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ON SPLIT EXACT SEQUENCES AND PROJECTIVE SEMIMODULES

S. K. BHAMBRI AND MANISH KANT DUBEY

Abstract. In this paper the notion of split exact sequences of semimodules is introduced. We also study some results on projective semimodules that are analogous to module theory.

1. Introduction and preliminaries

A semiring is a commutative monoid $(S,+)$ having additive identity zero 0_S and a semigroup $(S,\cdot)$ which are connected by ring like distributivity. Let S be a semiring. A left S-semimodule M is a commutative monoid $(M,+)$ which has a zero element 0_M, together with an operation $S\times M \longrightarrow M$; defined by $(a,x)\longrightarrow ax$ such that for all $a,b\in S$ and $x,y\in M$,

(i) $a(x+y)=ax+ay$,
(ii) $(a+b)x=ax+bx$,
(iii) $(ab)x=a(bx)$,
(iv) $0_Sx=0_M=a0_M$.

A right S-semimodule is defined in an analogous manner. A non empty subset A of S-semimodule M is a subsemimodule of M if A is closed under addition and scalar multiplication. Let M and N be left S-semimodules. A homomorphism from M to N is a map $f: M\longrightarrow N$ such that,

(i) $f(m_1+m_2)=f(m_1)+f(m_2)$,
(ii) $f(am)=af(m)$, for all $m,m_1,m_2\in M$ and for all $a\in S$.

Definition 1.1 ([6]). *Let A and B be S-semimodules and $f: A\longrightarrow B$ be S-semimodule homomorphism. Define*

$$K_f=\{(a,b)\in A\times A\,|\,f(a)+x=f(b)+x \text{ for some } x\in B\}$$
$$I_f=\{(c,d)\in B\times B\,|\,c+f(a)=d+f(b) \text{ for some } a,b\in A\}$$
$$\bar{I}_f=\{(c,d)\in B\times B\,|\,c+f(a)+x=d+f(b)+x \text{ for some } a,b\in A, \text{some } x\in B\}.$$

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Then f is said to be a monic if $K_f = \bar{\Delta}_A$, where $\Delta_A = \{(a,a)|a \in A\}$ and

$$\bar{\Delta}_A = \{(a,b) \in A \times A | a + x = b + x \quad \text{for some} \quad x \in A\}$$

and f is said to be an epic if for any $b \in B$ there exist some $a_i \in A$, $i = 1, 2$ and $x \in B$ such that $b + f(a_1) + x = f(a_2) + x$.

Definition 1.2 ([6])**.** *Let A, B be S-semimodules. Then an S-semimodules homomorphism $f : A \longrightarrow B$ is said to be a Z-homomorphism if for each $a \in A$ there exists $x \in B$ such that $f(a) + x = x$.*

Definition 1.3 ([6])**.** *A sequence of S-semimodules and S-semimodule homomorphism is a diagram of the form,*

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \ldots$$

Such a sequence is said to be exact if $\bar{I}_{f_{i-1}} = K_{f_i}$ for all i .

Definition 1.4. *Let A and B be S-semimodules and $f : A \longrightarrow B$ be S-semimodules homomorphism. Then f is said to be i-regular if for each $b \in B$ there exist $a_1, a_2 \in A$ such that $b + f(a_1) = f(a_2)$ and f is said to be k-regular if $f(a_1) + x = f(a_2) + x$ where $a_1, a_2 \in A$ and $x \in B$ implies $f(a_1) = f(a_2)$.*

Result 1.5 ([6])**.** Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be S-semimodule homomorphisms. Then $gf : A \longrightarrow C$ is a Z-homomorphism if and only if $\bar{I}_f \subseteq K_g$.

Definition 1.6 (Projective Semimodule)**.** *A S-semimodule P is called projective if it satisfies the following two properties.*

(i) *If a S-homomorphism $f : A \longrightarrow B$ is an epic and $g : P \longrightarrow B$ is S-semimodule homomorphism then there exists a S-homomorphism $\phi : P \longrightarrow A$ such that $f \circ \phi = g$.*

(ii) *To every k-regular S-homomorphism $f : A \longrightarrow B$ and to every S-homomorphisms $\psi_1, \psi_2 : P \longrightarrow A$ with $f \circ \psi_1 = f \circ \psi_2$ there exist S-homomorphisms $k_1, k_2 : P \longrightarrow A$ such that $f \circ k_1$ and $f \circ k_2$ are Z-homomorphisms and $\psi_1 + k_1 = \psi_2 + k_2$.*

Definition 1.7 ([7])**.** *Let $\{M_i\}_{i \in I}$ be a family of S-semimodules. The cartesian product $\prod_{i \in I} M_i$ forms a S-semimodule under usual operations called the direct product of $\{M_i\}$.*

In the direct product $\prod_i M_i$, the set of all elements whose components x_i are equal to 0 except for a finite number of i is denoted by $\bigoplus_{i \in I} M_i$ and is called the external direct sum of $\{M_i\}$. Then $\bigoplus_{i \in I} M_i$ is a subsemimodule of $\prod_i M_i$.

Definition 1.8 ([8])**.** *Let M be a S-semimodule. Then M is called cancellative if whenever $m + x = m' + x$ for $m, m', x \in M$, we have $m = m'$.*

2. Split exact sequences

In this section, we define split exact sequences of S-semimodules and prove some theorems that are analogous to module theory.

Definition 2.1. *An exact sequence of S-semimodules of the form*

$$\dots M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} M_0 \longrightarrow 0$$

is said to split if there exist S-semimodule homomorphisms $g_i : M_i \longrightarrow M_{i+1}$ such that

(i) $f_1 \circ g_0 = I_{M_0}$
(ii) $g_{i-1} f_i + f_{i+1} g_i = I_{M_i}$ *for all* $i \geq 1$

where addition of S-semimodule homomorphisms is defined in the usual manner.

Theorem 2.2. *If the sequences of S-semimodules*

$$0 \rightleftarrows M' \underset{q}{\overset{u}{\rightleftarrows}} M \underset{v}{\overset{p}{\rightleftarrows}} M'' \rightleftarrows 0$$

are such that $qu = I_{M'}$, $pv = I_{M''}$, $uq + vp = I_M$ then each of the above sequences are split exact.

Proof. We need only to show that the given sequences are exact. To show u is a monic let $(a, b) \in K_u$. Then $u(a)+m = u(b)+m$ for some $m \in M$ implies $qu(a) + q(m) = qu(b) + q(m)$ or, $a + q(m) = b + q(m)$ implies $(a, b) \in \bar{\Delta}_{M'}$.

So, $K_u \subseteq \bar{\Delta}_{M'}$. Hence u is a monic.

To show $K_p = \bar{I}_u$, let $(m_1, m_2) \in K_p$. Then $p(m_1) + m'' = p(m_2) + m''$ for some $m'' \in M''$ implies $vp(m_1) + v(m'') = vp(m_2) + v(m'')$ and $vp(m_1)+uq(m_1)+uq(m_2)+v(m'') = vp(m_2)+uq(m_1)+uq(m_2)+v(m'')$ or, $m_1 + u(q(m_2)) + v(m'') = m_2 + u(q(m_1)) + v(m'')$ as $vp + uq = I_M$ implies $(m_1, m_2) \in \bar{I}_u$. So, $K_p \subseteq \bar{I}_u$.

Again, let $(m_1, m_2) \in \bar{I}_u$. Then

$$m_1 + u(m_1') + m = m_2 + u(m_2') + m \quad \text{for some } m_1', m_2' \in M', \text{ and } m \in M \tag{1}$$

implies

$$p(m_1) + pu(m_1') + p(m) = p(m_2) + pu(m_2') + p(m). \tag{2}$$

Again, from (1) we have $q(m_1)+qu(m_1')+q(m) = q(m_2)+qu(m_2')+q(m)$ or, $q(m_1) + m_1' + q(m) = q(m_2) + m_2' + q(m)$ (as $qu = I_{M'}$) which implies $uq(m_1) + u(m_1') + uq(m) = uq(m_2) + u(m_2') + uq(m)$.

Adding $vp(m_1) + vp(m_2)$ on both sides, we get $m_1 + vp(m_2) + u(m_1') + uq(m) = m_2 + vp(m_1) + uq(m) + u(m_2')$ which implies $p(m_1) + p(m_2) +$

$pu(m_1') + puq(m) = p(m_2) + p(m_1) + puq(m) + pu(m_2')$ or implies $p(m_2) + p(m_1) + pu(m_1') + p(m) + puq(m) = p(m_2) + p(m_1) + puq(m) + pu(m_2') + p(m)$.

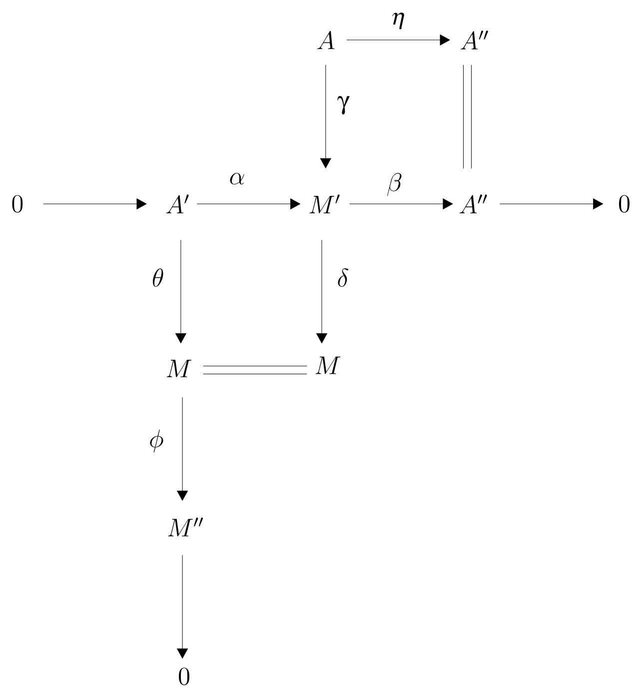
Using (2), we get $p(m_2) + p(m_2) + pu(m_2') + p(m) + puq(m) = p(m_2) + p(m_1) + pu(m_2') + puq(m) + p(m)$ which implies $(m_1, m_2) \in K_p$. So, $\bar{I}_u \subseteq K_p$. Hence $K_p = \bar{I}_u$.

Finally, p is an epic, because $pv(m'') = m''$ implies

$$m'' + p(0) + 0 = pv(m'') + 0.$$

Hence the given sequence is split exact. Similarly we can show that the other sequence is also split exact. $\square$

Theorem 2.3. *Consider a commutative diagram of S-semimodules*



and suppose that all columns are exact and the row is split exact (i.e. there exists a S-homomorphism $\beta' : A'' \longrightarrow M'$ such that $\beta\beta' = I_{A''}$ and a S-homomorphism $\alpha' : M' \longrightarrow A'$ such that $\alpha\alpha' + \beta'\beta = I_{M'}$), then $\bar{I}_\eta = K_{\phi\delta\beta'}$.

Proof. Let $(x, y) \in K_{\phi\delta\beta'}$. Then

$$\phi\delta\beta'(x) + m'' = \phi\delta\beta'(y) + m'' \text{ for some } m'' \in M''$$

implies $(\delta\beta'(x), \delta\beta'(y)) \in K_\phi = \bar{I}_\theta$ (by exactness).

Therefore $\delta\beta'(x) + \theta(q_1) + m = \delta\beta'(y) + \theta(q_2) + m$ for some $q_1, q_2 \in A'$ and $m \in M$ or, $\delta\beta'(x) + \delta\alpha(q_1) + m = \delta\beta'(y) + \delta\alpha(q_2) + m$ (as $\theta = \delta\alpha$) implies $(\beta'(x) + \alpha(q_1), \beta'(y) + \alpha(q_2)) \in K_\delta = \bar{I}_\gamma$ (by exactness).

Therefore $\beta'(x) + \alpha(q_1) + \gamma(a_1) + r = \beta'(y) + \alpha(q_2) + \gamma(a_2) + r$ for some $a_1, a_2 \in A$ and $r \in M'$ implies $\beta\beta'(x) + \beta\alpha(q_1) + \beta\gamma(a_1) + \beta(r) = \beta\beta'(y) + \beta\alpha(q_2) + \beta\gamma(a_2) + \beta(r)$.

Since $\beta\alpha$ is a Z-homomorphism we have $\beta\alpha(q_1) + a''_1 = a''_1$ and $\beta\alpha(q_2) + a''_2 = a''_2$ for some $a''_1, a''_2 \in A''$.

Since $\beta\beta' = I$ and $\beta\gamma = \eta$ we obtain

$$x + \eta(a_1) + a''_1 + a''_2 + \beta(r) = y + \eta(a_2) + a''_1 + a''_2 + \beta(r)$$

which implies $(x, y) \in \bar{I}_\eta$. Therefore $K_{\phi\delta\beta'} \subseteq \bar{I}_\eta$.

Again, since the given row splits we have $\alpha\alpha' + \beta'\beta = I_{M'}$.

Let $a \in A$. Then $\gamma(a) \in M'$. Therefore

$$\alpha\alpha'\gamma(a) + \beta'\beta\gamma(a) = \gamma(a)$$

which implies $\phi\delta\alpha\alpha'\gamma(a) + \phi\delta\beta'\eta(a) = \phi\delta\gamma(a)$ (as $\beta\gamma = \eta$) or,

$$\phi\theta\alpha'\gamma(a) + \phi\delta\beta'\eta(a) = \phi\delta\gamma(a) \ \text{ (as } \delta\alpha = \theta). \tag{3}$$

Since $\delta\gamma$ and $\phi\theta$ are Z-homomorphisms, we get $\phi\theta(\alpha'\gamma(a)) + u = u$ for some $u \in M''$ and $\delta\gamma(a) + m = m$ for some $m \in M$, which implies $\phi\delta\gamma(a) + \phi(m) = \phi(m)$.

From (3), $\phi\delta\beta'\eta(a) + u + \phi(m) = u + \phi(m)$ which implies $\phi\delta\beta'\eta$ is a Z-homomorphism, therefore by Result 1.5, $\bar{I}_\eta \subseteq K_{\phi\delta\beta'}$. Hence $K_{\phi\delta\beta'} = \bar{I}_\eta$. □

3. Projective semimodule

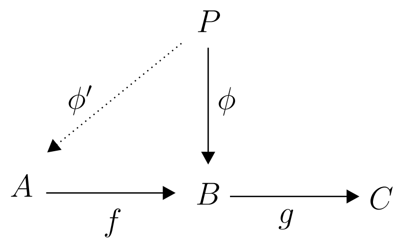
In this section we study some results on projective semimodule which are analogous to module theory.

Definition 3.1. *Let A and B be S-semimodules and $f : A \longrightarrow B$ be an S-homomorphism. Define*

$$\bar{J}_f = \{b \in B | b + f(a_1) + x = f(a_2) + x \quad \textit{for some } a_1, a_2 \in A, x \in B\}$$

Clearly, $\bar{J}_f$ is a subsemimodule of B.

Theorem 3.2. *Let P be projective S-semimodule. If $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of S-semimodules, then for every S-homomorphism $\phi : P \longrightarrow B$ such that $g \circ \phi$ is a Z-homomorphism there exists a S-homomorphism $\phi' : P \longrightarrow A$ with $f \circ \phi' = \phi$.*



Proof. Let $b \in \bar{J}_\phi$. Then

$$b + \phi(p_1) + u = \phi(p_2) + u, \quad \text{for some } p_1, p_2 \in P \text{ and } u \in B$$

implies

$$g(b) + g \circ \phi(p_1) + g(u) = g \circ \phi(p_2) + g(u). \tag{4}$$

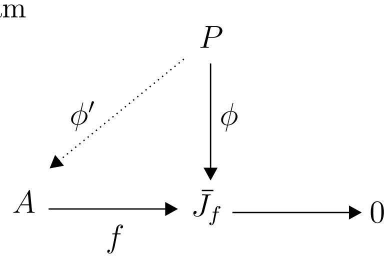
Since $g \circ \phi$ is a Z-homomorphism therefore

$$g \circ \phi(p_1) + c_1 = c_1 and g \circ \phi(p_2) + c_2 = c_2, \quad \text{for some } c_1, c_2 \in C\,.$$

From (4), $g(b) + c_1 + c_2 + g(u) = c_1 + c_2 + g(u)$ implies $(b, 0) \in K_g = \bar{I}_f$, which implies $b \in \bar{J}_f$.

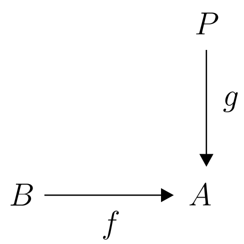
Hence, $\bar{J}_\phi \subseteq \bar{J}_f$.

Consider the diagram



Clearly f is an epic. Since P is projective there exists an S-homomorphism $\phi' : P \longrightarrow A$ such that $f \circ \phi' = \phi$. □

Theorem 3.3. *Consider the diagram*



of S-semimodules and S-homomorphisms. If P is projective S-semimodule then the following are equivalent:

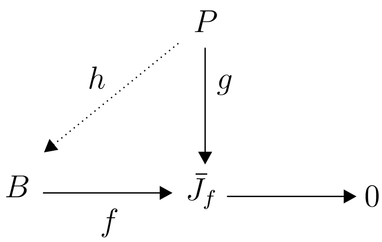
(i) *there is an S-homomorphism $h : P \longrightarrow B$ such that $f \circ h = g$,*
(ii) $\bar{J}_g \subseteq \bar{J}_f$.

Proof. (i)⇒(ii). Let $y \in \bar{J}_g$. Then

$$y + g(p_1) + b = g(p_2) + b, \quad \text{for some } p_1, p_2 \in P \text{ and } b \in A$$

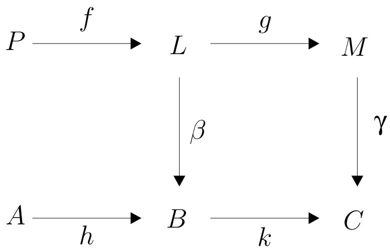
which implies $y + f \circ h(p_1) + b = f \circ h(p_2) + b$ (by (i)), hence $y \in \bar{J}_f$. Therefore $\bar{J}_g \subseteq \bar{J}_f$.

Conversely, let $\bar{J}_g \subseteq \bar{J}_f$. Consider



Since P is projective, there exists an S-homomorphism $h : P \longrightarrow B$ such that $f \circ h = g$. □

Theorem 3.4. *Suppose that*

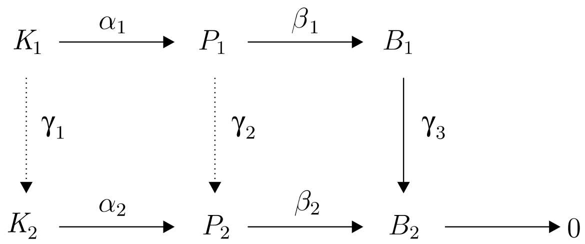


is a commutative diagram of S-semimodules and S-homomorphisms, that P is projective, $g \circ f$ is a Z-homomorphism and the lower row is exact then there exists an S-homomorphism $P \longrightarrow A$ which makes the diagram commutative.

Proof. Let $p \in P$. Since $g \circ f$ is a Z-homomorphism then there exists an $m \in M$ such that $g \circ f(p) + m = m$ which implies $\gamma \circ g \circ f(p) + \gamma(m) = \gamma(m)$ or, $k \circ \beta(f(p)) + \gamma(m) = \gamma(m)$, hence $(\beta f(p), 0) \in K_k = \bar{I}_h$ (as lower row is exact) which implies $\beta f(p) + h(a_1) + x = h(a_2) + x$ for some $a_1, a_2 \in A$ and $x \in B$. So, $\beta f(p) \in \bar{J}_h$, for all $p \in P$.

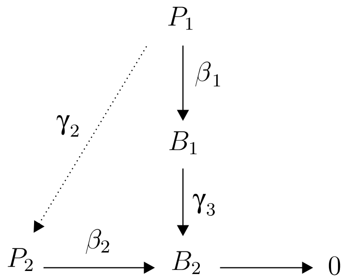
Define $\theta : P \longrightarrow \bar{J}_h$ such that $\theta(p) = \beta f(p)$. Clearly, θ is a S-homomorphism. Let $a \in A$. Then $h(a) \in \bar{J}_h$ therefore $h : A \longrightarrow \bar{J}_h$. By the definition of $\bar{J}_h$, $h : A \longrightarrow \bar{J}_h$ is an epic. Since P is projective there exists an S-homomorphism $\alpha : P \longrightarrow A$ such that $h\alpha = \theta = \beta f$. Hence the diagram commutes. □

Theorem 3.5. *Consider the diagram of S-semimodules and S-homomorphisms*



in which the rows are exact, P_1 is projective and α_1 is i-regular. Then there are S-homomorphisms $\gamma_2 : P_1 \longrightarrow P_2$ and $\gamma_1 : K_1 \longrightarrow K_2$ such that the completed diagram is commutative.

Proof. Consider the diagram



Since β_2 is an epic therefore by the projectivity of P_1 there exists an S-homomorphism $\gamma_2 : P_1 \longrightarrow P_2$ such that $\beta_2\gamma_2 = \gamma_3\beta_1$, i.e. the diagram commutes.

Again, let $y \in \bar{J}_{\gamma_2}$. Then

$$y + \gamma_2(p_1) + p_2 = \gamma_2(p_1') + p_2 \quad \text{for some } p_1, p_1' \in P_1 \ \text{ and } p_2 \in P_2$$

implies $\beta_2(y) + \beta_2\gamma_2(p_1) + \beta_2(p_2) = \beta_2\gamma_2(p_1') + \beta_2(p_2)$ or,

$$\beta_2(y) + \gamma_3\beta_1(p_1) + \beta_2(p_2) = \gamma_3\beta_1(p_1') + \beta_2(p_2) \tag{5}$$

Since α_1 is i-regular, we have

$$p_1 + \alpha_1(k_1) = \alpha_1(k_2) and p_1' + \alpha_1(k_3) = \alpha_1(k_4), \ \text{ for some } k_1, k_2, k_3, k_4 \in K_1$$

which implies $p_1 + \alpha_1(k_1 + k_4) = p_1' + \alpha_1(k_2 + k_3)$ or,

$$\gamma_3\beta_1(p_1) + \gamma_3\beta_1\alpha_1(k_5) = \gamma_3\beta_1(p_1') + \gamma_3\beta_1\alpha_1(k_6) \tag{6}$$

where $k_5 = k_1 + k_4$ and $k_6 = k_2 + k_3$.

Adding $\gamma_1\beta_1\alpha_1(k_5)$ on both sides of (5) and using (6), we get

$$\beta_2(y) + \gamma_3\beta_1(p_1') + \gamma_3\beta_1\alpha_1(k_6) + \beta_2(p_2) = \gamma_3\beta_1(p_1') + \beta_2(p_2) + \gamma_3\beta_1\alpha_1(k_5).$$

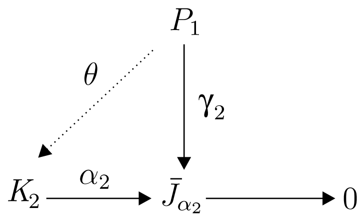
Since $\beta_1\alpha_1$ is a Z-homomorphism we have

$$\beta_1\alpha_1(k_5) + u_1 = u_1, u_1 \in B_1 \quad \text{implies} \quad \gamma_3\beta_1\alpha_1(k_5) + \gamma_3(u_1) = \gamma_3(u_1)$$
$$\beta_1\alpha_1(k_6) + u_2 = u_2, u_2 \in B_1 \quad \text{implies} \quad \gamma_3\beta_1\alpha_1(k_6) + \gamma_3(u_2) = \gamma_3(u_2)$$

So, $\beta_2(y) + \gamma_3\beta_1(p_1') + \gamma_3(u_1 + u_2) + \beta_2(p_2) = \gamma_3\beta_1(p_1') + \gamma_3(u_1 + u_2) + \beta_2(p_2)$ which implies $(y, 0) \in K_{\beta_2} = \bar{I}_{\alpha_2}$.

Therefore $y + \alpha_2(k_2') + x = \alpha_2(k_2'') + x$, for some $k_2', k_2'' \in K_2$ and $x \in P_2$. This implies $y \in \bar{J}_{\alpha_2}$. Therefore $\bar{J}_{\gamma_2} \subseteq \bar{J}_{\alpha_2}$.

Consider the following diagram



By the projectivity of P_1 there exists an S-homomorphism $\theta : P_1 \longrightarrow K_2$ such that $\gamma_2 = \alpha_2\theta$.

Define a map $\gamma_1 : K_1 \longrightarrow K_2$ such that $\gamma_1(k_1) = \theta\alpha_1(k_1)$, $k_1 \in K_1$. Clearly γ_1 is a S-homomorphism.

Now $\alpha_2\gamma_1(k_1) = \alpha_2\theta\alpha_1(k_1) = \gamma_2\alpha_1(k_1)$ or, $\alpha_2\gamma_1 = \gamma_2\alpha_1$. Hence the completed diagram is commutative. □

Proposition 3.6. *Suppose $\{P_i : i \in I\}$ is a family of projective S-semimodules. Then their direct sum $P = \bigoplus_i P_i$ is also projective.*

Proof. Let $f : A \longrightarrow B$ be an epic S-homomorphism of S-semimodules. Let$g : P \longrightarrow B$ be a S-homomorphism. Let $\pi_i : P \longrightarrow P_i$ be the canonical projection and $q_i : P_i \longrightarrow P$ be the canonical injection. Define $g_i : P_i \longrightarrow B$ such that $g_i = gq_i$ for each $i \in I$. Since P_i is projective, there exists an S-homomorphism $h_i : P_i \longrightarrow A$ such that $fh_i = g_i$ for each $i \in I$.

Define $h : P \longrightarrow A$ by $h = \sum\limits_i h_i\pi_i$. Then

$$\begin{aligned} fh &= f\left(\sum_i h_i\pi_i\right) = \sum_i fh_i\pi_i = \sum_i g_i\pi_i = \sum_i gq_i\pi_i \\ &= g\sum_i q_i\pi_i = g. \end{aligned}$$

So, P satisfies the property (i) of projective S-semimodule.

Let $f : A \longrightarrow B$ be k-regular S-homomorphism. Let $\psi_1, \psi_2 : P \longrightarrow A$ be S-homomorphisms with $f \circ \psi_1 = f \circ \psi_2$.

Define $\psi_i, \psi_i' : P_i \longrightarrow A$ by $\psi_i = \psi_1 \circ q_i$ and $\psi_i' = \psi_2 \circ q_i$ for all $i \in I$. Then

$$f \circ \psi_i = f \circ \psi_1 \circ q_i = f \circ \psi_2 \circ q_i = f \circ \psi_i'$$

$$\psi_1 = \psi_1 \circ \left(\sum_i q_i\pi_i\right) = \sum_i \psi_1 \circ q_i \circ \pi_i = \sum_i \psi_i\pi_i.$$

Similarly, $\psi_2 = \sum\limits_i \psi_i' \circ \pi_i$.

Since each P_i is projective, there exists an S-homomorphism $k_i, k_i' : P_i \longrightarrow A$ for each $i \in I$, such that $f \circ k_i$ and $f \circ k_i'$ are Z-homomorphisms. Also $\psi_i + k_i = \psi_i' + k_i'$.

Since f is k-regular, $f \circ k_i$ and $f \circ k_i'$ are Z-homomorphisms implies

$$f \circ k_i = 0 = f \circ k_i' \quad \text{for all} \quad i \in I.$$

Define $k_1 : P \longrightarrow A$ by $k_1 = \sum\limits_i k_i \circ \pi_i$ and $k_2 : P \longrightarrow A$ by $k_2 = \sum\limits_i k_i' \circ \pi_i$. Then $f \circ k_1 = 0 = f \circ k_2$ implies $f \circ k_1$ and $f \circ k_2$ are Z-homomorphisms and $\psi_1 + k_1 = \psi_2 + k_2$.

So, P satisfies the property (ii) of projective S-semimodule. □

Theorem 3.7. *Every diagram of S-semimodules and S-homomorphism of the form*

$$\begin{array}{ccccccccc}
 & & P_1 & & & & P_3 & & \\
 & & \downarrow{\scriptstyle \beta_1} & & & & \downarrow{\scriptstyle \beta_3} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

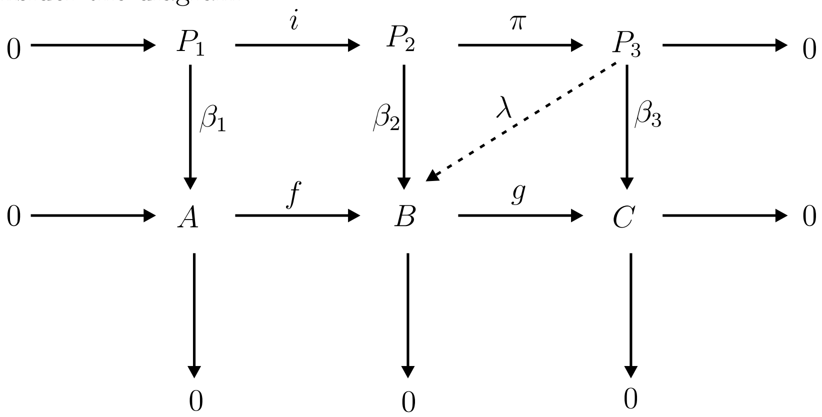
in which the row and the columns are exact, P_1 and P_3 are projective, g is k-regular can be extended to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{i} & P_2 & \xrightarrow{\pi} & P_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the top row and the second column are exact and P_2 is also projective.

Proof. Let $P_2 = P_1 \oplus P_3$. By Proposition 3.6, P_2 is also projective. Since g is an epic and $\beta_3 : P_3 \longrightarrow C$ is an S-homomorphism then by projectivity of P_3, there exists an S-homomorphism $\lambda : P_3 \longrightarrow B$ such that $\beta_3 = g\lambda$.

Consider the diagram



where $\pi : P_2 \longrightarrow P_3$ is given by $\pi(p_2) = \pi(p_1, p_3) = p_3$ is an S-homomorphism and $i : P_1 \longrightarrow P_2$ such that $i(p_1) = (p_1, 0)$ is an S-homomorphism.

Define a map $\beta_2 : P_2 \longrightarrow B$ such that

$$\beta_2(p_2) = f\beta_1(p_1) + \lambda(p_3) \quad \text{where } p_1 \in P_1 \text{ and } p_3 \in P_3.$$

Now,

$$\begin{aligned} g\beta_2(p_2) &= gf\beta_1(p_1) + g\lambda(p_3) \\ &= g\lambda(p_3) \text{ (as } g \text{ is } k\text{-regular and } g \circ f \text{ is a } Z\text{-homomorphism)} \\ &= \beta_3(p_3) = \beta_3\pi(p_2). \end{aligned}$$

Therefore, $g\beta_2 = \beta_3\pi$.

Now, $\beta_2 i(p_1) = \beta_2(p_1, 0) = f\beta_1(p_1) + \lambda(0) = f\beta_1(p_1)$. Therefore, $\beta_2 i = f\beta_1$. Hence the diagram commutes.

To show β_2 is an epic let $b \in B$. Then $g(b) \in C$. Since β_3 is an epic there exist $p_3, p_3' \in P_3$ such that $g(b) + \beta_3(p_3) + x = \beta_3(p_3') + x$, for some $x \in C$ or, $g(b) + g\lambda(p_3) + x = g\lambda(p_3') + x$ (as $\beta_3 = g\lambda$) or, $(b + \lambda(p_3), \lambda(p_3')) \in K_g = \bar{I}_f$.

Therefore there exist $a_1, a_2 \in A$ and some $b_1 \in B$ such that

$$b + \lambda(p_3) + f(a_1) + b_1 = \lambda(p_3') + f(a_2) + b_1$$

which implies

$$b + \beta_2(p_2) + f(a_1) + b_1 = \beta_2(p_2') + f(a_2) + b_1 \tag{7}$$

where $p_2 = (0, p_3)$ and $p_2' = (0, p_3')$.

Since β_1 is an epic there exist $p_1', p_1'', p_1''', p_1'''' \in P_1$ such that $a_1 + \beta_1(p_1') + a = \beta_1(p_1'') + a$ and $a_2 + \beta_1(p_1''') + a' = \beta_1(p_1'''') + a'$ for some $a, a' \in A$.

Adding the above, we get $a_1 + \beta_1(q_1) + a'' = a_2 + \beta_1(q_2) + a''$ where $q_1 = p_1' + p_1''''$, $q_2 = p_1'' + p_1'''$ and $a'' = a + a'$, which implies $f(a_1) + f\beta_1(q_1) + f(a'') = f(a_2) + f\beta_1(q_2) + f(a'')$ or,

$$f(a_1) + \beta_2 i(q_1) + f(a'') = f(a_2) + \beta_2 i(q_2) + f(a'') \tag{8}$$

Adding $\beta_2 i(q_1) + f(a'')$ on both sides of (7) we obtain

$b+\beta_2(p_2)+f(a_1)+\beta_2 i(q_1)+f(a'')+b_1 = \beta_2(p_2')+f(a_2)+\beta_2 i(q_1)+b_1+f(a'')$.

Using (8), we have $b + \beta_2(p_2) + f(a_2) + \beta_2 i(q_2) + f(a'') + b_1 = \beta_2(p_2') + f(a_2) + \beta_2 i(q_1) + f(a'') + b_1$ or, $b + \beta_2(p_2 + i(q_2)) + f(a_2) + f(a'') + b_1 = \beta_2(p_2' + i(q_1)) + f(a'') + f(a_2) + b_1$ which implies that β_2 is an epic.

To show that top row is exact, we first show i is a monic. Let $(x, y) \in K_i$. Then $i(x) + (p_1, p_3) = i(y) + (p_1, p_3)$ for some $(p_1, p_3) \in P_2$ or, $(x, 0) + (p_1, p_3) = (y, 0) + (p_1, p_3)$ which implies $x + p_1 = y + p_1$, for some $p_1 \in P_1$. Therefore $(x, y) \in \bar{\Delta}_{P_1}$. So, $K_i \subseteq \bar{\Delta}_{P_1}$.

Hence i is a monic. Clearly, π is an epic as π is surjective.

Finally, we will show $K_\pi = \bar{I}_i$. Since $\pi \circ i(p_1) = \pi(i(p_1)) = \pi(p_1, 0) = 0$ for all $p_1 \in P_1$, $\pi \circ i$ is a Z-homomorphism. Therefore by Result 1.5, $\bar{I}_i \subseteq K_\pi$.

Again, let $(x, y) \in K_\pi$ where $x = (p_1, p_3)$ and $y = (p_1', p_3')$. Then $\pi(x) + u = \pi(y) + u$ for some $u \in P_3$ or,

$$p_3 + u = p_3' + u. \tag{9}$$

From equation (9) we have $(p_1, p_3)+(p_1', 0)+(0, u) = (p_1', p_3')+(p_1, 0)+(0, u)$ or, $(p_1, p_3) + i(q_1) + z = (p_1', p_3') + i(q_2) + z$, where $q_1 = (p_1', 0)$, $q_2 = (p_1, 0)$ and $z = (0, u)$, or, $x+i(q_1)+z = y+i(q_2)+z$, which implies that $(x, y) \in \bar{I}_i$.

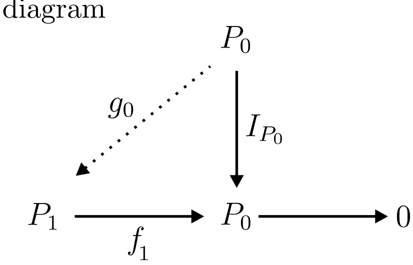
Therefore $K_\pi \subseteq \bar{I}_i$. So, $K_\pi = \bar{I}_i$. □

Theorem 3.8. *Consider an exact sequence of S-semimodules*

$$0 \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \longrightarrow 0$$

such that P_0 and P_1 are projective S-semimodules and P_2 is cancellative and every element in P_1 has an additive inverse. Then the above sequence splits.

Proof. Consider the diagram



Since f_1 is an epic then there exists $g_0 : P_0 \longrightarrow P_1$ such that

$$f_1 g_0 = I_{P_0}. \tag{10}$$

Let $f = (I_{P_1} - g_0 f_1) : P_1 \longrightarrow P_1$ be an S-homomorphism.

Then $f + g_0 f_1 = I_{P_1}$ implies $f_1 f + f_1 g_0 f_1 = f_1$ or,

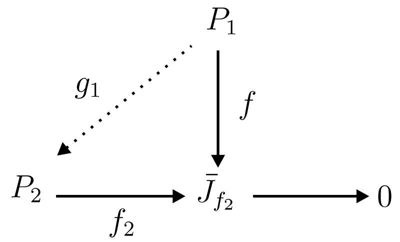
$$f_1 f + f_1 = f_1. \tag{11}$$

Let $y \in \bar{J}_f$. Then $y + f(p_1) + z = f(p_1') + z$, for some $p_1, p_1' \in P_1$ and $z \in P_1$ implies $f_1(y) + f_1f(p_1) + f_1(z) = f_1f(p_1') + f_1(z)$.

Adding $f_1(p_1) + f(p_1')$ on both sides, we get $f_1(y) + f_1f(p_1) + f_1(p_1) + f_1(p_1') + f_1(z) = f_1f(p_1') + f_1(p_1') + f_1(p_1) + f_1(z)$. Using (11), we have $f_1(y)+f_1(p_1)+f_1(z)+f_1(p_1') = f_1(p_1')+f_1(p_1)+f'(z)$ which implies $(y, 0) \in K_{f_1} = \bar{I}_{f_2}$ (By exactness).

Therefore, $y + f_2(p_2) + u = f_2(p_2') + u$, for some $p_2, p_2' \in P_2$ and $u \in P_1$ which implies $y \in \bar{J}_{f_2}$. Hence $\bar{J}_f \subseteq \bar{J}_{f_2}$.

Consider



Since f_2 is an epic then by projectivity of P_1, there exists S-homomorphism $g_1 : P_1 \longrightarrow P_2$ such that $f_2g_1 = f$, which implies $f + g_0f_1 = f_2g_1 + g_0f_1$ or,

$$I_{P_1} = f_2g_1 + g_0f_1. \tag{12}$$

From (12), $f_2 = f_2g_1f_2 + g_0f_1f_2$ implies

$$f_2(p_2) = f_2g_1f_2(p_2) + g_0f_1f_2(p_2), \quad p_2 \in P_2 \tag{13}$$

Since f_1f_2 is a Z-homomorphism, we have $f_1f_2(p_2)+u = u$, for some $u \in P_0$ which implies $g_0f_1f_2(p_2) + g_0(u) = g_0(u)$.

From (13), we have $g_0(u) + f_2(p_2) = f_2g_1f_2(p_2) + g_0f_1f_2(p_2) + g_0(u)$ or, $f_2(p_2) + g_0(u) = f_2g_1f_2(p_2) + g_0(u)$, which implies $(p_2, g_1f_2(p_2)) \in K_{f_2} = \bar{\Delta}_{P_2}$ (as f_2 is a monic), or $g_1f_2(p_2) + v = p_2 + v$, for some $v \in P_2$, or $g_1f_2(p_2) = p_2$ for all $p_2 \in P_2$ (as P_2 is cancellative). Therefore

$$g_1f_2 = I_{P_2} \tag{14}$$

Hence (10), (12) and (14) implies that the given sequence splits. □

References

[1] Barry Mitchell, *Theory of Categories*, Academic Press, New York–London, 1965.
[2] John Dauns, *Modules and Rings*, Cambridge University Press, 1994.
[3] D. G. Northcott, *First course of homological algebra*, Cambridge Uniersity Press, 1973.
[4] I. S. Luther and I. B. S. Passi, *Alegebra*, Volume 3, Modules, Narosa Publications, New Delhi, 2002.

[5] J. S. Golan, *The theory of semirings with application in mathematics and theoretical computer sciences*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 54, Longman Sci. Tech. Horlow, 1992.
[6] M. R. Adhikari and P. Mukhophadhyay, *Exact sequences of semimodule*, Bull. Cal. Math. Society, 94 (1), (2002), 23–32.
[7] Michihiro Takahasi, *On the Bordism categories II*, Math. Semin. Notes, 9 (1981).
[8] Michihiro Takahasi, *Extensions of semimodules II*, Math. Semin. Notes, 11 (1983), 83–118.
[9] S. K. Bhambri and Manish Kant Dubey, *Some results on exact sequences of semimodules analogues to module thoery*, Soochow J. Math., 32 (4) (2006), 485–498.
[10] T. S. Blyth, *Modules Theory*, Second edition, Clarendon Press, Oxford, 1990.

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S. K. Bhambri
Department of Mathematics
Kirorimal College
University of Delhi
Delhi 110007, India
E–mail: skbhambri@rediffmail.com

M. K. Dubey
Department of Mathematics
University of Delhi
Delhi 110 007, India
E–mail: kantmanish@yahoo.com