SARAJEVO JOURNAL OF MATHEMATICS
Vol.3 (15) (2007), 41–45

A NOTE ON VERY WEAK SOLUTIONS FOR A CLASS OF NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We prove a new *a priori* estimate for very weak solutions of a class of nonlinear elliptic equations.

1. Introduction

Let Ω be a bounded regular domain in R^n. We consider very weak solutions $w \in W_0^{1,r}(\Omega, R^m)$ with $r > \max\{1, p-1\}$ of the nonlinear system

$$-div\Big(|g+\nabla w|^{p-2}(g+\nabla w)\Big) + b(x) = 0 \tag{1}$$

where $g \in L^r(\Omega, R^{m\times n})$, $b(x) \in L^{\frac{r}{p-1}}(\Omega, R^m)$, the natural exponent $p > 1$ throughout in this paper. Equation (1) is understood in the weak sense, that is

$$\int_\Omega |g+\nabla w|^{p-2}(g+\nabla w)\cdot\nabla\varphi dx + \int_\Omega b(x)\varphi(x)dx = 0 \tag{2}$$

for every $\varphi \in W_0^{1,\frac{r}{r-p+1}}(\Omega, R^n)$.

Iwaniec and Sbordone studied the p–harmonic system in [2]

$$div(|\nabla u|^{p-2}\nabla u) = 0. \tag{3}$$

They have shown that there exist $r_1 = r_1(p, m, \Omega)$ and $r_2 = r_2(p, m, \Omega)$, satisfying

$$1 < r_1 < p < r_2 < \infty$$

such that every very weak p–harmonic mapping $u \in W_{loc}^{1,r_1}(\Omega, R^m)$ belongs to $W_{loc}^{1,r_2}(\Omega, R^m)$. They conjectured (Conjecture 1 in [2]) that every $r_1 > \max\{1, p-1\}$ would do for the regularity result for the p–harmonic system, but their estimate for r_1 was very close to p.

2000 *Mathematics Subject Classification.* 30C62, 35J60.

Key words and phrases. Very weak solution, Hodge decomposition, regular domain.

The first author is supported by youngteacher's foundation of north China electric power university.

Juha Kinnunen and Shulin Zhou studied the following nonhomogeneous system in [3]

$$div\Big(|g+\nabla w|^{p-2}(g+\nabla w)\Big)=div\, h \tag{4}$$

where $g\in L^r(\Omega,R^{m\times n})$ and $h\in L^{\frac{r}{p-1}}(\Omega,R^{m\times n})$ are matrix fields. They have shown that if $w(x)\in W_0^{1,r}(\Omega,R^m)$ with $r>\max\{1,p-1\}$ is the very weak solution of (4), r can be chosen arbitrarily close to one when p is close to two.

The objective of our note is to study equation (1) and we get a result similar to the one of [3].

2. Main results

For the convenience of the reader we recall the formulation of the Hodge decomposition (Theorem 3 in [2]).

Lemma 1. *Let Ω be a regular domain in R^n and $w\in W_0^{1,r}(\Omega,R^m)$ with $r>1$ and let $-1<\varepsilon<r-1$. Then there exist $\phi\in W_0^{1,\frac{r}{1+\varepsilon}}(\Omega,R^m)$ and a divergence free matrix field $H\in L^{\frac{r}{1+\varepsilon}}(\Omega,R^{m\times n})$ such that*

$$|\nabla w|^{\varepsilon}\nabla w=\nabla\phi+H \tag{5}$$

and

$$\|H\|_{\frac{r}{1+\varepsilon},\Omega}\le C_r(\Omega,m)|\varepsilon|\|\nabla w\|_{r,\Omega}^{1+\varepsilon}. \tag{6}$$

Remark. Fot the definition of a regular domain see [2], and there we notice that balls and cubes are regular domains in R^n.

The following Lemma comes from ([4] Theorem 2.3).

Lemma 2. (Poincare theorem) *Let Ω be a bounded domain in R^n. Then there exists a constant $C=C(p,n,\Omega)<\infty$ such that for every $u\in W_0^{1,p}(\Omega)$*

$$\|u\|_{p,\Omega}\le C\|\nabla u\|_{p,\Omega}. \tag{7}$$

Theorem 1. *Let $r>\max\{1,p-1\}$ and suppose that $w\in W_0^{1,r}(\Omega,R^m)$, with r satisfying* (1)*. Then there exists $\delta=\delta(m,\Omega)>0$ such that if* $\max\{|p-2|,|r-1|\}<\delta$*, then*

$$\int_\Omega|\nabla w|^r dx\le C(p,m,r,\Omega)\int_\Omega(|g|^r+|b(x)|^{\frac{r}{p-1}})dx. \tag{8}$$

Proof. Using Lemma 1 with $\varepsilon=r-p$, we obtain functions $\phi_1\in W_0^{1,\frac{r}{r-p+1}}$ (Ω,R^m) and $H_1\in L^{\frac{r}{r-p+1}}(\Omega,R^{m\times n})$ such that

$$|\nabla w|^{r-p}\nabla w=\nabla\phi_1+H_1 \tag{9}$$

$$\int_\Omega H_1 \cdot \nabla \varphi dx = 0, \text{ for every } \varphi \in W_0^{1,\frac{r}{p-1}}(\Omega, R^m) \tag{10}$$

and

$$\|H_1\|_{\frac{r}{r-p+1},\Omega} \leq C_1|r-p|\|\nabla w\|_{r,\Omega}^{r-p+1}, \quad C_1 = C_r(\Omega, m). \tag{11}$$

In particular, we have

$$\|\nabla \phi_1\|_{\frac{r}{r-p+1},\Omega} \leq (C_1+1)|r-p|\|\nabla w\|_{r,\Omega}^{r-p+1}. \tag{12}$$

Since ϕ_1 can be used as a test function in (2), we obtain

$$\int_\Omega |g+\nabla w|^{p-2}(g+\nabla w)\cdot \nabla \phi_1 dx + \int_\Omega b(x)\phi_1(x)dx = 0.$$

Inserting (9) we arrive at

$$\begin{aligned}\int_\Omega |\nabla w|^r dx &= \int_\Omega |\nabla w|^{p-2}\nabla w \cdot H_1 dx - \int_\Omega b(x)\phi_1(x)dx \\ &\quad + \int_\Omega (|\nabla w|^{p-2}\nabla w - |g+\nabla w|^{p-2}(g+\nabla w))\cdot \nabla \phi_1 dx \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{13}$$

We begin with estimating I_1. By using Lemma 1 again with $\varepsilon = p-2$, we obtain $\phi_2 \in W_0^{1,\frac{r}{p-1}}(\Omega, R^m)$ and $H_2 \in L^{\frac{r}{p-1}}(\Omega, R^{m\times n})$ such that

$$|\nabla w|^{p-2}\nabla w = \nabla \phi_2 + H_2 \tag{14}$$

$$\int_\Omega H_2 \cdot \nabla \varphi dx = 0, \quad \text{for every } \varphi \in W_0^{1,\frac{r}{r-p+1}}(\Omega, R^m) \tag{15}$$

and

$$\|H_2\|_{\frac{r}{p-1},\Omega} \leq C_1|p-2|\|\nabla w\|_{r,\Omega}^{p-1}, \quad C_1 = C_r(\Omega, m). \tag{16}$$

Using (14), (15), (9), (10) and (16), we have

$$\begin{aligned} I_1 &= \int_\Omega (\nabla \phi_2 + H_2)\cdot H_1 dx = \int_\Omega H_1 \cdot H_2 \\ &= \int_\Omega (|\nabla w|^{r-p}\nabla w - \nabla \phi_1)\cdot H_2 dx \\ &= \int_\Omega |\nabla w|^{r-p}\nabla w \cdot H_2 dx \\ &\leq C_1|p-2|\|\nabla w\|_{r,\Omega}^r. \end{aligned}$$

The same reasoning shows that

$$I_1 \le C_1|r-p|\|\nabla w\|_{r,\Omega}^r$$

and hence

$$I_1 \le C_1 \min\{|p-2|, |r-p|\}\|\nabla w\|_{r,\Omega}^r. \tag{17}$$

Then we estimate I_2, by *Hölder* inequality, we have

$$I_2 = -\int_\Omega b(x)\cdot\phi_1 dx \le \left(\int_\Omega |b(x)|^{\frac{r}{p-1}} dx\right)^{\frac{p-1}{r}} \cdot \left(\int_\Omega |\phi_1|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}}.$$

Since $\phi_1 \in W_0^{1,\frac{r}{r-p+1}}(\Omega, R^m)$, using Lemma 2

$$\|\phi_1\|_{\frac{r}{r-p+1},\Omega} \le C\|\nabla\phi_1\|_{\frac{r}{r-p+1},\Omega}$$

and hence

$$I_2 \le C\|b(x)\|_{\frac{r}{p-1},\Omega}\|\nabla\phi_1\|_{\frac{r}{r-p+1},\Omega} \le C\|b(x)\|_{\frac{r}{p-1},\Omega}\|\nabla w\|_{r,\Omega}^{r-p+1}. \tag{18}$$

By virtue of (12), we may estimate I_3 in the same way as in [2], by using the Lipschitz property of $|\nabla w|^{p-2}\nabla w$ and we have

$$\begin{aligned} I_3 &\le C\int_\Omega |g|(|\nabla w| + |g|)^{p-2}|\nabla\phi_1| dx \\ &\le C\Big(\|g\|_{r,\Omega}\||g| + |\nabla w|\|_{r,\Omega}^{p-2}\|\nabla\phi_1\|_{\frac{r}{r-p+1},\Omega}\Big) \\ &\le C\Big(\|g\|_{r,\Omega}\||g| + |\nabla w|\|_{r,\Omega}^{p-2}\|\nabla w\|_{r,\Omega}^{r-p+1}\Big) \\ &\le C\Big(\|g\|_{r,\Omega}^{p-1}\|\nabla w\|_{r,\Omega}^{r-p+1} + \|g\|_{r,\Omega}\|\nabla w\|_{r,\Omega}^{r-1}\Big). \end{aligned} \tag{19}$$

Using (13), (17), (18) and (19) we get

$$\begin{aligned} &\Big(1 - C_1\min\{|p-2|, |r-p|\}\Big)\|\nabla w\|_{r,\Omega}^r \\ &\le C\Big(\|b(x)\|_{\frac{r}{p-1},\Omega}\|\nabla w\|_{r,\Omega}^{r-p+1} + \|g\|_{r,\Omega}^{p-1}\|\nabla w\|_{r,\Omega}^{r-p+1} + \|g\|_{r,\Omega}\|\nabla w\|_{r,\Omega}^{r-1}\Big). \end{aligned}$$

The only point remaining is to separate $\|\nabla w\|_r$ from the terms in the right hand side. This can be done routinely with the aid of Young's inequality. We continue in this fashion obtaining the estimate, for every $\theta > 0$

$$\Big(1 - C_1\min\{|p-2|, |r-p|\} - \theta\Big)\|\nabla w\|_{r,\Omega}^r \le C_\theta\Big(\|g\|_{r,\Omega}^r + \|b\|_{\frac{r}{p-1},\Omega}^{\frac{r}{p-1}}\Big).$$

In particular, if $C_1|p-2| < 1$, then (2) holds. Estimates for the constant $C_1 = C_r(\Omega, m)$ can be found in [1] and formula (11) in [2]. Using these estimates it is easy to see that we may choose $C_1 = C(m, p, r, \Omega)$. This completes the proof of the theorem. □

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(Received: September 28, 2005)
(Revised: September 23, 2006)

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