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SOLUTIONS OF NEUMANN BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH p–LAPLACIAN

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Abstract. Sufficient conditions for the existence of at least one solution of Neumann boundary value problems for higher order nonlinear functional difference equations with p–Laplacian are established. We allow f to be at most linear, superlinear or sublinear in the obtained results.

1. Introduction

Solvability of boundary value problems for functional difference equations were studied by many authors, one may see the text books [1, 2] for general theory, papers [4, 7, 11, 13, 14, 15, 20-26] for boundary value problems of second order difference equations, papers [5, 6, 8, 9, 10, 12] for boundary value problems of higher order difference equations, and papers [16-19] for periodic boundary value problems of difference equations.

In this paper, we study the boundary value problems for the higher order nonlinear functional difference equation with p-Laplacian, i.e. the equation

$$\Delta[\phi(\Delta x(n))] = f(n, x(n+1), x(n-\tau_1(n)), \dots, x(n-\tau_m(n)), \quad n \in [0, T-1], \tag{1}$$

subject to the Neumann type boundary value conditions

$$\begin{cases} \Delta x(0) = 0, \quad x(i) = \gamma(i), \quad i \in [-\tau, \dots, -1], \\ \Delta x(T) = 0, \quad x(i) = \psi(i), \quad i \in [T+2, \dots, T+\delta], \end{cases} \tag{2}$$

where $\phi : R \to R$ is a homeomorphism with its inverse defined by ϕ^{-1}, ϕ satisfies the condition $\phi(x)x \geq 0$ for all $x \in R$, $T \geq 1$, Z is the set of integers,

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$\tau_i : [0, T-1] \to Z$, $i = 1, \cdots, m$,

$$\tau = \max\{0, \max_{n\in[0,T-1]}\{\tau_i(n)\} : \ i = 1, \ldots, m\},$$
$$\delta = -\min\{0, \min_{n\in[0,T-1]}\{\tau_i(n)\} : \ i = 1, \ldots, m\},$$

$f : [0, T-1] \times R^{m+1} \to R$ is continuous, $\gamma : [-\tau, -1] \to R$, and $\psi : [T+2, T+\delta] \to R$. We suppose that

$$p = \min\{\min_{n\in[0,T-1]}\{n - \tau_i(n)\} : \ i = 1, \ldots, m\} \le T - 1,$$
$$q = \max\{\max_{n\in[0,T-1]}\{n - \tau_i(n)\} : \ i = 1, \ldots, m\} \ge 1.$$

Then the set $E = \{n \in [0, T-1] : \ 1 \le n - \tau_i(n) \le T - 1, \ \ i = 1, \cdots, m\} \ne \emptyset$.

The motivation of this paper is mainly due to papers [4-7, 11, 13, 24].

In [24], Cabada and Otero-Espinar studied a class of boundary value problems consisting of the second order difference equation and Neumann boundary conditions

$$\begin{cases} u_{k+2} = f(k, u_{k+1}, u_k), & k \in [0, N-1], \\ \Delta u_0 = A, \quad \Delta u_N = B. \end{cases}$$

Assuming the existence of of a pair of ordered lower and upper solutions γ and β, existence results for solutions of the above problems were established.

Henderson and Thompson [4], using lower and upper solutions methods, and Avery and Peterson [11], using fixed point theorem, proved the results on existence of at least three solutions of the problem

$$\begin{cases} \Delta^2 x(k-1) + f(k, x(k), x(k) - x(k-1)), & k = 1, \ldots, n-1, \\ x(0) = x(n) = 0. \end{cases}$$

In [13], the authors studied the existence of positive solutions of the problem

$$\begin{cases} \Delta^2 x(k) + g(x(k)), & k = 0, \ldots, N, \\ x(0) = \Delta x(N+1) = 0 \end{cases}$$

by using fixed point theorems in cones in Banach spaces. Liu and Ge in [7] studied the problem

$$\begin{cases} \Delta^2 x(k) + a(k) f(k, x(k)), & k = 0, \ldots, N, \\ \Delta x(N+1) = 0, \quad x(0) - \alpha \Delta x(0) = 0. \end{cases}$$

Agarwal and Henderson in [6] established the existence results for positive solutions of boundary value problem for the third order difference equation

$$\begin{cases} \Delta^3 x(k) + a(k) f(x(k)) = 0, & k = 0, \ldots, T, \\ x(0) = x(1) = 0, \quad x(T+3) = 0. \end{cases}$$

In [5], Kong, Kong and Zhang studied the existence of positive solutions of the boundary value problems for third order functional difference equations

$$\begin{cases} \Delta^3 x(k) + a(k)a(k)f(k, x(\omega(k))) = 0, \quad k = 0, \dots, T, \\ x(i) = \phi(i), \quad i \in [n_1, 0], \quad x(1) = 0, \\ x(i) = \psi(i), \quad i \in [T+3, n_2], \\ \phi(0) = \psi(T+3) = 0. \end{cases}$$

Another motivation of this paper is due to the papers [27, 28] and the references therein. In [27], Cabada and Otero-Espinar established the existence and comparison results for difference ϕ–Laplacian boundary value problems consisting of the equation

$$-\Delta[\phi(\Delta x(k))] = f(k, x(k+1)), \quad k \in \{0, 1, \dots, N-1\}, \tag{3}$$

and one of the following boundary conditions

$$\Delta x(0) = N_0, \ \Delta x(N) = N_1, \tag{4}$$

and

$$x(0) - x(N) = C_0, \quad \Delta x(0) - \Delta x(N) = C_1. \tag{5}$$

The methods used in papers [27, 28] and the references cited there are lower and upper solutions methods and monotone iterative technique and comparison principles. In [27], the following assumptions are used.

(H_1). $\phi : R \to R$ is a strictly increasing homeomorphism and ϕ^{-1} is a H–Lipschitzian function on R; i.e.,

$$|\phi^{-1}(x) - \phi^{-1}(y)| \leq H|x - y|, \quad x, y \in R.$$

(H_1^*). $\phi : R \to R$ is a strictly increasing homeomorphism and ϕ^{-1} is a locally H–Lipschitzian function on R; i.e., for every compact interval $[h_1, h_2]$ there exists $H > 0$ such that for all $x, y \in [h_1, h_2]$

$$|\phi^{-1}(x) - \phi^{-1}(y)| \leq H|x - y|.$$

(H_3). There exists $M < 0$ for which

$$f(k, x) - f(k, y) \leq M(y - x), \quad \beta(k) \leq y \leq x \leq \alpha(k), \quad k \in I = [0, \dots, N-1].$$

Boundary value problem (1) and (2) is called Neumann type boundary value conditions since Neumann boundary value problem for the second order difference equation

$$\begin{cases} \Delta^2 x(k) + g(k, x(k+1), x(k)), \quad k = 0, \dots, T-1, \\ \Delta x(0) = \Delta x(T) = 0 \end{cases} \tag{6}$$

is a special case of the boundary value problem (1) and (2). We know of no other paper concerned with the solvability of problem (1) and (2).

The purposes of this paper are to establish sufficient conditions for the existence of at least one solutions of BVP (1) and (2) by a different method. It is interesting that we allow that f be sublinear, at most linear or superlinear.

This paper is organized as follows. In Section 2, we give the main results, and in Section 3, the examples to illustrate the main results will be presented.

2. Main Results

To get existence results for solutions of BVP (1) and (2), we need the following fixed point theorem, which was used to solve multi-point boundary value problems for differential equations in many papers.

Let X and Y be Banach spaces, $L:\ \text{Dom}\ L\subset X\to Y$ be a Fredholm operator of index zero, $P:\ X\to X$, $Q:\ Y\to Y$ be projectors such that

$$\text{Im}\ P=\text{Ker}\ L,\ \text{Ker}\ Q=\text{Im}\ L,\ \ X=\text{Ker}\ L\oplus\text{Ker}\ P,\ \ Y=\text{Im}\ L\oplus\text{Im}\ Q.$$

It follows that

$$L|_{\text{Dom}\ L\cap\text{Ker}\ P}:\ \text{Dom}\ L\cap\text{Ker}\ P\to\text{Im}\ L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $\text{Dom}\ L\cap\overline{\Omega}\neq\emptyset$, the map $N:\ X\to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact.

Theorem GM. [3] *Let L be a Fredholm operator of index zero and let N be L–compact on Ω. Assume that the following conditions are satisfied:*

- (i) *$Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(\text{Dom}L\setminus\text{Ker}L)\cap\partial\Omega]\times(0,1)$;*
- (ii) *$Nx\notin\text{Im}L$ for every $x\in\text{Ker}L\cap\partial\Omega$;*
- (iii) *$\deg(\wedge QN|_{\text{Ker}L},\ \Omega\cap\text{Ker}L,0)\neq 0$, where $\wedge:\ \text{Ker}L\to Y/\text{Im}L$ is the isomorphism.*

Then the equation $Lx=Nx$ has at least one solution in $\text{Dom}L\cap\overline{\Omega}$.

Let $X=R^{T+\tau+\delta+1}\times R^{T+1}$ be endowed with the norm

$$||(x,y)||_X=\max\left\{\max_{n\in[0,T]}|y(n)|,\ \max_{n\in[-\tau,T+\delta]}|x(n)|\right\},$$

$Y=R^{T+1}\times R^T$ be endowed with the norm

$$||(u,v)||_Y=\max\left\{\max_{n\in[0,T]}|u(n)|,\ \max_{n\in[0,T-1]}|v(n)|\right\}.$$

It is easy to see that X and Y are Banach spaces. Choose

$$\mathrm{Dom}L=\left\{x\in X:\begin{array}{c}x(i)=0,\ i\in[-\tau,\dots,-1],\\ x(i)\in R,\ i\in[0,T+1],\\ x(i)=0,\ i\in[T+2,\dots,T+\delta],\end{array}\right\}\times\left\{y\in R^{T+1}:y(0)=y(T)=0\right\}.$$

Let

$$L:\mathrm{Dom}L\cap X\to Y,\quad L\bullet\begin{pmatrix}x(n)\\ y(n)\end{pmatrix}=\begin{pmatrix}\Delta x(n)\\ \Delta y(n)\end{pmatrix},\quad (x,y)\in\mathrm{Dom}L,$$

and $N:X\to Y$ by

$$N\bullet\begin{pmatrix}x(n)\\ y(n)\end{pmatrix}=\begin{pmatrix}\phi^{-1}(y(n))\\ f\big(n,w(n+1),w(n-\tau_1(n)),\dots,w(n-\tau_m(n))\big)\end{pmatrix}$$

where $w(n)=x(n)+x_0(n)$, for all $(x,y)\in X$, where

$$x_0(n)=\begin{cases}\gamma(n),\ n\in[-\tau,-1],\\ 0,\ n\in[0,T+1],\\ \psi(n),\ n\in[T+2,T+\delta],\end{cases}$$

and $y_0(n)=0$ for all $n\in[0,T]$. It is easy to show that

$$L\bullet\begin{pmatrix}x_0(n)\\ y_0(n)\end{pmatrix}=0$$

and that $(x,y)\in\mathrm{Dom}L$ is a solution of $L\bullet(x,y)=N\bullet(x,y)$ imply that $x+x_0$ is a solution of problem (1) and (2).

It is easy to check the following results.

(i). $\mathrm{Ker}L=\left\{x\in R^{T+\delta+\tau+1}:x_c(n)=\begin{cases}0,\ n\in[-\tau,\dots,-1],\\ c,\ n\in[0,T+1], & c\in R\\ 0,\ n\in[T+2,\dots,T+\delta],\end{cases}\right\}$ $\times\{(0,\dots,0)\in R^{T+1}\}$.

(ii). $\mathrm{Im}L=\left\{(u,v)\in R^{T+1}\times R^T:\ \sum_{n=0}^{T-1}v(n)=0\right\}$.

(iii). L is a Fredholm operator of index zero.

(iv). There are projectors $P:X\to X$ and $Q:Y\to Y$ such that $\mathrm{Ker}L=\mathrm{Im}P$, $\mathrm{Ker}Q=\mathrm{Im}L$. Furthermore, let $\Omega\subset X$ be an open bounded subset with $\overline{\Omega}\cap\mathrm{Dom}L\neq\emptyset$, then N is L–compact on $\overline{\Omega}$.

The projectors $P : X \to X$ and $Q : Y \to Y$, the isomorphism $\wedge : \text{Ker}L \to Y/\text{Im}L$ and the generalized inverse $K_p : \text{Im}L \to \text{Dom}L \cap \text{Im}P$ are as follows:

$$
\begin{aligned}
P \bullet \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} &= \begin{pmatrix} x_0(n) \\ y_0(n) \end{pmatrix}, \\
x_0(n) &= \begin{cases} 0, & n \in [-\tau, -1], \\ x(0), & n \in [0, T+1], \\ 0, & n \in [T+2, T+\delta], \end{cases} \qquad y_0(n) = 0, \text{ for } n \in [0, T], \\
Q \bullet \begin{pmatrix} u(n) \\ v(n) \end{pmatrix} &= \begin{pmatrix} u_0(n) \\ v_0(n) \end{pmatrix}, \\
u_0(n) &= 0, \ n \in [0, T], \ \ v_0(n) = \frac{1}{T} \sum_{n=0}^{T-1} y(n), \quad n \in [0, T-1], \\
\wedge \begin{pmatrix} x_c \\ 0 \end{pmatrix} &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \\
(x_c, 0) &= (\overbrace{0, \dots, 0}^{\tau}, \overbrace{c, \dots, c}^{T+2}, \overbrace{0, \dots, 0}^{\delta-1}, \overbrace{0, \dots, 0}^{T+1}) \in \text{Ker}L, \\
u_0(n) &= 0, \ n \in [0, T], \ v_0(n) = c, \ n \in [0, T-1], \\
K_p \bullet \begin{pmatrix} u(n) \\ v(n) \end{pmatrix} &= \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}, \\
x(n) &= \begin{cases} 0, & n \in [-\tau, -1], \\ \sum_{j=0}^{n-1} u(j), & n \in [0, T+1], \\ 0, & n \in [T+2, T+\delta], \end{cases} \\
y(n) &= \sum_{j=0}^{n-1} v(j), n \in [0, T-1].
\end{aligned}
$$

Suppose
(B)**.** Let

$$
x_{\tau_i,c,0}(n) = \begin{cases} \gamma(n - \tau_i(n)), & n - \tau_i(n) \in [-\tau, -1], \\ \psi(n - \tau_i(n)), & n - \tau_i(n) \in [T+2, T+\delta], \\ c, & n - \tau_i(n) \in [0, T+1]. \end{cases}
$$

There is a constant $M > 0$ such that

$$
c \left[\sum_{n=0}^{T-1} f(n, c, x_{\tau_1,c,0}(n), \dots, x_{\tau_m,c,0}(n)) \right] > 0
$$

for all $|c| > M$ or

$$c\left[\sum_{n=0}^{T-1} f(n, c, x_{\tau_1,c,0}(n), \ldots, x_{\tau_m,c,0}(n))\right] < 0$$

for all $|c| > M$.

Theorem L. *Suppose that* (B) *holds and that there exist numbers* $\beta > 0$, $\theta > 1$, *nonnegative sequences* $p_i(n), r(n) (i = 0, \ldots, m)$, *functions* $g(n, x_0, \ldots, x_m)$, $h(n, x_0, \ldots, x_m)$ *such that* $f(n, x_0, \ldots, x_m) = g(n, x_0, \ldots, x_m) + h(n, x_0, \ldots, x_m)$ *and*

$$g(n, x_0, x_1, \ldots, x_m)x_0 \geq \beta|x_0|^{\theta+1},$$

and

$$|h(n, x_0, \ldots, x_m)| \leq \sum_{s=0}^{m} p_i(n)|x_i|^{\theta} + r(n),$$

for all $n \in \{1, \ldots, T\}$, $(x_0, x_1, \ldots, x_m) \in R^{m+1}$. *Then problems* (1) *and* (2) *have at least one solution if*

$$||p_0|| + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| < \beta. \tag{7}$$

Proof. To apply Theorem GM, we want to define an open bounded subset Ω of X such that (i), (ii) and (iii) of Theorem GM hold.

Step 1. Let $\Omega_1 = \{(x, y) : L(x, y) = \lambda N(x, y),\ ((x, y), \lambda) \in [(\mathrm{Dom}L \setminus \mathrm{Ker}L)] \times (0, 1)\}$. For $x \in \Omega_1$, we have $L \bullet (x, y) = \lambda N \bullet (x, y)$, $\lambda \in (0, 1)$, so

$$\begin{cases} \Delta x(n) = \lambda\phi^{-1}(y(n)) \\ \Delta y(n) = \lambda f(n, w(n+1), w(n - \tau_1(n)), \ldots, w(n - \tau_m(n))). \end{cases} \tag{8}$$

So

$$\Delta[\phi(\Delta w(n))]w(n+1) = \lambda\phi(\lambda) f(n, w(n+1), w(n - \tau_1(n)), \ldots, \\ w(n - \tau_m(n))w(n+1).$$

It is easy to see from (2) and the definition of $w(n) = x(n) + x_0(n)$ that

$$
\begin{aligned}
&\sum_{n=0}^{T-1} \Delta[\phi(\Delta w(n))]w(n+1) \\
&= \sum_{n=0}^{T-1} [\phi(\Delta w(n+1)) - \phi(\Delta w(n))][w(n+2) - \Delta w(n+1)] \\
&= \sum_{n=0}^{T-1} [\phi(\Delta w(n+1))w(n+2) - \phi(\Delta w(n))w(n+1)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sum_{n=0}^{T-1} \phi(\Delta w(n+1))\Delta w(n+1) \\
&= \phi(\Delta w(T))w(T+1) - \phi(\Delta w(0))w(1) - \sum_{n=0}^{T-1} \phi(\Delta w(n+1))\Delta w(n+1) \\
&= -\sum_{n=0}^{T-1} \phi(\Delta w(n+1))\Delta w(n+1) \leq 0.
\end{aligned}
$$

So, we get

$$
\sum_{n=0}^{T-1} f(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1) \leq 0.
$$

It follows that

$$
\begin{aligned}
&\beta \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \\
&\leq \sum_{n=0}^{T-1} g(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1) \\
&\leq -\sum_{n=0}^{T-1} h(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))w(n+1) \\
&\leq \sum_{n=0}^{T-1} |h(n, w(n+1), w(n-\tau_1(n)), \ldots, w(n-\tau_m(n))| \, |w(n+1)| \\
&\leq \sum_{n=0}^{T-1} p_0(n)|w(n+1)|^{\theta+1} + \sum_{i=1}^{m} \sum_{n=0}^{T-1} p_i(n)|w(n-\tau_i(n))|^{\theta}|w(n+1)|
\end{aligned}
$$

$$+\sum_{n=0}^{T-1} r(n)|w(n+1)|$$

$$\leq ||p_0||\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}+\sum_{i=1}^{m}||p_i||\sum_{n=0}^{T-1}|w(n-\tau_i(n))|^{\theta}|w(n+1)|$$

$$+||r||\sum_{n=0}^{T-1}|w(n+1)|.$$

For $x_i \geq 0$, $y_i \geq 0$, the Holder's inequality implies

$$\sum_{i=1}^{s} x_i y_i \leq \left(\sum_{i=1}^{s} x_i^p\right)^{1/p}\left(\sum_{i=1}^{s} y_i^q\right)^{1/q}, \quad 1/p+1/q=1, \quad q>0, \ p>0.$$

It follows that

$$\beta\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}$$

$$\leq ||p_0||\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

$$+\sum_{i=1}^{m}||p_i||\left(\sum_{n=0}^{T-1}|w(n-\tau_i(n))|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

$$=||p_0||\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

$$+\sum_{i=1}^{m}||p_i||\left(\sum_{\substack{u\in\{n-\tau_i(n)-1:\\ n=0,\dots,T-1\}}}|w(u+1)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

$$\leq ||p_0||\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}+T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||$$

$$\times\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}+\sum_{n=T}^{T+\delta}|\psi(n+1)|^{\theta+1}+\sum_{n=-\tau}^{-1}|\gamma(n+1)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}$$

$$\times\left(\sum_{n=0}^{T-1}|w(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}$$

$$= ||p_0|| \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1} \right)^{\frac{1}{\theta+1}} + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||$$

$$\times \left(1 + \frac{\sum_{n=T}^{T+\delta} |\psi(n+1)|^{\theta+1} + \sum_{n=-\tau}^{-1} |\gamma(n+1)|^{\theta+1}}{\sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}} \right)^{\frac{\theta}{\theta+1}} \sum_{n=0}^{T-1} |w(n+1)|^{\theta+1}.$$

It follows from (4) that there is $M_1 > 0$ such that $\sum_{u=0}^{T-1} |w(u+1)|^{\theta+1} \le M_1$.

Hence $|w(n+1)| \le (M_1/T)^{1/(\theta+1)}$ for all $n \in \{0, \dots, T-1\}$. Thus we get

$$|x(n+1)| \le |w(n+1)| + |x_0(n+1)| \le (M_1/T)^{1/(\theta+1)} + ||x_0||, \quad n \in [0, \dots, T-1].$$

Hence $||x|| \le (M_1/T)^{1/(\theta+1)} + ||x_0||$. Then

$$|y(n)| = \left| \sum_{s=0}^{n-1} \Delta y(s) \right| \le \sum_{s=0}^{n-1} |\Delta y(s)| \le T \max_{\substack{n \in [0, T-1], \\ |w_i| \le (M_1/T)^{1/(\theta+1)}, \\ i=0,\dots,m}} |f(n, w_0, \dots, w_m)|.$$

It follows that

$$||y|| \le \sum_{s=0}^{n-1} |\Delta y(s)| \le T \max_{\substack{n \in [0, T-1], |w_i| \le (M_1/T)^{1/(\theta+1)}, \\ i=0,\dots,m}} |f(n, x_0, \dots, x_m)|.$$

So Ω_1 is bounded. This completes Step 1.

Step 2. Prove that the set $\Omega_2 = \{(x, y) \in \mathrm{Ker}L : \ N(x, y) \in \mathrm{Im}L\}$ is bounded.

For $(x, y) \in \mathrm{Ker}L$, we have $x(n) = (\overbrace{0, \dots, 0}^{\tau}, \overbrace{c, \dots, c}^{T+2}, \overbrace{0, \dots, 0}^{\delta-1})$ and $y(n) = 0$ for $n \in [0, T]$. Thus we have

$$N \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} \phi^{-1}(y(n)) \\ f(n, w(n+1), w(n-\tau_1(n)), \dots, w(n-\tau_m(n))) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ f(n, c, x_{\tau_1,c,0}, \dots, x_{\tau_m,c,0}) \end{pmatrix},$$

where

$$x_{\tau_i,c,0} = \begin{cases} \gamma(n-\tau_i(n)), & n-\tau_i(n) \in [-\tau, -1], \\ \psi(n-\tau_i(n)), & n-\tau_i(n) \in [T+2, T+\delta], \\ c, & n-\tau_i(n) \in [0, T+1]. \end{cases}$$

$N(x, y) \in \mathrm{Im}L$ implies that

$$\sum_{n=0}^{T-1} f(n, c, x_{\tau_1,c,0}, \dots, x_{\tau_m,c,0}) = 0.$$

It follows from condition (B) that $|c| \leq M$. Thus Ω_2 is bounded.

Step 3. Prove that the set $\Omega_3 = \{(x,y) \in \mathrm{Ker}L : \pm\lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}$ is bounded.

If the first inequality of (B) holds, let

$$\Omega_3 = \{(x,y) \in \mathrm{Ker}L : \lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}.$$

We will prove that Ω_3 is bounded. For $x(n) = (\overbrace{0,\ldots,0}^{\tau}, \overbrace{c,\ldots,c}^{T+2}, \overbrace{0,\ldots,0}^{\delta-1})$ and $y(n) = 0$ for $n \in [0,T]$ such that $(x,y) \in \Omega_3$, and $\lambda \in [0,1]$, we have

$$-(1-\lambda)\sum_{n=0}^{T-1} f(n, c, x_{\tau_1,c,0}, \ldots, x_{\tau_m,c,0}) = \lambda c T.$$

If $\lambda = 1$, then $c = 0$. If $\lambda \neq 1$, then

$$0 \geq -(1-\lambda)c\sum_{n=0}^{T-1} f(n, c, x_{\tau_1,c,0}, \ldots, x_{\tau_m,c,0}) = \lambda c^2 T > 0,$$

is a contradiction.

If the second inequality of (B) holds, let

$$\Omega_3 = \{(x,y) \in \mathrm{Ker}L : -\lambda \wedge (x,y) + (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\},$$

Similarly, we can get a contradiction. So Ω_3 is bounded.

Step 4. Obtain an open bounded set Ω such that (i), (ii) and (iii) in Theorem GM hold.

In the following, we shall show that all conditions of Theorem GM are satisfied. Set Ω be an open bounded subset of X such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$. We know that L is a Fredholm operator of index zero and N is L–compact on $\overline{\Omega}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus $L(x,y) \neq \lambda N(x,y)$ for $x \in (\mathrm{Dom}L) \setminus \mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $N(x,y) \notin \mathrm{Im}L$ for $(x,y) \in \mathrm{Ker}L \cap \partial\Omega$.

In fact, let $H(x,\lambda) = \pm\lambda \wedge (x,y) + (1-\lambda)QN(x,y)$. According the definition of Ω, we know that $\Omega \supset \overline{\Omega_3}$, thus $H((x,y),\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$, thus by the homotopy property of degree,

$$\begin{aligned}\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\pm\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0\end{aligned}$$

since $0 \in \Omega$. Thus by Theorem GM, $L(x,y) = N(x,y)$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$, which is a solution of problem (1) and (2). The proof is completed. □

3. An Example

In this section, we present an example to illustrate the main results in Section 2.

Example 3.1. Consider the following problem

$$\begin{cases} \Delta[(\Delta x(n))^3] = \beta[x(n+1)]^{2k+1} + \sum_{i=1}^{m} p_i(n)[x(n-i)]^{2k+1} + r(n), \text{for} \\ n \in [0, T-1], \\ \Delta x(0) = \Delta x(T) = 0, \\ x(i) = i, \ \ i \in [-m, -1], \end{cases} \tag{9}$$

where k is a nonnegative integer, $\beta > 0$, $p_i(n), r(n)$ are sequences.

Corresponding to the assumptions of Theorem L, we set $\phi(x) = |x|^2 x$, $g(n, x_0, \dots, x_m) = \beta[x_0]^{2k+1}$, $h(, x_0, \dots, x_m) = \sum_{i=1}^{m} p_i(n) x_i^{2k+1} + r(n)$ with $\theta = 2k+1$. It is easy to see that assumptions in Theorem L hold, and

$$f(n, c, x_{\tau_1,c,0}, \dots, x_{\tau_m,c,0}) = c^{2k+1}\beta + \sum_{i=1}^{m} p_i(n)(n-i)^{2k+1} + r(n)$$

implies that there is $M > 0$ such that $c \sum_{n=0}^{T-1} f(n, c, x_{\tau_1,c,0}, \dots, x_{\tau_m,c,0}) > 0$ for all $n \in Z$ and $|c| > M$.

It follows from Theorem L that (9) has at least one solution if

$$||p_0|| + T^{\frac{2k+1}{2k+2}} \sum_{i=1}^{m} ||p_i|| < \beta.$$

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