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ON SPECIAL WEAKLY RICCI-SYMMETRIC KENMOTSU MANIFOLDS

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ABSTRACT. In this paper, we have studied special weakly Ricci symmetric Kenmotsu manifolds. We show that if a special weakly Ricci-symmetric Kenmotsu manifold admits a cyclic parallel Ricci tensor then the associate 1–form α must be zero. On the other hand we show that a special weakly Ricci-symmetric Kenmotsu manifold can not be an Einstein manifold if the associate 1–form $\alpha \neq 0$ and Ricci tensor of a special weakly Ricci-symmetric Kenmotsu manifold is parallel.

1. Introduction

In 1971, K. Kenmotsu studied a class of contact Riemann manifolds satisfying some special conditions. We call them Kenmotsu manifolds [6]. Several authors have studied some properties of Kenmotsu manifolds since then. In the recent years, J-B. Jun, U. C. De and G. Pathak (see [7]) partially classified the Kenmotsu manifolds and considered manifolds admitting the transformation which keeps the Riemannian curvature tensor and Ricci tensor invariant, U. C. De and C. Özgür (see [9]) studied the quasi-conformal curvature tensor of a Kenmotsu manifold and S Hong, C. Özgür and M. M. Tripathi (see [5]) obtained some results on the concircular curvature tensor of Kenmotsu manifolds.

As a generalization of Chaki's pseudosymmetric and pseudo Ricci symmetric manifolds (see [2] and [3]), the notion of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamássy and T. Q. Binh (see [12] and [13]). These type manifolds were studied with different structures by several authors (see [4], [10] and [12]). Recently in [11], C. Özgür studied weakly symmetric Kenmotsu manifolds. The notion of special weakly Ricci symmetric manifolds was introduced and studied by H. Singh, and Q. Khan in [8].

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In this paper, we have studied some geometric properties of special weakly Ricci-symmetric Kenmotsu manifolds. The paper is organized as follows. In Section 2, we give a brief account of almost contact metric manifolds and Kenmotsu manifolds. In Section 3, we consider a special weakly Ricci-symmetric Kenmotsu manifold admits a cyclic parallel Ricci tensor and we show that under these conditions the 1–form α must vanish. On the other hand we show that the Ricci tensor of a special weakly Ricci-symmetric Kenmotsu manifold is parallel and we find necessary conditions for a special weakly Ricci-symmetric Kenmotsu manifold to be an Einstein manifold.

2. Preliminaries

Let M^n be an n–dimensional differentiable manifold equipped with a triple (ϕ,ξ,η), where ϕ is a $(1,1)$–tensor field, ξ is a vector field, η is a 1–form on M^n such that

$$\eta(\xi)=1,\ \phi^2=-I+\xi\otimes\eta \tag{1}$$

which implies

$$\phi\xi=0,\ \eta\circ\phi=0,\ rank(\phi)=n-1. \tag{2}$$

If M^n admits a Riemannian metric g, such that

$$g(\phi X,\phi Y)=g(X,Y)-\eta\left(X\right)\eta\left(Y\right), \tag{3}$$

$$g(\xi,X)=\eta(X), \tag{4}$$

then M^n is said to admit a (ϕ,ξ,η,g)–structure. If moreover

$$\left(\bigtriangledown_X\phi\right)Y=-g(X,\phi Y)\xi-\eta\left(Y\right)\phi X. \tag{5}$$

and

$$\bigtriangledown_X\xi=-\phi^2X \tag{6}$$

where $\bigtriangledown$ denotes the Riemannian connection of g, then (M^n,ϕ,ξ,η,g) (where $n=2m+1$) is called a *Kenmotsu manifold*.

In a Kenmotsu manifold M^n, besides these relations the following relations also hold ([7]):

$$S(X,\xi)=-\left(n-1\right)\eta(X) \tag{7}$$

$$R(X,\xi)Y=g(X,Y)\xi-\eta(Y)X \tag{8}$$

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X \tag{9}$$

$$g(R(\xi,X)Y,\xi)=-g(X,Y)+\eta(X)\eta(Y) \tag{10}$$

$$R(\xi,X)\xi=X-\eta(X)\xi \tag{11}$$

for any vector fields X,Y of M^n.

3. On special weakly Ricci-symmetric Kenmotsu manifolds

An n–dimensional Riemannian manifold (M^n, g) is called a *special weakly Ricci-symmetric manifold* $(SWRS)_n$ if

$$(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X), \tag{12}$$

for any vector fields X, Y on M^n, where α is a 1–form and is defined by

$$\alpha(X) = g(X,\rho), \tag{13}$$

where ρ is the associated vector field [8].

Theorem 3.1. *If a special weakly Ricci-symmetric Kenmotsu manifold admits a cyclic parallel Ricci tensor then the* 1*–form* α *must be vanish.*

Proof. Let (12) and (13) be satisfied in a Kenmotsu manifold M^n. Taking the cyclic sum in (12), we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = 4(\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X)). \tag{14}$$

Let M^n admit a cyclic Ricci tensor. Then (14) reduces to

$$\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X) = 0. \tag{15}$$

Taking $Z = \xi$ in (15) and using (7) and (13), we get

$$-(n-1)\,\alpha(X)\eta(Y) - (n-1)\,\alpha(Y)\eta(X) + \eta(\rho)S(Y,X) = 0. \tag{16}$$

Now putting $Y = \xi$ in (16) and using (1), (7) and (13), we get

$$-(n-1)\,\alpha(X) - (n-1)\,\eta(\rho)\eta(X) - (n-1)\,\eta(\rho)\eta(X) = 0. \tag{17}$$

Taking $X = \xi$ in (17) and using (1) and (13), we get

$$\eta(\rho) = 0. \tag{18}$$

So by the use of (18) in (17), we have $\alpha(X) = 0$, for any vector fields X on M^n. This completes the proof of the theorem. □

Theorem 3.2. *A special weakly Ricci-symmetric Kenmotsu manifold can not be an Einstein manifold if the* 1*–form* $\alpha \neq 0$.

Proof. For an Einstein manifold $(\nabla_X S)(Y,Z) = 0$ and $S(Y,Z) = kg(Y,Z)$, then (12) gives

$$2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X) = 0. \tag{19}$$

Taking $Z = \xi$ in (19) and using (7) and (13), we get

$$-2\,(n-1)\,\alpha(X)\eta(Y) - (n-1)\,\alpha(Y)\eta(X) + \eta(\rho)S(Y,X) = 0. \tag{20}$$

Taking $X = \xi$ in (20) and using (1), (7) and (13), we get

$$-2\,(n-1)\,\eta(\rho)\eta(Y) - (n-1)\,\alpha(Y) - (n-1)\,\eta(\rho)\eta(Y) = 0. \tag{21}$$

Taking $Y = \xi$ in (21) and using (1) and (13), we get

$$\eta(\rho) = 0. \tag{22}$$

Using (22) in (21), we get $\alpha(Y) = 0$, for any vector fields Y on M^n, which completes the proof. □

Theorem 3.3. *The Ricci tensor of a special weakly Ricci-symmetric Kenmotsu manifold is parallel.*

Proof. Taking $Z = \xi$ in (12), we have

$$(\nabla_X S)(Y,\xi) = 2\alpha(X)S(Y,\xi) + \alpha(Y)S(X,\xi) + \alpha(\xi)S(Y,X). \tag{23}$$

The left-hand side can be written in the form

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi). \tag{24}$$

Then, in view of (7), (13), and (24), equation (23) becomes

$$\begin{aligned} &\nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi) \\ &\quad = -2\,(n-1)\,\alpha(X)\eta(Y) - (n-1)\,\alpha(Y)\eta(X) + \eta(\rho)S(Y,X). \end{aligned} \tag{25}$$

Taking $Y = \xi$ in (25) and using (1), (6), (7) and (13), we get

$$-2\,(n-1)\,\alpha(X) - (n-1)\,\eta(\rho)\eta(X) - (n-1)\,\eta(\rho)\eta(X) = 0. \tag{26}$$

Putting $X = \xi$ in (26) we obtain

$$\eta(\rho) = 0. \tag{27}$$

Using (27) in (26) we get

$$\alpha(X) = 0, \tag{28}$$

for any vector fields Y on M^n. Hence in view of (28) in (12), we get $\nabla_X S = 0$, which is proves the result. □

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