : P \rightarrow J_h$ such that $\theta(p) = \beta f(p)$. Clearly, $\theta$ is a $S$-homomorphism. Let $a \in A$. Then $h(a) \in J_h$ therefore $h : A \rightarrow J_h$. By the definition of $J_h$, $h : A \rightarrow J_h$ is an epic. Since $P$ is projective there exists an $S$-homomorphism $\alpha : P \rightarrow A$ such that $h\alpha = \theta = \beta f$. Hence the diagram commutes. \hfill \Box

**Theorem 3.5.** Consider the diagram of $S$-semimodules and $S$-homomorphisms

\[ \begin{array}{cccccc}
K_1 & \xrightarrow{\alpha_1} & P_1 & \xrightarrow{\beta_1} & B_1 \\
\downarrow{\gamma_1} & & \downarrow{\gamma_2} & & \downarrow{\gamma_3} \\
K_2 & \xrightarrow{\alpha_2} & P_2 & \xrightarrow{\beta_2} & B_2 & \rightarrow 0
\end{array} \]
in which the rows are exact, \( P_1 \) is projective and \( \alpha_1 \) is \( i \)-regular. Then there are \( S \)-homomorphisms \( \gamma_2 : P_1 \to P_2 \) and \( \gamma_1 : K_1 \to K_2 \) such that the completed diagram is commutative.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\beta_1} & P_2 \\
\gamma_2 & & \beta_2 \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{\gamma_3} & B_2 \\
\end{array}
\]

Since \( \beta_2 \) is an epic therefore by the projectivity of \( P_1 \) there exists an \( S \)-homomorphism \( \gamma_2 : P_1 \to P_2 \) such that \( \beta_2 \gamma_2 = \gamma_3 \beta_1 \), i.e. the diagram commutes.

Again, let \( y \in \bar{J}_{\gamma_2} \). Then

\[ y + \gamma_2(p_1) + p_2 = \gamma_2(p_1') + p_2 \quad \text{for some} \quad p_1, p_1' \in P_1 \quad \text{and} \quad p_2 \in P_2 \]

implies \( \beta_2(y) + \beta_2 \gamma_2(p_1) + \beta_2(p_2) = \beta_2 \gamma_2(p_1') + \beta_2(p_2) \) or,

\[ \beta_2(y) + \gamma_3 \beta_1(p_1) + \beta_2(p_2) = \gamma_3 \beta_1(p_1') + \beta_2(p_2) \] (5)

Since \( \alpha_1 \) is \( i \)-regular, we have

\[ p_1 + \alpha_1(k_1) = \alpha_1(k_2) \text{and} \alpha_1(k_3) = \alpha_1(k_4), \quad \text{for some} \quad k_1, k_2, k_3, k_4 \in K_1 \]

which implies \( p_1 + \alpha_1(k_1 + k_4) = p_1' + \alpha_1(k_2 + k_3) \) or,

\[ \gamma_3 \beta_1(p_1) + \gamma_3 \beta_1 \alpha_1(k_5) = \gamma_3 \beta_1(p_1') + \gamma_3 \beta_1 \alpha_1(k_6) \] (6)

where \( k_5 = k_1 + k_4 \) and \( k_6 = k_2 + k_3 \).

Adding \( \gamma_1 \beta_1 \alpha_1(k_5) \) on both sides of (5) and using (6), we get

\[ \beta_2(y) + \gamma_3 \beta_1(p_1') + \gamma_3 \beta_1 \alpha_1(k_6) + \beta_2(p_2) = \gamma_3 \beta_1(p_1') + \beta_2(p_2) + \gamma_3 \beta_1 \alpha_1(k_5). \]

Since \( \beta_1 \alpha_1 \) is a \( Z \)-homomorphism we have

\[ \beta_1 \alpha_1(k_5) + u_1 = u_1, \quad u_1 \in B_1 \quad \text{implies} \quad \gamma_3 \beta_1 \alpha_1(k_5) + \gamma_3(u_1) = \gamma_3(u_1) \]

\[ \beta_1 \alpha_1(k_6) + u_2 = u_2, \quad u_2 \in B_1 \quad \text{implies} \quad \gamma_3 \beta_1 \alpha_1(k_6) + \gamma_3(u_2) = \gamma_3(u_2) \]

So, \( \beta_2(y) + \gamma_3 \beta_1(p_1') + \gamma_3(u_1 + u_2) + \beta_2(p_2) = \gamma_3 \beta_1(p_1') + \gamma_3(u_1 + u_2) + \beta_2(p_2) \) which implies \( (y, 0) \in K_{\beta_2} = \bar{I}_{\alpha_2} \).

Therefore \( y + \alpha_2(k') + x = \alpha_2(k'') + x \), for some \( k', k'' \in K_2 \) and \( x \in P_2 \). This implies \( y \in \bar{J}_{\alpha_2} \). Therefore \( \bar{J}_{\gamma_2} \subseteq \bar{J}_{\alpha_2} \).

Consider the following diagram
By the projectivity of $P_1$ there exists an $S$-homomorphism $\theta: P_1 \rightarrow K_2$ such that $\gamma_2 = \alpha_2 \theta$.

Define a map $\gamma_1 : K_1 \rightarrow K_2$ such that $\gamma_1(k_1) = \theta \alpha_1(k_1)$, $k_1 \in K_1$.

Clearly $\gamma_1$ is an $S$-homomorphism.

Now $\alpha_2 \gamma_1(k_1) = \alpha_2 \theta \alpha_1(k_1) = \gamma_2 \alpha_1(k_1)$ or, $\alpha_2 \gamma_1 = \gamma_2 \alpha_1$. Hence the completed diagram is commutative.

**Proposition 3.6.** Suppose $\{P_i : i \in I\}$ is a family of projective $S$-semimodules. Then their direct sum $P = \bigoplus P_i$ is also projective.

**Proof.** Let $f : A \rightarrow B$ be an epic $S$-homomorphism of $S$-semimodules. Let $g : P \rightarrow B$ be a $S$-homomorphism. Let $\pi_i : P \rightarrow P_i$ be the canonical projection and $q_i : P_i \rightarrow P$ be the canonical injection. Define $g_i : P_i \rightarrow B$ such that $g_i = g q_i$ for each $i \in I$. Since $P_i$ is projective, there exists an $S$-homomorphism $h_i : P_i \rightarrow A$ such that $f h_i = g_i$ for each $i \in I$.

Define $h : P \rightarrow A$ by $h = \sum h_i \pi_i$. Then

$$fh = f \left( \sum h_i \pi_i \right) = \sum_i f h_i \pi_i = \sum_i g_i \pi_i = \sum_i g q_i \pi_i = g \sum_i q_i \pi_i = g.$$

So, $P$ satisfies the property (i) of projective $S$-semimodule.

Let $f : A \rightarrow B$ be $k$-regular $S$-homomorphism. Let $\psi_1, \psi_2 : P \rightarrow A$ be $S$-homomorphisms with $f \circ \psi_1 = f \circ \psi_2$.

Define $\psi_i, \psi_i' : P_i \rightarrow A$ by $\psi_i = \psi_1 \circ q_i$ and $\psi_i' = \psi_2 \circ q_i$ for all $i \in I$.

Then

$$f \circ \psi_i = f \circ \psi_1 \circ q_i = f \circ \psi_2 \circ q_i = f \circ \psi_i'$$

$$\psi_1 = \psi_1 \circ \left( \sum_i q_i \pi_i \right) = \sum_i \psi_1 \circ q_i \circ \pi_i = \sum_i \psi_i \pi_i.$$

Similarly, $\psi_2 = \sum_i \psi_i' \circ \pi_i$.

Since each $P_i$ is projective, there exists an $S$-homomorphism $k_i, k_i' : P_i \rightarrow A$ for each $i \in I$, such that $f \circ k_i$ and $f \circ k_i'$ are $Z$-homomorphisms. Also $\psi_i + k_i = \psi_i' + k_i'$. 
Since $f$ is $k$-regular, $f \circ k_i$ and $f \circ k'_i$ are $Z$-homomorphisms implies
\[ f \circ k_i = 0 = f \circ k'_i \text{ for all } i \in I. \]

Define $k_1 : P \to A$ by $k_1 = \sum_i k_i \circ \pi_i$ and $k_2 : P \to A$ by $k_2 = \sum_i k'_i \circ \pi_i$.
Then $f \circ k_1 = 0 = f \circ k_2$ implies $f \circ k_1$ and $f \circ k_2$ are $Z$-homomorphisms and
\[ \psi_1 + k_1 = \psi_2 + k_2. \]

So, $P$ satisfies the property (ii) of projective $S$-semimodule. □

**Theorem 3.7.** Every diagram of $S$-semimodules and $S$-homomorphism of the form

\[
\begin{array}{ccccccccc}
0 & \rightarrow & P_1 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\
& & \downarrow{\beta_1} & \downarrow{f} & \downarrow{g} & \downarrow{} & & \downarrow{} & & \downarrow{} \\
& & 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

in which the row and the columns are exact, $P_1$ and $P_3$ are projective, $g$ is $k$-regular can be extended to a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & P_1 & \rightarrow & P_2 & \rightarrow & P_3 & \rightarrow & 0 \\
& & \downarrow{\beta_1} & \downarrow{i} & \downarrow{\beta_2} & \downarrow{\pi} & \downarrow{\beta_3} & & \downarrow{} & & \downarrow{} \\
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\
& & & & & & & & & & \\
& & & & & & & & & & \\
\end{array}
\]

in which the top row and the second column are exact and $P_2$ is also projective.

**Proof.** Let $P_2 = P_1 \oplus P_3$. By Proposition 3.6, $P_2$ is also projective. Since $g$ is an epic and $\beta_3 : P_3 \to C$ is an $S$-homomorphism then by projectivity of $P_3$, there exists an $S$-homomorphism $\lambda : P_3 \to B$ such that $\beta_3 = g\lambda$. 
Consider the diagram

\[
\begin{array}{ccccccc}
0 & \to & P_1 & \xrightarrow{i} & P_2 & \xrightarrow{\pi} & P_3 & \to & 0 \\
\downarrow{\beta_1} & & \downarrow{\beta_2} & & \downarrow{\beta_3} & & & \\
0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0
\end{array}
\]

where \( \pi : P_2 \to P_3 \) is given by \( \pi(p_2) = \pi(p_1, p_3) = p_3 \) is an \( S \)-homomorphism and \( i : P_1 \to P_2 \) such that \( i(p_1) = (p_1, 0) \) is an \( S \)-homomorphism.

Define a map \( \beta_2 : P_2 \to B \) such that

\[
\beta_2(p_2) = f\beta_1(p_1) + \lambda(p_3)
\]

where \( p_1 \in P_1 \) and \( p_3 \in P_3 \).

Now,

\[
g\beta_2(p_2) = gf\beta_1(p_1) + g\lambda(p_3)
\]

\[
= g\lambda(p_3) \quad \text{(as \( g \) is \( k \)-regular and \( g \circ f \) is a \( Z \)-homomorphism)}
\]

\[
= \beta_3(p_3) = \beta_3\pi(p_2).
\]

Therefore, \( g\beta_2 = \beta_3\pi \).

Now, \( \beta_2 i(p_1) = i(p_1, 0) = f\beta_1(p_1) + \lambda(0) = f\beta_1(p_1) \). Therefore, \( \beta_2 i = f\beta_1 \). Hence the diagram commutes.

To show \( \beta_2 \) is an epic let \( b \in B \). Then \( g(b) \in C \). Since \( \beta_3 \) is an epic there exist \( p_3, p_3' \in P_3 \) such that \( g(b) + \beta_3(p_3) = \beta_3(p_3') + x \), for some \( x \in C \) or, \( g(b) + g\lambda(p_3) + x = g\lambda(p_3') + x \) (as \( \beta_3 = g\lambda \) or, \( (b + \lambda(p_3), \lambda(p_3')) \in K_3 = \bar{1} \)).

Therefore there exist \( a_1, a_2 \in A \) and some \( b_1 \in B \) such that

\[
b + \lambda(p_3) + f(a_1) + b_1 = \lambda(p_3') + f(a_2) + b_1
\]

which implies

\[
b + \beta_2(p_2) + f(a_1) + b_1 = \beta_2(p_2') + f(a_2) + b_1 \quad (7)
\]

where \( p_2 = (0, p_3) \) and \( p_2' = (0, p_3') \).

Since \( \beta_1 \) is an epic there exist \( p_1', p_1'', p_1''' \in P_1 \) such that \( a_1 + \beta_1(p_1') + a = \beta_1(p_1'') + a \) and \( a_2 + \beta_1(p_1''') + a' = \beta_1(p_1''') + a' \) for some \( a, a' \in A \).

Adding the above, we get \( a_1 + \beta_1(q_1) + a'' = a_2 + \beta_1(q_2) + a'' \) where \( q_1 = p_1' + p_1'', q_2 = p_1'' + p_1''' \) and \( a'' = a + a' \), which implies \( f(a_1) + f\beta_1(q_1) + f(a'') = f(a_2) + f\beta_1(q_2) + f(a'') \) or,

\[
f(a_1) + \beta_2 i(q_1) + f(a'') = f(a_2) + \beta_2 i(q_2) + f(a'') \quad (8)
\]
Adding $\beta_2i(q_1) + f(a'')$ on both sides of (7) we obtain
\[ b + \beta_2(p_2) + f(a_1) + \beta_2i(q_1) + f(a'') + b_1 = \beta_2(p_2') + f(a_2) + \beta_2i(q_1) + b_1 + f(a''). \]
Using (8), we have $b + \beta_2(p_2) + f(a_2) + \beta_2i(q_2) + f(a'') + b_1 = \beta_2(p_2) + f(a_2) + \beta_2i(q_1) + f(a'') + b_1$ or, $b + \beta_2(p_2 + i(q_2)) + f(a_2) + f(a'') + b_1 = \beta_2(p_2 + i(q_1)) + f(a'') + f(a_2) + b_1$ which implies that $\beta_2$ is an epic.

To show that top row is exact, we first show $i$ is a monic. Let $(x, y) \in K_i$. Then $i(x) + (p_1, p_3) = i(y) + (p_1, p_3)$ for some $(p_1, p_3) \in P_2$ or, $(x, 0) + (p_1, p_3) = (y, 0) + (p_1, p_3)$ which implies $x + p_1 = y + p_1$, for some $p_1 \in P_1$. Therefore $(x, y) \in \Delta P_i$. So, $K_i \subseteq \Delta P_i$.

Hence $i$ is a monic. Clearly, $\pi$ is an epic as $\pi$ is surjective.

Finally, we will show $K_\pi = \bar{I_i}$. Since $\pi \circ i(p_1) = \pi(i(p_1)) = \pi(p_1, 0) = 0$ for all $p_1 \in P_1$, $\pi \circ i$ is a $Z$-homomorphism. Therefore by Result 1.5, $\bar{I_i} \subseteq K_\pi$.

Again, let $(x, y) \in K_\pi$ where $x = (p_1, p_3)$ and $y = (p_1', p_3')$. Then $\pi(x) + u = \pi(y) + u$ for some $u \in P_3$ or,
\[ p_3 + u = p_3' + u. \]
From equation (9) we have $(p_1, p_3) + (p_1', 0) + (0, u) = (p_1', p_3') + (p_1, 0) + (0, u)$ or, $(p_1, p_3) + i(q_1) + z = (p_1', p_3') + i(q_2) + z$, where $q_1 = (p_1, 0)$, $q_2 = (p_1, 0)$ and $z = (0, u)$, or, $x + i(q_1) + z = y + i(q_2) + z$, which implies that $(x, y) \in \bar{I_i}$.

Therefore $K_\pi \subseteq \bar{I_i}$. So, $K_\pi = \bar{I_i}$. □

**Theorem 3.8.** Consider an exact sequence of $S$-semimodules
\[ 0 \longrightarrow P_2 \overset{f_2}{\longrightarrow} P_1 \overset{f_1}{\longrightarrow} P_0 \longrightarrow 0 \]
such that $P_0$ and $P_1$ are projective $S$-semimodules and $P_2$ is cancellative and every element in $P_1$ has an additive inverse. Then the above sequence splits.

**Proof.** Consider the diagram
\[
\begin{array}{ccc}
& & P_0 \\
& g_0 \swarrow & \\
P_1 & \overset{f_1}{\longrightarrow} & P_0 \\
\downarrow & & \downarrow \\
I_{P_0} & & \\
\end{array}
\]
Since $f_1$ is an epic then there exists $g_0 : P_0 \longrightarrow P_1$ such that
\[ f_1g_0 = I_{P_0}. \]
\[ (10) \]
Let $f = (I_{P_0} - g_0f_1) : P_1 \longrightarrow P_1$ be an $S$-homomorphism.
Then $f + g_0f_1 = I_{P_0}$ implies $f_1f + f_1g_0f_1 = f_1$ or,
\[ f_1f = f_1. \]
\[ (11) \]
Let $y \in \mathcal{J}_f$. Then $y + f(p_1) + z = f(p_1') + z$, for some $p_1, p_1' \in P_1$ and $z \in P_1$ implies $f_1(y) + f_1f(p_1) + f_1(z) = f_1f(p_1') + f_1(z)$.

Adding $f_1(p_1) + f(p_1')$ on both sides, we get $f_1(y) + f_1f(p_1) + f_1(p_1) + f_1(p_1') + f_1(z) = f_1f(p_1') + f_1(p_1) + f_1(z)$. Using (11), we have $f_1(y) + f_1(p_1) + f_1(z) + f_1(p_1') = f_1(p_1') + f_1(p_1) + f'(z)$ which implies $(y, 0) \in K_{f_1} = P_{f_2}$ (By exactness).

Therefore, $y + f_2(p_2) + u = f_2(p_2') + u$, for some $p_2, p_2' \in P_2$ and $u \in P_1$ which implies $y \in \mathcal{J}_{f_2}$. Hence $\mathcal{J}_f \subseteq \mathcal{J}_{f_2}$.

Consider

\[
\begin{array}{ccc}
P_1 & \xrightarrow{f} & \mathcal{J}_{f_2} \\
\downarrow & & \downarrow \\
P_2 & \xrightarrow{f_2} & 0
\end{array}
\]

Since $f_2$ is an epic then by projectivity of $P_1$, there exists $S$-homomorphism $g_1 : P_1 \rightarrow P_2$ such that $f_2g_1 = f$, which implies $f + g_0f_1 = f_2g_1 + g_0f_1$ or,

\[
I_{P_1} = f_2g_1 + g_0f_1. \tag{12}
\]

From (12), $f_2 = f_2g_1f_2 + g_0f_1f_2$ implies

\[
f_2(p_2) = f_2g_1f_2(p_2) + g_0f_1f_2(p_2), \quad p_2 \in P_2 \tag{13}
\]

Since $f_1f_2$ is a $Z$-homomorphism, we have $f_1f_2(p_2) + u = u$, for some $u \in P_0$ which implies $g_0f_1f_2(p_2) + g_0(u) = g_0(u)$.

From (13), we have $g_0(u) + f_2(p_2) = f_2g_1f_2(p_2) + g_0f_1f_2(p_2) + g_0(u)$ or, $f_2(p_2) + g_0(u) = f_2g_1f_2(p_2) + g_0(u)$, which implies $(p_2, g_1f_2(p_2)) \in K_{f_2} = \Delta_{p_2}$ (as $f_2$ is a monic), or $g_1f_2(p_2) + v = p_2 + v$, for some $v \in P_2$, or $g_1f_2(p_2) = p_2$ for all $p_2 \in P_2$ (as $P_2$ is cancellative). Therefore

\[
g_1f_2 = I_{P_2} \tag{14}
\]

Hence (10), (12) and (14) implies that the given sequence splits. $\square$

**References**


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