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ON WEAKLY W_2-SYMMETRIC MANIFOLDS

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ABSTRACT. In the present paper we introduce a type of non-flat Riemannian manifold called *weakly W_2-symmetric manifolds* and study their geometric properties. The existence of such manifolds is shown by several non-trivial examples.

1. INTRODUCTION

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by L. Tamássy and T. Q. Binh [8]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) &+ \beta(Y)R(X, Z, U, V)\\ &+ \gamma(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V)\\ &+ \sigma(V)R(Y, Z, U, X) \qquad (1.1)\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. Recently U. C. De and S. Bandyopadhyay [2] established the existence of a $(WS)_n$ by an example and proved that in a $(WS)_n$, the associated 1-forms $\beta = \gamma$ and $\delta = \sigma$. Hence (1.1) reduces to the following form:

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) &+ \beta(Y)R(X, Z, U, V)\\ &+ \beta(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V)\\ &+ \delta(V)R(Y, Z, U, X) \qquad (1.2)\end{aligned}$$

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Also De and Bandyopadhyay [3] studied weakly conformally symmetric manifolds. Then the first author and his co-author [7] introduced the notion of weakly quasi-conformally symmetric manifold with several examples. In this connection it may be mentioned that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric manifold introduced by Chaki [1], the defining condition of a $(WS)_n$ is little weaker than that of a generalized pseudo-symmetric manifold. That is, if in (1.1) the 1-form α is replaced by 2α and σ is replaced by α then the manifold will be a generalized pseudo-symmetric manifold [1].

In 1970 G. P. Pokhariyal and R. S. Mishra [5] introduced the notion of a new curvature tensor, denoted by W_2 and studied its relativistic significance. The W_2-curvature tensor of type $(0, 4)$ is defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)] \tag{1.3}$$

where S is the Ricci tensor of type (0, 2) and r is the scalar curvature of the manifold.

The present paper deals with a Riemannian manifold $(M^n, g)(n > 2)$ (the condition $(n > 2)$ is assumed throughout this paper) whose W_2-curvature tensor is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_X W_2)(Y,Z,U,V) = {} & \alpha(X)W_2(Y,Z,U,V) + \beta(Y)W_2(X,Z,U,V) \\ & + \gamma(Z)W_2(Y,X,U,V) + \delta(U)W_2(Y,Z,X,V) \\ & + \sigma(V)W_2(Y,Z,U,X),\end{aligned} \tag{1.4}$$

where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (not simultaneously zero). Such a manifold will be called a *"weakly W_2-symmetric manifold"* and denoted by $(WW_2S)_n$; where the first 'W' stands for *' weakly'* and 'W_2' stands for the '*W_2-curvature tensor'*.

Section 2 is concerned with preliminaries. It is shown that in a $(WW_2S)_n$ the associated 1-forms $\beta = \gamma$ but $\delta \neq \sigma$ and hence the defining condition (1.4) of a $(WW_2S)_n$ takes the following form:

$$\begin{aligned}(\nabla_X W_2)(Y,Z,U,V) = {} & \alpha(X)W_2(Y,Z,U,V) + \beta(Y)W_2(X,Z,U,V) \\ & + \beta(Z)W_2(Y,X,U,V) + \delta(U)W_2(Y,Z,X,V) \\ & + \sigma(V)W_2(Y,Z,U,X)\end{aligned} \tag{1.5}$$

where α, β, δ and σ are 1-forms (not simultaneously zero).

In Section 3 we investigate the nature of scalar curvature of a $(WW_2S)_n$. It is proved that if in a $(WW_2S)_n$ the Ricci tensor is of Codazzi type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X, L) = \lambda(X)$. Also it is shown that in a $(WW_2S)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector associated with the 1-form β. Every $(WS)_n$ is a $(WW_2S)_n$ but not conversely. However we obtain a sufficient condition for a $(WW_2S)_n$ to be a $(WS)_n$.

Section 4 deals with an Einstein $(WW_2S)_n$ and it is proved that if such a manifold is $(WS)_n$ then the scalar curvature vanishes provided that $\alpha+\beta+\delta$ is non-vanishing everywhere. Also a sufficient condition for an Einstein $(WW_2S)_n$ to be a $(WS)_n$ is obtained. Finally it is shown that if the vector field L defined by $g(X,L) = \lambda(X)$ is a concurrent vector field in an Einstein $(WW_2S)_n$ then it reduces to a $(WS)_n$. The last section provides the existence of a $(WW_2S)_n$ with several non-trivial examples of both vanishing and non-vanishing scalar curvature.

2. Preliminaries

In this section we derive some formulas which will be needed to the study of a $(WW_2S)_n$. Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then from (1.3) we have the following:

$$\sum_{i=1}^{n} W_2(e_i, Y, Z, e_i) = \frac{n}{n-1}[S(Y,Z) - \frac{r}{n} g(Y,Z)] = \frac{n}{n-1} P(Y,Z), \quad (2.1)$$

where $P(Y,Z) = S(Y,Z) - \frac{r}{n} g(Y,Z)$,

$$\sum_{i=1}^{n} W_2(X, Y, e_i, e_i) = 0 = \sum_{i=1}^{n} W(e_i, e_i, Z, U), \quad (2.2)$$

$$\sum_{i=1}^{n} W_2(X, e_i, e_i, U) = 0. \quad (2.3)$$

Also from (1.3) it follows that

$$\begin{aligned} &\text{(i)} \quad W_2(X,Y,Z,U) = -W_2(Y,X,Z,U), \\ &\text{(ii)} \quad W_2(X,Y,Z,U) \neq -W_2(X,Y,U,Z), \\ &\text{(iii)} \quad W_2(X,Y,Z,U) \neq W_2(Z,U,X,Y), \\ &\text{(iv)} \quad W_2(X,Y,Z,U) + W_2(Y,Z,X,U) + W_2(Z,X,Y,U) = 0. \quad (2.4) \end{aligned}$$

In view of (1.3) we obtain by virtue of Bianchi identity that

$$\begin{aligned} &(\nabla_X W_2)(Y,Z,U,V) + (\nabla_Y W_2)(Z,X,U,V) + (\nabla_Z W_2)(X,Y,U,V) \\ &= \frac{1}{n-1}[g(Y,U)(\nabla_X S)(Z,V) - g(Z,U)(\nabla_X S)(Y,V) \\ &+ g(Z,U)(\nabla_Y S)(X,V) - g(X,U)(\nabla_Y S)(Z,V) \\ &+ g(X,U)(\nabla_Z S)(Y,V) - g(Y,U)(\nabla_Z S)(X,V)]. \quad (2.5) \end{aligned}$$

We now suppose that in a Riemannian manifold the Ricci tensor is of Codazzi type [4]. Then we have

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) = (\nabla_Z S)(X,Y)$$

for all vector fields X, Y, Z on the manifold. Hence (2.5) yields

$$(\nabla_X W_2)(Y,Z,U,V) + (\nabla_Y W_2)(Z,X,U,V) + (\nabla_Z W_2)(X,Y,U,V) = 0. \tag{2.6}$$

Conversely, if a Riemannian manifold satisfies the relation (2.6), then (2.5) yields

$$g(Y,U)(\nabla_X S)(Z,V) - g(Z,U)(\nabla_X S)(Y,V) + g(Z,U)(\nabla_Y S)(X,V)$$
$$- g(X,U)(\nabla_Y S)(Z,V) + g(X,U)(\nabla_Z S)(Y,V) - g(Y,U)(\nabla_Z S)(X,V) = 0.$$

Setting $Y = V = e_i$ in the above relation and then taking summation over i, $1 \leq i \leq n$ we get

$$[(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)] - \frac{1}{2}[dr(X)g(Z,U) - dr(Z)g(X,U)] = 0,$$

which yields on contraction over Z and U that $dr(X) = 0$ for all X and consequently the last relation reduces to

$$(\nabla_X S)(Z,U) = (\nabla_Z S)(X,U)$$

for all $X, Z, U \in \chi(M)$. Hence the Ricci tensor is of Codazzi type. Thus we can state the following:

Proposition 2.1. *In a Riemannian manifold* (M^n, g) $(n > 2)$, *the Ricci tensor is of Codazzi type if and only if the relation* (2.6) *holds.*

Proposition 2.2. *The defining condition of a* $(WW_2S)_n$ *can always be expressed in the form* (1.5).

Proof. Interchanging Y and Z in (1.4) we get

$$\begin{aligned}(\nabla_X W_2)(Z,Y,U,V) = {} & \alpha(X)W_2(Z,Y,U,V) + \beta(Z)W_2(X,Y,U,V) \\ & + \gamma(Y)W_2(Z,X,U,V) + \delta(U)W_2(Z,Y,X,V) \\ & + \sigma(V)W_2(Z,Y,U,X).\end{aligned} \tag{2.7}$$

Adding (1.4) and (2.7) we obtain by virtue of (2.4)(i) that

$$A(Y)W_2(X,Z,U,V) + A(Z)W_2(X,Y,U,V) = 0, \tag{2.8}$$

where $A(X) = \beta(X) - \gamma(X)$ for all X.
If we now choose a particular vector field ρ such that $A(\rho) \neq 0$ then putting $Y = Z = \rho$ in (2.8) we get $W_2(X, \rho, U, V) = 0$.
Again setting $Z = \rho$ in (2.8) we obtain $W_2(X,Y,U,V) = 0$ for all vector fields X, Y, U and V, which contradicts to our assumption that the manifold is not W_2-flat. Hence we must have $A(X) = 0$ for all X and consequently

$\beta(X) = \gamma(X)$ for all X.

But in view of (2.4)(ii) it follows that the relation $\delta = \sigma$ does not hold in a $(WW_2S)_n$. Hence the defining condition of a $(WW_2S)_n$ can be written as (1.5). This proves the proposition. □

3. Nature of scalar curvature of a $(WW_2S)_n$

Let Q be the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the Ricci tensor S i.e., $g(QX,Y) = S(X,Y)$.

In view of (1.5), the relation (2.6) reduces to the following:

$$\begin{aligned}&\lambda(X)W_2(Y,Z,U,V) + \lambda(Y)W_2(Z,X,U,V) + \lambda(Z)W_2(X,Y,U,V)\\&+ \sigma(V)[W_2(Y,Z,U,X) + W_2(Z,X,U,Y) + W_2(X,Y,U,Z)] = 0 \end{aligned}\tag{3.1}$$

where $\lambda(X) = \alpha(X) - 2\beta(X)$ for all X.

Setting $Y = V = e_i$ in (3.1) and taking summation over i, $1 \le i \le n$, we get by virtue of (2.1) and (2.4) that

$$\frac{n}{n-1}[\lambda(X)P(Z,U) - \lambda(Z)P(X,U)] + \lambda(W_2(Z,X)U) = 0. \tag{3.2}$$

Again putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \le i \le n$, we get

$$\lambda(QZ) = \frac{r}{n}\lambda(Z). \tag{3.3}$$

That is,

$$S(Z,L) = \frac{r}{n}g(Z,L).$$

This leads to the following:

Theorem 3.1. *If in a $(WW_2S)_n$ the Ricci tensor is of Codazzi type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X,L) = \lambda(X)$ for all X.*

Next by virtue of (1.5), the relation (2.5) takes the form

$$\begin{aligned}&\lambda(X)W_2(Y,Z,U,V) + \lambda(Y)W_2(Z,X,U,V) + \lambda(Z)W_2(X,Y,U,V)\\&+ \sigma(V)[W_2(Y,Z,U,X) + W_2(Z,X,U,Y) + W_2(X,Y,U,Z)]\\&= \frac{1}{n-1}[\{(\nabla_Y S)(X,V) - (\nabla_X S)(Y,V)\}g(Z,U) + \{(\nabla_Z S)(Y,V)\\&- (\nabla_Y S)(Z,V)\}g(X,U) + \{(\nabla_X S)(Z,V) - (\nabla_Z S)(X,V)\}g(Y,U)]. \end{aligned}\tag{3.4}$$

Setting $Y = V = e_i$ in (3.4) and taking summation over i, $1 \le i \le n$, we obtain

$$\frac{n}{n-1}[\lambda(X)P(Z,U) - \lambda(Z)P(X,U)] + \lambda(W_2(Z,X)U)$$

$$= \frac{1}{n-1}[(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)$$
$$+ \frac{1}{2}\{dr(Z)g(X,U) - dr(X)g(Z,U)\}]. \tag{3.5}$$

Putting $X = U = e_i$ in (3.5) and taking summation over i, $1 \le i \le n$, we get

$$\frac{n-2}{2n}dr(Z) = \lambda(QZ) - \frac{r}{n}\lambda(Z). \tag{3.6}$$

If the manifold is of constant scalar curvature then (3.6) reduces to (3.3). Hence we can state the following:

Theorem 3.2. *If a $(WW_2S)_n$ is of constant scalar curvature then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X,L) = \lambda(X)$ for all X.*

Using (1.3) in the left hand side of (1.5) yields

$$(\nabla_X R)(Y,Z,U,V) + \frac{1}{n-1}[(\nabla_X S)(Z,V)g(Y,U) - (\nabla_X S)(Y,V)g(Z,U)]$$
$$= \alpha(X)W_2(Y,Z,U,V) + \beta(Y)W_2(X,Z,U,V) + \beta(Z)W_2(Y,X,U,V)$$
$$+ \delta(U)W_2(Y,Z,X,V) + \sigma(V)W_2(Y,Z,U,X). \tag{3.7}$$

Setting $X = V = e_i$ in (3.7) and taking summation over i, $1 \le i \le n$, we have

$$(divR)(Y,Z)U + \frac{1}{2(n-1)}[dr(Z)g(Y,U) - dr(Y)g(Z,U)]$$
$$= \alpha(W_2(Y,Z)U) + \sigma(W_2(Y,Z)U) + \frac{1}{n-1}\beta(Y)[nS(Z,U)$$
$$- rg(Z,U)] - \frac{1}{n-1}\beta(Z)[nS(Y,U) - rg(Y,U)], \tag{3.8}$$

where '*div*' denotes the divergence.
Again replacing Z and U by e_i in (3.8) and taking summation over i, $1 \le i \le n$, we get by virtue of (2.2) and (2.3) that

$$\beta(QY) = \frac{r}{n}\beta(Y),$$

which can be written as

$$S(Y,\rho_2) = \frac{r}{n}g(Y,\rho_2), \tag{3.9}$$

where $S(Y,\rho_2) = \beta(QY)$ and $g(Y,\rho_2) = \beta(Y)$. This leads to the following:

Theorem 3.3. *In a $(WW_2S)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2 defined by $g(X,\rho_2) = \beta(X)$ for all X.*

We now obtain a sufficient condition for a $(WW_2S)_n$ to be a $(WS)_n$. Let us consider a $(WW_2S)_n$ such that the Ricci tensor vanishes i.e., $S(X,Y) = 0$ for all X and Y. Then from (1.3) it follows that

$$W_2(X,Y,Z,U) = R(X,Y,Z,U). \tag{3.10}$$

By virtue of (1.5) and (3.10) it can be easily seen that the relation (1.2) holds. Thus we have the following:

Theorem 3.4. *If in a $(WW_2S)_n$ the Ricci tensor vanishes then it is a $(WS)_n$.*

4. Einstein $(WW_2S)_n$

This section deals with a $(WW_2S)_n$ which is an Einstein manifold. So we have

$$S(X,Y) = \frac{r}{n} g(X,Y) \tag{4.1}$$

from which it follows that

$$dr(X) = 0 \quad \text{and} \quad (\nabla_Z S)(X,Y) = 0 \quad \text{for all} \quad X,Y,Z. \tag{4.2}$$

Using (4.1) and (4.2) in (1.3) we have

$$(\nabla_X W_2)(Y,Z,U,V) = (\nabla_X R)(Y,Z,U,V). \tag{4.3}$$

In view of (1.5), the relation (4.3) takes the form

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\ &+ \beta(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\ &+ \sigma(V)R(Y,Z,U,X) + \frac{1}{n-1}[\alpha(X)\{g(Y,U)S(Z,V) \\ &- g(Z,U)S(Y,V)\} + \beta(Y)\{g(X,U)S(Z,V) \\ &- g(Z,U)S(X,V)\} + \beta(Z)\{g(Y,U)S(X,V) \\ &- g(X,U)S(Y,V)\} + \delta(U)\{g(X,Y)S(Z,V) \\ &- g(Z,X)S(Y,V)\} + \sigma(V)\{g(Y,U)S(Z,X) \\ &- g(Z,U)S(X,Y)\}].\end{aligned} \tag{4.4}$$

Now if the Einstein $(WW_2S)_n$ is a $(WS)_n$, then using (1.2) and (4.1) in (4.4) we obtain

$$\begin{aligned}&[\delta(V) - \sigma(V)]R(Y,Z,U,X) \\ &= \frac{r}{n(n-1)}[\alpha(X)\{g(Y,U)g(Z,V) - g(Z,U)g(Y,V)\} \\ &+ \beta(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\} \\ &+ \beta(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}\end{aligned}$$

$$+ \delta(U)\{g(X,Y)g(Z,V) - g(Z,X)g(Y,V)\}$$
$$+ \sigma(V)\{g(Y,U)g(Z,X) - g(Z,U)g(X,Y)\}]. \tag{4.5}$$

Setting $X = U = e_i$ in (4.5) and then taking summation over i, $1 \le i \le n$ we get

$$r[\alpha(Y)g(Z,V) - \alpha(Z)g(Y,V) + (n-1)\{\beta(Y)g(Z,V) - \beta(Z)g(Y,V)\}$$
$$+ \delta(Y)g(Z,V) - \delta(Z)g(Y,V)] = 0. \tag{4.6}$$

Further putting $Z = V = e_i$ in (4.6) and taking summation over i, $1 \le i \le n$, we have

$$r[\alpha(Y) + (n-1)\beta(Y) + \delta(Y)] = 0. \tag{4.7}$$

Again contracting (4.5) over X and Y we obtain

$$r[\alpha(U)g(Z,V) - \alpha(V)g(Z,U) + \beta(U)g(Z,V) - \beta(V)g(Z,U)$$
$$+ (n-1)\{\delta(U)g(Z,V) - \delta(V)g(Z,U)\}] = 0, \tag{4.8}$$

which yields, on further contraction with respect to Z and V,

$$r[\alpha(U) + \beta(U) + (n-1)\delta(U)] = 0 \quad \text{for all} \quad U.$$

Replacing U by Y in the above equation we have

$$r[\alpha(Y) + \beta(Y) + (n-1)\delta(Y)] = 0. \tag{4.9}$$

Also replacing Y and V by e_i in (4.5) and taking summation over i, $1 \le i \le n$, we obtain

$$\sigma(R(X,U)Z) - \delta(R(X,U)Z) = \frac{r}{n(n-1)}[(n-1)\alpha(X)g(Z,U) + \beta(X)g(Z,U)$$
$$+ (n-2)\beta(Z)g(X,U) + (n-1)\delta(U)g(Z,X)$$
$$- \sigma(U)g(Z,X) + \sigma(X)g(Z,U)],$$

which yields, on contraction with respect to Z and U, that

$$r[n\alpha(X) + 2\beta(X) + 2\delta(X)] = 0 \quad \text{for all} \quad X.$$

Interchanging X and Y in the above equation we have

$$r[n\alpha(Y) + 2\beta(Y) + 2\delta(Y)] = 0. \tag{4.10}$$

Adding (4.7), (4.9) and (4.10) we obtain

$$r = 0, \quad \text{if} \quad \alpha(Y) + \beta(Y) + \delta(Y) \neq 0.$$

This leads to the following:

Theorem 4.1. *If an Einstein $(WW_2S)_n$ is a $(WS)_n$ then the scalar curvature of the manifold vanishes provided that $\alpha + \beta + \delta$ is not zero everywhere on the manifold.*

Again in an Einstein $(WW_2S)_n$ if $r = 0$, then we have $S(X,Y) = 0$ for all X, Y and hence (4.4) yields

$$(\nabla_X R)(Y,Z,U,V) = \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \tag{4.11}$$
$$+ \beta(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V)$$
$$+ \sigma(V)R(Y,Z,U,X). \tag{4.12}$$

Thus we can state the following:

Theorem 4.2. *An Einstein $(WW_2S)_n$ with vanishing scalar curvature is a $(WS)_n$.*

Definition 4.1. *In a Riemannian manifold a vector field P is said to be parallel if it satisfies the following condition:*

$$\nabla_X P = 0 \quad \textit{for all} \;\; X.$$

Let us now consider an Einstein $(WW_2S)_n$ in which the vector field L defined by $g(X,L) = \alpha(X) - 2\beta(X)$ is a parallel vector field. Then we have

$$\nabla_X L = 0 \quad \text{for all} \quad X. \tag{4.13}$$

Therefore, using Ricci identity we get

$$R(X,Y,L,U) = 0, \quad \text{which yields}$$
$$S(Y,L) = 0. \tag{4.14}$$

From (4.14) and (3.9), it follows that $r = 0$ if $||L||^2 \neq 0$.
Again, if $r = 0$ then (4.1) implies that $S(X,Y) = 0$ and consequently from (4.4), it follows that the manifold is a $(WS)_n$. Thus we can state the following:

Theorem 4.3. *If in an Einstein $(WW_2S)_n$ the vector field L defined by $g(X,L) = \lambda(X)$ is a parallel vector field , then it is a $(WS)_n$ provided that $||L||^2 \neq 0$.*

Definition 4.2. *A vector field T on a Riemannian manifold is said to be concurrent* [6] *if $\nabla_X T = kX$ where k is a constant.*

In particular, if $k = 0$ then T is said to be a parallel vector field.
Next we suppose that in an Einstein $(WW_2S)_n$ the vector field L defined by $g(X,L) = \lambda(X) = \alpha(X) - 2\beta(X)$ is a concurrent vector field.
Then we have

$$\nabla_X L = kX \quad \text{where} \quad k \quad \text{is a constant.} \tag{4.15}$$

Making use of Ricci identity we have

$$R(X,Y,L,U) = 0 \quad \text{which implies that}$$
$$S(Y,L) = 0 \quad \text{for all} \quad Y. \tag{4.16}$$

Now (4.16) yields $r = 0$ provided that $||L||^2 \neq 0$.
Thus arguing as in the case of parallel vector field we obtain that the manifold under consideration is a $(WS)_n$. Hence we can state the following:

Theorem 4.4. *If in an Einstein $(WW_2S)_n$ the vector field L defined by $g(X, L) = \lambda(X)$ is a concurrent vector field, then it is a $(WS)_n$ provided that $||L||^2 \neq 0$.*

5. Examples of $(WW_2S)_n$

This section deals with several examples of $(WW_2S)_n$. On the real number space R^n (with coordinates $x^1, x^2, \ldots, x^n$) we define suitable Riemannian metric g such that R^n becomes a Riemannian manifold (M^n, g). We calculate the components of the curvature tensor, the Ricci tensor, W_2-curvature tensor and its covariant derivatives and then we verify the defining relation (1.5).

Example 1. We define a Riemannian metric on the 4-dimensional real number space R^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (kx^1)^2 v(x^4)(dx^4)^2, \quad (i, j = 1, 2, 3, 4) \tag{5.1}$$

where $f = p_0 + p_1 x^3 + p_2(x^3)^2$, p_0, p_1 and p_2 are non-constant functions of x^1 only, v is a function of x^4 and k is a non-zero arbitrary constant.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^2_{11} = \frac{1}{2}f_{\cdot 1}, \quad \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_{\cdot 3}, \quad \Gamma^4_{14} = \frac{1}{x^1}, \quad \Gamma^2_{44} = -k^2x^1v,$$
$$\Gamma^4_{44} = \frac{(v)_{\cdot 4}}{2v}, \quad R_{1331} = \frac{1}{2}f_{\cdot 33}, \quad S_{11} = \frac{1}{2}f_{\cdot 33},$$

and the components which can be obtained from these by the symmetry properties. Here '.' denotes the partial differentiation with respect to the coordinates. Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore R^4 with the considered metric is a Riemannian manifold M^4 whose scalar curvature is zero. In view of the above relations the non-zero components of the W_2-curvature tensor and their covariant derivatives are obtained as follows:

$$(W_2)_{1211} = -\frac{1}{6}f_{\cdot 33} = -\frac{1}{3}p_2, \tag{5.2}$$

$$(W_2)_{1331} = \frac{1}{3}f_{\cdot 33} = \frac{2}{3}p_2, \tag{5.3}$$

$$(W_2)_{1313} = -\frac{1}{2}f_{\cdot 33} = -p_2, \tag{5.4}$$

$$(W_2)_{1441} = -\frac{1}{3}(kx^1)^2 v(x^4) p_2, \tag{5.5}$$

$$(W_2)_{1211,1} = -\frac{1}{3}(p_2)_{.1}, \tag{5.6}$$

$$(W_2)_{1331,1} = \frac{2}{3}(p_2)_{.1}, \tag{5.7}$$

$$(W_2)_{1313,1} = -(p_2)_{.1}, \tag{5.8}$$

$$(W_2)_{1441,1} = -\frac{1}{3}(kx^1)^2 v(x^4)(p_2)_{.1}, \tag{5.9}$$

and the components which can be obtained from (5.2)-(5.9) by the symmetry properties, where '.' denotes the covariant differentiation with respect to the metric tensor g. Hence our (M^4, g) is neither W_2-flat nor W_2-symmetric.

We shall now show that this M^4 is a $(WW_2S)_4$, i.e., it satisfies (1.5). Let us now consider the 1-forms

$$\begin{cases} \alpha_i(x) = \frac{(p_2)_{.1}}{3p_2} & \text{for } i = 1, \\ \qquad\quad = 0 & \text{otherwise}, \\ \beta_i(x) = \frac{2(p_2)_{.1}}{3p_2} & \text{for } i = 1, \\ \qquad\quad = 0 & \text{otherwise}, \\ \delta_i(x) \ = 0 & \text{for all } i, \\ \sigma_i(x) \ = 0 & \text{for all } i \end{cases} \tag{5.10}$$

at any point $x \in M$. In our M^4, (1.5) reduces with these 1-forms to the following equations

(i) $(W_2)_{1211,1} = \alpha_1(W_2)_{1211} + \beta_1(W_2)_{1211} + \beta_2(W_2)_{1111} + \delta_1(W_2)_{1211} + \sigma_1(W_2)_{1211}$,

(ii) $(W_2)_{1331,1} = \alpha_1(W_2)_{1331} + \beta_1(W_2)_{1331} + \beta_3(W_2)_{1131} + \delta_3(W_2)_{1311} + \sigma_1(W_2)_{1331}$,

(iii) $(W_2)_{1313,1} = \alpha_1(W_2)_{1313} + \beta_1(W_2)_{1313} + \beta_3(W_2)_{1113} + \delta_1(W_2)_{1313} + \sigma_3(W_2)_{1311}$,

(iv) $(W_2)_{1441,1} = \alpha_1(W_2)_{1441} + \beta_1(W_2)_{1441} + \beta_4(W_2)_{1141} + \delta_4(W_2)_{1411} + \sigma_1(W_2)_{1441}$,

since for the cases other than (i)-(iv) the components of each term of (1.5) vanishes identically and the relation (1.5) holds trivially. Now from (5.2) and (5.6) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

$$\begin{aligned} \text{R.H.S. of (i)} &= (\alpha_1 + \beta_1 + \delta_1 + \sigma_1)(W_2)_{1211} \\ &= -\frac{1}{3}(p_2)_{.1} \\ &= \text{L.H.S. of (i)}. \end{aligned}$$

By a similar argument as in (i) it can be shown that the relations (ii)-(iv) are true. Hence we can state the following:

Theorem 5.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + (kx^1)^2 v(x^4)(dx^4)^2, \quad (i, j = 1, 2, 3, 4)$$

where $f = p_0 + p_1 x^3 + p_2(x^3)^2$, p_0, p_1 and p_2 are non-constant functions of x^1 only, v is a function of x^4 and k is a non-zero arbitrary constant. Then (M^4, g) is a weakly W_2-symmetric manifold with vanishing scalar curvature which is neither W_2-symmetric nor W_2-recurrent.

In particular, if we take $p_2 = e^{x^1}$, then (5.2) to (5.9) are respectively reduce to the following

$$(W_2)_{1211} = -\frac{1}{3}e^{x^1}, \tag{5.11}$$

$$(W_2)_{1331} = \frac{2}{3}e^{x^1}, \tag{5.12}$$

$$(W_2)_{1313} = -e^{x^1}, \tag{5.13}$$

$$(W_2)_{1441} = -\frac{1}{3}(kx^1)^2 v(x^4)e^{x^1}, \tag{5.14}$$

$$(W_2)_{1211,1} = -\frac{1}{3}e^{x^1}, \tag{5.15}$$

$$(W_2)_{1331,1} = \frac{2}{3}e^{x^1}, \tag{5.16}$$

$$(W_2)_{1313,1} = -e^{x^1}, \tag{5.17}$$

$$(W_2)_{1441,1} = -\frac{1}{3}(kx^1)^2 v(x^4)e^{x^1}, \tag{5.18}$$

and hence the manifold under consideration is neither W_2-symmetric nor W_2-recurrent. If we consider the 1-forms

$$\begin{cases} \alpha_i(x) = \frac{1}{3} & \text{for } i = 1, \\ \qquad\ \ = 0 & \text{otherwise}, \\ \beta_i(x) = \frac{2}{3} & \text{for } i = 1, \\ \qquad\ \ = 0 & \text{otherwise}, \\ \delta_i(x) = 0 & \text{for all } i, \\ \sigma_i(x) = 0 & \text{for all } i \end{cases} \tag{5.19}$$

then proceeding similarly as the previous case it can be easily shown by virtue of (5.11)-(5.19) that the manifold under consideration satisfies (i)-(iv) and hence is a $(WW_2S)_4$. Thus we have the following:

Theorem 5.2. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (kx^1)^2 v(x^4)(dx^4)^2, \quad (i, j = 1, 2, 3, 4)$$

where $f = p_0 + p_1x^3 + e^{x^1}(x^3)^2$, p_0, p_1 are non-constant functions of x^1 only, v is a function of x^4 and k is a non-zero arbitrary constant. Then (M^4, g) is a weakly W_2-symmetric manifold with vanishing scalar curvature which is neither W_2-symmetric nor W_2-recurrent.

Example 2. Let $M^n = \{(x^1, x^2, \ldots, x^n) \in R^n : 0 < x^3 < 1\}$ $(n \geq 4)$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + \sum_{k=3}^{n}(dx^k)^2, \quad (i, j = 1, 2, \ldots, n) \tag{5.20}$$

where $f = a_0 + a_1x^3 + e^{x^1}\left[\frac{1}{2}(x^3)^2 + \frac{1}{6}(x^3)^3 + \cdots + \frac{1}{(n-2)(n-3)}(x^3)^{n-2}\right]$, a_0, a_1 are functions of x^1 only.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the W_2-curvature tensor and their covariant derivatives are $\Gamma^2_{11} = \frac{1}{2}f_{\cdot 1}$, $\Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_{\cdot 3}$, $R_{1331} = \frac{1}{2}f_{\cdot 33}$, $S_{11} = \frac{1}{2}f_{\cdot 33}$,

$(W_2)_{1211} = -\frac{1}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1331} = \frac{n-2}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1313} = -\frac{n-2}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1pp1} = -\frac{1}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$ for $4 \leq p \leq n$,

$(W_2)_{1211,1} = -\frac{1}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1211,3} = -\frac{1}{2(n-1)}e^{x^1}[\frac{1-(n-3)(x^3)^{n-4}(1-x^3)}{(1-x^3)^2}]$,

$(W_2)_{1331,1} = \frac{n-2}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1331,3} = \frac{n-2}{2(n-1)}e^{x^1}[\frac{1-(n-3)(x^3)^{n-4}(1-x^3)}{(1-x^3)^2}]$,

$(W_2)_{1313,1} = -\frac{n-2}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1313,3} = -\frac{n-2}{2(n-1)}e^{x^1}[\frac{1-(n-3)(x^3)^{n-4}(1-x^3)}{(1-x^3)^2}]$,

$(W_2)_{1pp1,1} = -\frac{1}{2(n-1)}e^{x^1}[\frac{1-(x^3)^{n-3}}{1-x^3}]$,

$(W_2)_{1pp1,3} = -\frac{1}{2(n-1)} e^{x^1} [\frac{1-(n-3)(x^3)^{n-4}(1-x^3)}{(1-x^3)^2}]$, for $4 \leq p \leq n$
and the components which can be obtained from these by the symmetry properties, where '.' denotes the partial differentiation with respect to the coordinates, ',' denotes the covariant differentiation and S_{ij} denotes the components of the Ricci tensor. Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore our M^n with the considered metric is a Riemannian manifold which is neither W_2-symmetric nor W_2-recurrent.

We shall now show that this M^n is a $(WW_2S)_n$, i.e., it satisfies (1.5). If we consider the 1-forms

$$\begin{aligned}
\alpha_i(x) &= \frac{7}{8} \quad \text{for} \quad i = 1, \\
&= \frac{1-(n-3)(x^3)^{n-4}(1-(x^3)}{(1-x^3)\{1-(x^3)^{n-3}\}} \quad \text{for} \quad i = 3, \\
&= 0 \quad \text{otherwise}, \\
\beta_i(x) &= \frac{1}{8} \quad \text{for} \quad i = 1, \\
&= -\frac{(n-4)(x^3)^{n-4}}{1-(x^3)^{n-3}} \quad \text{for} \quad i = 3, \\
&= 0 \quad \text{otherwise}, \\
\delta_i(x) &= \frac{(n-4)(x^3)^{n-4}}{1-(x^3)^{n-3}} \quad \text{for} \quad i = 3, \\
&= 0 \quad \text{otherwise}, \\
\sigma_i(x) &= \frac{(n-4)(x^3)^{n-4}}{1-(x^3)^{n-3}} \quad \text{for} \quad i = 3, \\
&= 0 \quad \text{otherwise},
\end{aligned}$$

at any point $x \in M$ then proceeding similarly as in Example 1, it can be easily shown that the manifold under consideration is a $(WW_2S)_n$. Hence we can state the following:

Theorem 5.3. *Let $M^n (n \geq 4)$ be a Riemannian manifold endowed with the metric given in* (5.20). *Then (M^n, g) is a weakly W_2-symmetric manifold with vanishing scalar curvature which is neither W_2-symmetric nor W_2-recurrent.*

Example 3. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in R^4 : e^{x^1} \neq \frac{K^2}{4}, K \neq 0\}$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (1+2\gamma)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\right], \quad (i,j = 1,2,3,4) \qquad (5.21)$$

where $\gamma = \frac{e^{x^1}}{K^2}$ and K is a non-zero constant.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the W_2-curvature tensor and their covariant derivatives are

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{\gamma}{1+2\gamma}, \quad \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{\gamma}{1+2\gamma},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{\gamma}{1+2\gamma},$$

$$S_{11} = \frac{3\gamma}{(1+2\gamma)^2}, S_{22} = S_{33} = S_{44} = \frac{\gamma}{(1+2\gamma)^2}, r = \frac{6\gamma}{(1+2\gamma)^3} \neq 0,$$

$$(W_2)_{1212} = (W_2)_{1313} = (W_2)_{1414} = -\frac{2\gamma}{3(1+2\gamma)}, \tag{5.22}$$

$$(W_2)_{2332} = (W_2)_{2442} = (W_2)_{3443} = -\frac{\gamma}{3(1+2\gamma)}, \tag{5.23}$$

$$(W_2)_{2323} = (W_2)_{2424} = (W_2)_{3434} = \frac{\gamma}{3(1+2\gamma)}, \tag{5.24}$$

$$(W_2)_{1212,1} = (W_2)_{1313,1} = (W_2)_{1414,1} = -\frac{2\gamma}{3(1+2\gamma)^2}, \tag{5.25}$$

$$(W_2)_{2332,1} = (W_2)_{2442,1} = (W_2)_{3443,1} = \frac{\gamma(4\gamma-1)}{3(1+2\gamma)^2}, \tag{5.26}$$

$$(W_2)_{2323,1} = (W_2)_{2424,1} = (W_2)_{3434,1} = -\frac{\gamma(4\gamma-1)}{3(1+2\gamma)^2}, \tag{5.27}$$

and the components which can be obtained from these by the symmetry properties, where ‘,’ denotes the covariant differentiation with respect to the metric tensor g. Therefore our M^4 with the considered metric is a Riemannian manifold of non-vanishing scalar curvature and the manifold is neither W_2-symmetric nor W_2-recurrent.

We shall now show that this M^4 is a $(WW_2S)_4$, i.e., it satisfies (1.5). If we consider the 1-forms

$$\begin{aligned} \alpha_i(x) &= \tfrac{1-4\gamma}{1+2\gamma} \text{ for } i = 1, \\ &= 0 \text{ otherwise}, \\ \beta_i(x) &= \tfrac{\gamma}{1+2\gamma} \text{ for } i = 1, \\ &= 0 \text{ otherwise}, \\ \delta_i(x) &= \tfrac{3\gamma}{1+2\gamma} \text{ for } i = 1, \\ &= 0 \text{ otherwise}, \\ \sigma_i(x) &= \tfrac{1}{1+2\gamma} \text{ for } i = 1, \\ &= 0 \text{ otherwise}, \end{aligned}$$

at any point $x \in M$. In our M^4, (1.5) reduces with these 1-forms to the following equations

(i) $(W_2)_{1212,1} = \alpha_1(W_2)_{1212} + \beta_1(W_2)_{1212} + \beta_2(W_2)_{1112} + \delta_1(W_2)_{1212} + \sigma_2(W_2)_{1211}$,

(ii) $(W_2)_{1313,1} = \alpha_1(W_2)_{1313} + \beta_1(W_2)_{1313} + \beta_3(W_2)_{1113} + \delta_1(W_2)_{1313} + \sigma_3(W_2)_{1311}$,

(iii) $(W_2)_{1414,1} = \alpha_1(W_2)_{1414} + \beta_1(W_2)_{1414} + \beta_4(W_2)_{1114} + \delta_1(W_2)_{1414} + \sigma_4(W_2)_{1411}$,

(iv) $(W_2)_{2332,1} = \alpha_1(W_2)_{2332} + \beta_2(W_2)_{1332} + \beta_3(W_2)_{2132} + \delta_3(W_2)_{2312} + \sigma_2(W_2)_{2331}$,

(v) $(W_2)_{2442,1} = \alpha_1(W_2)_{2442} + \beta_2(W_2)_{1442} + \beta_4(W_2)_{2142} + \delta_4(W_2)_{2412} + \sigma_2(W_2)_{2441}$,

(vi) $(W_2)_{3443,1} = \alpha_1(W_2)_{3443} + \beta_3(W_2)_{1443} + \beta_4(W_2)_{3143} + \delta_4(W_2)_{3413} + \sigma_3(W_2)_{3441}$,

(vii) $(W_2)_{2323,1} = \alpha_1(W_2)_{2323} + \beta_2(W_2)_{1323} + \beta_3(W_2)_{2123} + \delta_2(W_2)_{2313} + \sigma_3(W_2)_{2321}$,

(viii) $(W_2)_{2424,1} = \alpha_1(W_2)_{2424} + \beta_2(W_2)_{1424} + \beta_4(W_2)_{2124} + \delta_2(W_2)_{2414} + \sigma_4(W_2)_{2421}$,

(ix) $(W_2)_{3434,1} = \alpha_1(W_2)_{3434} + \beta_3(W_2)_{1434} + \beta_4(W_2)_{3134} + \delta_3(W_2)_{3414} + \sigma_4(W_2)_{3431}$,

since for the cases other than (i)-(ix) the components of each term of (1.5) vanishes identically and the relation (1.5) holds trivially. Now from (5.22) and (5.25) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

$$\begin{aligned}\text{R.H.S. of (i)} &= (\alpha_1 + \beta_1 + \delta_1)(W_2)_{1212} \\ &= -\frac{2\gamma}{3(1+2\gamma)^2} \\ &= \text{L.H.S. of (i)}.\end{aligned}$$

By a similar argument as in (i) it can be shown that the relations (ii)-(ix) are true. Hence we can state the following:

Theorem 5.4. *Let M^4 be a Riemannian manifold endowed with the metric given in* (5.21). *Then (M^4, g) is a weakly W_2-symmetric manifold with non-vanishing scalar curvature which is neither W_2-symmetric nor W_2-recurrent.*

Example 4. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in R^4 : 0 < x^4 < 1\}$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2, \quad (i, j = 1, 2, 3, 4). \tag{5.28}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the W_2-curvature tensor and their covariant derivatives are given by

$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}$, $\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}$, $R_{1441} = R_{2442} = R_{3443} = \frac{2}{9(x^4)^{\frac{2}{3}}}$,
$S_{11} = S_{22} = S_{33} = \frac{2}{9(x^4)^{\frac{2}{3}}}$, $S_{44} = \frac{2}{3(x^4)^2}$, $r = \frac{4}{3(x^4)^2} \neq 0$,
$(W_2)_{1221} = (W_2)_{1331} = (W_2)_{2332} = -\frac{2}{27}(x^4)^{\frac{2}{3}}$,
$(W_2)_{1441} = (W_2)_{2442} = (W_2)_{3443} = \frac{4}{27(x^4)^{\frac{2}{3}}}$,
$(W_2)_{1212} = (W_2)_{1313} = (W_2)_{2323} = \frac{2}{27}(x^4)^{\frac{2}{3}}$,
$(W_2)_{1221,4} = (W_2)_{1331,4} = (W_2)_{2332,4} = \frac{4}{27(x^4)^{\frac{1}{3}}}$,
$(W_2)_{1441,4} = (W_2)_{2442,4} = (W_2)_{3443,4} = -\frac{8}{27(x^4)^{\frac{5}{3}}}$,
$(W_2)_{1212,4} = (W_2)_{1313,4} = (W_2)_{2323,4} = -\frac{4}{27(x^4)^{\frac{1}{3}}}$,
and the components which can be obtained from these by the symmetry properties, where ‘,’ denotes the covariant differentiation with respect to the metric tensor g. Therefore our M^4 with the considered metric is a Riemannian manifold which is neither W_2-symmetric nor W_2-recurrent and of non-vanishing and non-constant scalar curvature.

We shall now show that this M^4 is a $(WW_2S)_4$, i.e., it satisfies (1.5). If we consider the 1-forms

$$\begin{aligned} \alpha_i(x) &= -1 \text{ for } i = 1,2,3, \\ &= -\tfrac{2}{x^4} \text{ for } i = 4, \\ \beta_i(x) &= \tfrac{1}{5} \text{ for } i = 1,2,\dots,4, \\ \delta_i(x) &= -\tfrac{1}{5} \text{ for } i = 1,2,\dots,4, \\ \sigma_i(x) &= -\tfrac{1}{5} \text{ for } i = 1,2,3, \\ &= 0 \text{ for } i = 4, \end{aligned}$$

at any point $x \in M$ then proceeding similarly as in Example 3, it can be easily shown that the manifold under consideration is a $(WW_2S)_4$. Hence we can state the following:

Theorem 5.5. *Let M^4 be a Riemannian manifold endowed with the metric given in* (5.28). *Then (M^4, g) is a weakly W_2-symmetric manifold with non-vanishing scalar curvature which is neither W_2-symmetric nor W_2-recurrent.*

Example 5. Let $M^n = \{(x^1, x^2, \dots, x^n) \in R^n : 0 < x^4 < 1\}$ $(n \geq 4)$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = \left[(x^4)^{\frac{4}{3}} - 1\right]\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + \delta_{ab}dx^a dx^b, \quad (5.29)$$

$0 < x^4 < 1$, where δ_{ab} is the Kronecker delta and a, b run from 1 to n.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the W_2-curvature tensor and their covariant derivatives are given by

$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}$, $\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}$,
$R_{1441} = R_{2442} = R_{3443} = \frac{2}{9(x^4)^{\frac{2}{3}}}$,
$S_{11} = S_{22} = S_{33} = \frac{2}{9(x^4)^{\frac{2}{3}}}$, $S_{44} = \frac{2}{3(x^4)^2}$, $r = \frac{4}{3(x^4)^2} \neq 0$,
$(W_2)_{1221} = (W_2)_{1331} = (W_2)_{2332} = -\frac{2}{9(n-1)}(x^4)^{\frac{2}{3}}$,
$(W_2)_{1441} = (W_2)_{2442} = (W_2)_{3443} = \frac{2(n-2)}{9(n-1)(x^4)^{\frac{2}{3}}}$,
$(W_2)_{1212} = (W_2)_{1313} = (W_2)_{2323} = \frac{2}{9(n-1)}(x^4)^{\frac{2}{3}}$,
$(W_2)_{1414} = (W_2)_{2424} = (W_2)_{3434} = -\frac{2(n-4)}{9(n-1)(x^4)^{\frac{2}{3}}}$,
$(W_2)_{1qq1} = (W_2)_{2qq2} = (W_2)_{3qq3} = -\frac{2}{9(n-1)(x^4)^{\frac{2}{3}}}$,
$(W_2)_{4qq4} = -\frac{2}{3(n-1)(x^4)^{\frac{2}{3}}}$ for $5 \leq q \leq n$,
$(W_2)_{1221,4} = (W_2)_{1331,4} = (W_2)_{2332,4} = \frac{4}{9(n-1)(x^4)^{\frac{1}{3}}}$,
$(W_2)_{1441,4} = (W_2)_{2442,4} = (W_2)_{3443,4} = -\frac{4(n-2)}{9(n-1)(x^4)^{\frac{5}{3}}}$,
$(W_2)_{1212,4} = (W_2)_{1313,4} = (W_2)_{2323,4} = -\frac{4}{9(n-1)(x^4)^{\frac{1}{3}}}$,
$(W_2)_{1414,4} = (W_2)_{2424,4} = (W_2)_{3434,4} = \frac{4(n-4)}{9(n-1)(x^4)^{\frac{5}{3}}}$,
$(W_2)_{1qq1,4} = (W_2)_{2qq2,4} = (W_2)_{3qq3,4} = \frac{4}{9(n-1)(x^4)^{\frac{5}{3}}}$,
$(W_2)_{4qq4,4} = \frac{4}{9(n-1)(x^4)^{\frac{5}{3}}}$ for $5 \leq q \leq n$,
and the components which can be obtained from these by the symmetry properties where ',' denotes the covariant differentiation with respect to the metric tensor g. Therefore our M^n with the considered metric is a Riemannian manifold which is neither W_2-symmetric nor W_2-recurrent and of non-vanishing and non-constant scalar curvature.

We shall now show that this M^n is a $(WW_2S)_n$, i.e., it satisfies (1.5). If we consider the 1-forms

$$\alpha_i(x) = -\frac{2}{3x^4} \text{ for } i = 4, \quad = 0 \text{ otherwise,}$$
$$\beta_i(x) = \frac{1}{x^4} \text{ for } i = 4, \quad = 0 \text{ otherwise,}$$
$$\delta_i(x) = -\frac{1}{x^4} \text{ for } i = 4, \quad = 0 \text{ otherwise,}$$
$$\sigma_i(x) = \frac{1}{3x^4} \text{ for } i = 4, \quad = 0 \text{ otherwise,}$$

at any point $x \in M$ then proceeding similarly as in Example 3, it can be easily shown that the manifold under consideration is a $(WW_2S)_n$. Hence we can state the following:

Theorem 5.6. *Let M^n be a Riemannian manifold endowed with the metric given in* (5.29). *Then (M^n, g) is a weakly W_2-symmetric manifold with non-vanishing scalar curvature which is neither W_2-symmetric nor W_2-recurrent.*

References

[1] M. C. Chaki, *On generalized pseudo-symmetric manifolds*, Publ. Math. Debrecen, 45 (1994), 305–312.

[2] U. C. De and S. Bandyopadhyay, *On weakly symmetric Riemannian spaces*, Publ. Math. Debrecen, 54 (3–4) (1999), 377–381.

[3] U. C. De and S. Bandyopadhyay, *On weakly conformally symmetric spaces*, Publ. Math. Debrecen, 57 (1–2) (2000), 71–78.

[4] D. Ferus, *A Remark on Codazzi Tensors on Constant Curvature Space*, Lecture Note Math., 838, Global Differential Geometry and Global Analysis, Springer-Verlag, New York, 1981.

[5] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistic significance*, Yokohama Math. J., 18 (1970), 105–108.

[6] J. A. Schouten, *Ricci Calculus* (2nd Edn.), Springer-Verlag, Berlin, 1954.

[7] A. A. Shaikh and K. K. Baishya, *On weakly quasi-conformally symmetric manifolds*, Soochow J. Math., 31 (4) (2005), 581–595.

[8] L. Tamássy and T. Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc., J. Bolyai, 50 (1989), 663–670.

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