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ON (g, s)-CONTINUOUS AND $(\pi g, s)$-CONTINUOUS FUNCTIONS

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ABSTRACT. New generalizations of contra-continuity called $(\pi g, s)$-continuity and (g, s)-continuity are presented. Characterizations and properties of $(\pi g, s)$-continuous functions are discussed.

1. INTRODUCTION

It is well known that the concept of closedness is fundamental with respect to the investigation of general topological spaces. Levine [24] initiated the study of generalized closed sets. The concept of πg-closed sets was introduced by Dontchev and Noiri [12]. In 2000, Dontchev and Noiri [12] obtained new characterizations of quasi-normal spaces [42] by using πg-closed sets. Initiation of contra-continuity was due to Dontchev [9]. In 1996, Dontchev proved that contra-continuous images of strongly S-closed spaces are compact. In this paper, we introduce and investigate a generalization of contra-continuity by utilizing πg-closed sets. The notions of $(\pi g, s)$-continuous functions and (g, s)-continuous functions are introduced. Also, we obtain characterizations and properties of $(\pi g, s)$-continuous functions.

2. PRELIMINARIES

In this paper, spaces X and Y mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $cl(A)$ and $int(A)$ represent the closure of A and the interior of A, respectively.

A subset A of a space X is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$) [38]. The δ-interior [40] of a subset A of X is the union of all regular open sets of X contained in A and it is denoted by $\delta\text{-}int(A)$. A subset A is called δ-open [40] if $A = \delta\text{-}int(A)$. The complement of δ-open set is called δ-closed. The δ-closure of a set A in a

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space (X, τ) is defined by δ-$cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau$ and $x \in U\}$ and it is denoted by δ-$cl(A)$.

The finite union of regular open sets is said to be π-open [42]. The complement of π-open set is said to be π-closed. A subset A of a space X is said to be generalized closed (briefly, g-closed) [24] (resp. πg-closed [12]) if $cl(A) \subset U$ whenever $A \subset U$ and U is open (resp. π-open) in X. If the complement of A is πg-closed (resp. g-closed), A is said to be πg-open (resp. g-open). The union (resp. intersection) of all πg-open (resp. πg-closed) sets, each contained in (resp. containing) a set S in a topological space X is called the πg-interior (resp. πg-closure) of S and it is denoted by πg-$int(S)$ (resp. πg-$cl(S)$) [17]. For any subset K of a topological space X, $X \backslash \pi g$-$cl(K) = \pi g$-$int(X \backslash K)$ [17]. If a subset A is πg-closed in a space X, then $A = \pi g$-$cl(A)$ [17].

A subset A is said to be semi-open [23] if $A \subset cl(int(A))$. The complement of a semi-open set is called semi-closed [8]. The intersection of all semi-closed sets containing A is called the semi-closure [8] of A and is denoted by $scl(A)$. The semi-interior of A is defined by the union of all semi-open sets contained in A and is denoted by $sint(A)$. A point $x \in X$ is said to be a θ-semi-cluster point [21] of a subset A of X if $cl(U) \cap A \neq \emptyset$ for every semi-open set U containing x. The set of all θ-semi-cluster points of A is called the θ-semi-closure of A and is denoted by θ-s-$cl(A)$. A subset A is called θ-semi-closed [21] if $A = \theta$-s-$cl(A)$. The complement of a θ-semi-closed set is called θ-semi-open.

A subset A of a space X is said to be (1) α-open [28] (resp. preopen [27] or locally dense [7], β-open [1] or semi-preopen [2]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$, $A \subset cl(int(cl(A)))$). If the complement of A is α-open (resp. preopen, β-open), then A is said to be α-closed (resp. preclosed, β-closed). The union of all α-open (resp. preopen) sets, each contained in a set S in a topological space X is called the α-interior (resp. preinterior) of S and it is denoted by $\alpha int(S)$ (resp. $pint(S)$). The intersection of all α-closed sets, each containing a set S in a topological space X is called the α-closure (resp. preclosure) of S and it is denoted by $\alpha cl(S)$ (resp. $pcl(S)$).

The family of all δ-open (resp. πg-open, πg-closed, regular open, regular closed, semi-open, closed) sets of X containing a point $x \in X$ is denoted by $\delta O(X, x)$ (resp. $\pi GO(X, x)$, $\pi GC(X, x)$, $RO(X, x)$, $RC(X, x)$, $SO(X, x)$, $C(X, x)$). The family of all δ-open (resp. πg-open, πg-closed, regular open, regular closed, semi-open, β-open, preopen) sets of X is denoted by $\delta O(X)$ (resp. $\pi GO(X)$, $\pi GC(X)$, $RO(X)$, $RC(X)$, $SO(X)$, $\beta O(X)$, $PO(X)$).

Definition 1. *A space X is said to be*

(1) *s-Urysohn* [3] *if for each pair of distinct points x and y in X, there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $cl(U) \cap cl(V) = \emptyset$,*

(2) *weakly Hausdorff* [36] *if each element of* X *is an intersection of regular closed sets.*

Definition 2. *A space* X *said to be*

(1) *S-closed* [39] *if every regular closed cover of* X *has a finite subcover,*
(2) *countably S-closed* [1] *if every countable cover of* X *by regular closed sets has a finite subcover,*
(3) *S-Lindelof* [25] *if every cover of* X *by regular closed sets has a countable subcover.*

Definition 3. [17] *A space* (X,τ) *is called* πg-$T_{1/2}$ *if every* πg*-closed set is closed.*

Definition 4. [13] *Let* B *be a subset of a space* X*. The set* $\cap\{A \in RO(X) : B \subset A\}$ *is called the r-kernel of* B *and is denoted by* $r\text{-}ker(B)$*.*

Proposition 5. [13] *The following properties hold for subsets* A*,* B *of a space* X*:*

(1) $x \in r\text{-}ker(A)$ *if and only if* $A \cap K \neq \varnothing$ *for any regular closed set* K *containing* x*.*
(2) $A \subset r\text{-}ker(A)$ *and* $A = r\text{-}ker(A)$ *if* A *is regular open in* X*.*
(3) $A \subset B$*, then* $r\text{-}ker(A) \subset r\text{-}ker(B)$*.*

Lemma 6. [26] *If* V *is an open set, then* $scl(V) = int(cl(V))$*.*

The subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of a function $f : X \to Y$ and is denoted by $G(f)$.

3. (g,s)-continuous and $(\pi g,s)$-continuous functions

Definition 7. *A function* $f : X \to Y$ *is called* $(\pi g,s)$*-continuous (resp.* (g,s)*-continuous) if the inverse image of each regular open set of* Y *is* πg*-closed (resp. g-closed) in* X*.*

Theorem 8. *The following are equivalent for a function* $f : X \to Y$*:*

(1) f *is* $(\pi g,s)$*-continuous,*
(2) *the inverse image of a regular closed set of* Y *is* πg*-open,*
(3) $f^{-1}(int(cl(V)))$ *is* πg*-closed for every open subset* V *of* Y*,*
(4) $f^{-1}(cl(int(F)))$ *is* πg*-open for every closed subset* F *of* Y*,*
(5) $f^{-1}(cl(U))$ *is* πg*-open in* X *for every* $U \in \beta O(Y)$*,*
(6) $f^{-1}(cl(U))$ *is* πg*-open in* X *for every* $U \in SO(Y)$*,*
(7) $f^{-1}(int(cl(U)))$ *is* πg*-closed in* X *for every* $U \in PO(Y)$*.*

Proof. $(1) \Leftrightarrow (2)$: Obvious.

$(1) \Leftrightarrow (3)$: Let V be an open subset of Y. Since $int(cl(V))$ is regular open, $f^{-1}(int(cl(V)))$ is πg-closed. The converse is similar.

(2) $\Leftrightarrow$ (4) : Similar to (1) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (5) : Let U be any β-open set of Y. By Theorem 2.4 of [2] that $cl(U)$ is regular closed. Then by (2) $f^{-1}(cl(U)) \in \pi GO(X)$.

(5) $\Rightarrow$ (6) : Obvious from the fact that $SO(Y) \subset \beta O(Y)$.

(6) $\Rightarrow$ (7) : Let $U \in PO(Y)$. Then $Y \backslash int(cl(U))$ is regular closed and hence it is semiopen. Then, we have $X \backslash f^{-1}(int(cl(U)) = f^{-1}(Y \backslash int(cl(U)))$ $= f^{-1}(cl(Y \backslash int(cl(U)))) \in \pi GO(X)$. Hence $f^{-1}(int(cl(U)))$ is πg-closed in X.

(7) $\Rightarrow$ (1) : Let U be any regular open set of Y Then $U \in PO(Y)$ and hence $f^{-1}(U) = f^{-1}(int(cl(U)))$ is πg-closed in X. □

Lemma 9. [32] *For a subset A of a topological space (Y, σ), the following properties hold:*

(1) $\alpha cl(A) = cl(A)$ *for every* $A \in \beta O(Y)$,
(2) $pcl(A) = cl(A)$ *for every* $A \in SO(Y)$,
(3) $scl(A) = int(cl(A))$ *for every* $A \in PO(Y)$.

Corollary 10. *The following are equivalent for a function $f : X \to Y$:*

(1) f *is* $(\pi g, s)$*-continuous,*
(2) $f^{-1}(\alpha cl(V))$ *is* πg*-open in* X *for every* $V \in \beta O(Y)$,
(3) $f^{-1}(pcl(V))$ *is* πg*-open in* X *for every* $V \in SO(Y)$,
(4) $f^{-1}(scl(V))$ *is* πg*-closed in* X *for every* $V \in PO(Y)$.

Proof. It follows from Lemma 9. □

Theorem 11. *Suppose that $\pi GC(X)$ is closed under arbitrary intersections. The following are equivalent for a function $f : X \to Y$:*

(1) f *is* $(\pi g, s)$*-continuous,*
(2) *the inverse image of a* θ*-semi-open set of* Y *is* πg*-open,*
(3) *the inverse image of a* θ*-semi-closed set of* Y *is* πg *-closed,*
(4) $f(\pi g\text{-}cl(U)) \subset r\text{-}ker(f(U))$ *for every subset* U *of* X,
(5) $\pi g\text{-}cl(f^{-1}(V)) \subset f^{-1}(r\text{-}ker(V))$ *for every subset* V *of* Y,
(6) *for each* $x \in X$ *and each* $V \in SO(Y, f(x))$, *there exists a* πg *-open set* U *in* X *containing* x *such that* $f(U) \subset cl(V)$,
(7) $f^{-1}(V) \subset \pi g\text{-}int(f^{-1}(cl(V)))$ *for every* $V \in SO(Y)$,
(8) $f(\pi g\text{-}cl(A)) \subset \theta\text{-}s\text{-}cl(f(A))$ *for every subset* A *of* X,
(9) $\pi g\text{-}cl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}s\text{-}cl(B))$ *for every subset* B *of* Y,
(10) $\pi g\text{-}cl(f^{-1}(V)) \subseteq f^{-1}(\theta\text{-}s\text{-}cl(V))$ *for every open subset* V *of* Y,
(11) $\pi g\text{-}cl(f^{-1}(V)) \subseteq f^{-1}(scl(V))$ *for every open subset* V *of* Y,
(12) $\pi g\text{-}cl(f^{-1}(V)) \subseteq f^{-1}(int(cl(V)))$ *for every open subset* V *of* Y.

Proof. (1) $\Rightarrow$ (2) : Since any θ-semi-open set is a union of regular closed sets, by using Theorem 8, (2) holds.

$(2)\Rightarrow(6)$: Let $x\in X$ and $V\in SO(Y)$ containing $f(x)$. Since $cl(V)$ is θ-semi-open in Y, there exists a πg-open set U in X containing x such that $x\in U\subset f^{-1}(cl(V))$. Hence $f(U)\subset cl(V)$.

$(6)\Rightarrow(7)$: Let $V\in SO(Y)$ and $x\in f^{-}(V)$. Then $f(x)\in V$. By (6), there exists a πg-open set U in X containing x such that $f(U)\subset cl(V)$. It follows that $x\in U\subset f^{-1}(cl(V))$. Hence $x\in \pi g\text{-}int(f^{-1}(cl(V)))$. Thus, $f^{-1}(V)\subset \pi g\text{-}int(f^{-1}(cl(V)))$.

$(7)\Rightarrow(1)$: Let F be any regular closed set of Y. Since $F\in SO(Y)$, then by (7), $f^{-1}(F)\subset \pi g\text{-}int(f^{-1}(F))$. This shows that $f^{-1}(F)$ is πg-open in X. Hence, by Theorem 8, (1) holds.

$(2)\Leftrightarrow(3)$: Obvious.

$(1)\Rightarrow(4)$: We shall use Theorem 8. Let U be any subset of X. Let $y\notin$r-$ker(f(U))$. Then there exists a regular closed set F containing y such that $f(U)\cap F=\varnothing$. Hence, we have $U\cap f^{-1}(F)=\varnothing$ and $\pi g\text{-}cl(U)\cap f^{-1}(F)=\varnothing$. Therefore, we obtain $f(\pi g\text{-}cl(U))\cap F=\varnothing$ and $y\notin f(\pi g\text{-}cl(U))$. Thus, $f(\pi g\text{-}cl(U))\subset$r-$ker(f(U))$.

$(4)\Rightarrow(5)$: Let V be any subset of Y. By (4), $f(\pi g\text{-}cl(f^{-1}(V)))\subset$r-$ker(V)$ and $\pi g\text{-}cl(f^{-1}(V))\subset f^{-1}($ r-$ker(V))$.

$(5)\Rightarrow(1)$: Let V be any regular open set of Y. By Proposition 5, $\pi g\text{-}cl(f^{-1}(V)\subset f^{-1}(\text{r-}ker(V))=f^{-1}(V)$ and $\pi g\text{-}cl((f^{-1}(V))=f^{-1}(V)$. We obtain that $f^{-1}(V)$ is πg-closed in X.

$(6)\Rightarrow(8)$: Let A be any subset of X. Suppose that $x\in \pi g\text{ -}cl(A)$ and G is any semiopen set of Y containing $f(x)$. By (6), there exists $U\in \pi GO(X,x)$ such that $f(U)\subset cl(G)$. Since $x\in \pi g\text{-}cl(A)$, $U\cap A\neq\emptyset$ and hence $\emptyset\neq f(U)\cap f(A)\subset cl(G)\cap f(A)$. Therefore, we obtain $f(x)\in\theta\text{-}s\text{-}cl(f(A))$ and hence $f(\pi g\text{-}cl(A))\subset\theta\text{-}s\text{-}cl(f(A))$.

$(8)\Rightarrow(9)$: Let B be any subset of Y. Then $f(\pi g\text{-}cl(f^{-1}(B)))\subset\theta\text{-}s\text{-}cl(f(f^{-1}(B)))\subset\theta\text{-}s\text{-}cl(B)$ and $\pi g\text{-}cl(f^{-1}(B))\subset f^{-1}(\theta\text{-}s\text{-}cl(B))$.

$(9)\Rightarrow(6)$: Let V be any semiopen set of Y containing $f(x)$. Since $cl(V)\cap(Y\backslash cl(V))=\emptyset$, we have $f(x)\notin\theta\text{-}s\text{-}cl(Y\backslash cl(V))$ and $x\notin f^{-1}(\theta\text{-}s\text{-}cl(Y\backslash cl(V)))$. By (9), $x\notin\pi g\text{-}cl(f^{-1}(Y\backslash cl(V)))$. Hence, there exists $U\in\pi GO(X,x)$ such that $U\cap f^{-1}(Y\backslash cl(V))=\varnothing$ and $f(U)\cap(Y\backslash cl(V))=\varnothing$. It follows that $f(U)\subset cl(V)$. Thus, (6) holds.

$(9)\Rightarrow(10)$: Obvious.

$(10)\Rightarrow(11)$: Obvious from the fact that $\theta\text{-}s\text{-}cl(V)=scl(V)$ for an open set V.

$(11)\Rightarrow(12)$: Obvious from Lemma 6.

$(12)\Rightarrow(1)$: Let $V\in RO(Y)$. Then by (12) $\pi g\text{-}cl(f^{-1}(V))\subseteq f^{-1}(int(cl(V)))=f^{-1}(V)$. Hence, $f^{-1}(V)$ is πg-closed which proves that f is $(\pi g,s)$-continuous. □

Corollary 12. *Assume that $\pi GC(X)$ is closed under arbitrary intersections. The following are equivalent for a function $f : X \to Y$:*

(1) *f is $(\pi g, s)$-continuous,*
(2) *πg-cl$(f^{-1}(B)) \subset f^{-1}(\theta$-s-cl$(B))$ for every $V \in SO(Y)$,*
(3) *πg-cl$(f^{-1}(B)) \subset f^{-1}(\theta$-s-cl$(B))$ for every $V \in PO(Y)$,*
(4) *πg-cl$(f^{-1}(B)) \subset f^{-1}(\theta$-s-cl$(B))$ for every $V \in \beta O(Y)$.*

4. The related functions with $(\pi g, s)$-continuous functions

Definition 13. *A function $f : X \to Y$ is said to be:*

(1) *perfectly continuous* [30] *if $f^{-1}(V)$ is clopen in X for every open set V of Y,*
(2) *regular set-connected* [11, 15] *if $f^{-1}(V)$ is clopen in X for every $V \in RO(Y)$,*
(3) *almost s-continuous* [6, 33] *if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subset s$-cl(V),*
(4) *strongly continuous* [22] *if $f(cl(A)) \subset f(A)$ for every subset A of X or equivalently if the inverse image of every set in Y is clopen in X,*
(5) *RC-continuous* [10] *if $f^{-1}(V)$ is regular closed in X for each open set V of Y,*
(6) *contra R-map* [16] *if $f^{-1}(V)$ is regular closed in X for every regular open set V of Y,*
(7) *contra-super-continuous* [20] *if for each $x \in X$ and each $F \in C(Y, f(x))$, there exists a regular open set U in X containing x such that $f(U) \subset F$,*
(8) *almost contra-super-continuous* [14] *if $f^{-1}(V)$ is δ-closed in X for every regular open set V of Y,*
(9) *contra-continuous* [9] *if $f^{-1}(V)$ is closed in X for every open set V of Y,*
(10) *contra g-continuous* [5] *if $f^{-1}(V)$ is g-closed in X for every open set V of Y,*
(11) *(θ,s)-continuous* [21, 34] *if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subset cl(V)$,*
(12) *contra πg-continuous* [18] *if $f^{-1}(V)$ is πg-closed in X for every open set V of Y.*

Remark 14. The following diagram holds for a function $f : X \to Y$:

$$\begin{array}{ccc}
\text{strongly continuous} & \Rightarrow & \text{almost s-continuous} \\
\Downarrow & & \Downarrow \\
\text{perfectly continuous} & \Rightarrow & \text{regular set-connected} \\
\Downarrow & & \Downarrow \\
\text{RC-continuous} & \Rightarrow & \text{contra R-map} \\
\Downarrow & & \Downarrow \\
\text{contra-super-continuous} & \Rightarrow & \text{almost contra-super-continuous} \\
\Downarrow & & \Downarrow \\
\text{contra-continuous} & \Rightarrow & (\theta, s)\text{-continuous} \\
\Downarrow & & \Downarrow \\
\text{contra } g\text{-continuous} & \Rightarrow & (g, s)\text{-continuous} \\
\Downarrow & & \Downarrow \\
\text{contra } \pi g\text{-continuous} & \Rightarrow & (\pi g, s) \text{ -continuous}
\end{array}$$

None of these implications is reversible as shown in the following examples and in the related papers.

Example 15. Let X be the real numbers with the cofinite topology and $Y = \{a, b, c\}$ with the topology $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. We define the function $f : X \to Y$ such as

$$f(x) = \begin{cases} a, & x \in X\backslash\{0\} \\ c, & x = 0 \end{cases}$$

Then f is $(\pi g, s)$-continuous but it is not (g, s)-continuous.

Example 16. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{X, \varnothing, \{c\}, \{a, d\}, \{a, c, d\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ which is defined as $f(a) = a$, $f(b) = d$, $f(c) = b$, $f(d) = a$ is $(\pi g, s)$-continuous but it is not contra πg-continuous.

Example 17. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ which is defined as $f(a) = d$, $f(b) = c$, $f(c) = d$, $f(d) = b$ is (g, s)-continuous but it is not contra g-continuous.

Example 18. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{X, \varnothing, \{c\}, \{a, d\}, \{a, c, d\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ which is defined as $f(a) = c$, $f(b) = c$, $f(c) = b$, $f(d) = b$ is (g, s)-continuous but it is not (θ, s)-continuous.

A topological space (X, τ) is said to be extremely disconnected [4] if the closure of every open set of X is open in X.

Definition 19. *A function $f : X \to Y$ is said to be:*

(1) *πg-continuous* [12] *if $f^{-1}(V)$ is πg-open in X for every open set V of Y.*
(2) *almost πg-continuous* [12] *if $f^{-1}(V)$ is πg-open in X for every regular open set V of Y.*

Theorem 20. *Let (Y, σ) be extremely disconnected. Then, the following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

(1) *f is $(\pi g, s)$-continuous,*
(2) *f is almost πg-continuous.*

Proof. (1) $\Rightarrow$ (2) : Let $x \in X$ and U be any regular open set of Y containing $f(x)$. Since (Y, σ) is extremely disconnected, by Lemma 5.6 of [35] U is clopen and hence U is regular closed. Then $f^{-1}(U)$ is πg-open in X. Thus, f is almost πg-continuous.

(2) $\Rightarrow$ (1) : Let K be any regular closed set of Y. Since (Y, σ) is extremely disconnected, K is regular open and $f^{-1}(K)$ is πg-open in X. Thus, f is $(\pi g, s)$-continuous. □

Theorem 21. *Let $f : X \to Y$ be a function from a πg-$T_{1/2}$ space (X, τ) to a topological space (Y, υ). Then the following are equivalent:*

(1) *f is $(\pi g, s)$-continuous,*
(2) *f is (g, s)-continuous,*
(3) *f is (θ, s)-continuous.*

Definition 22. *A space is said to be P_Σ* [41] *or strongly s-regular* [19] *if for any open set V of X and each $x \in V$, there exists $K \in RC(X, x)$ such that $x \in K \subset V$.*

Theorem 23. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, if f is $(\pi g, s)$-continuous, X is πg-$T_{1/2}$ and Y is P_Σ, then f is continuous.*

Proof. Let G be any open set of Y. Since Y is P_Σ, there exists a subfamily Φ of $RC(Y)$ such that $G = \cup\{A : A \in \Phi\}$. Since X is πg-$T_{1/2}$ and f is $(\pi g, s)$-continuous, $f^{-1}(A)$ is open in X for each $A \in \Phi$ and $f^{-1}(G)$ is open in X. Thus, f is continuous. □

Theorem 24. *Let Y be a regular space and $f : X \to Y$ be a function. Suppose that the collection of πg-closed sets of X is closed under arbitrary intersections. Then if f is $(\pi g, s)$-continuous, f is πg-continuous.*

Proof. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is regular, there exists an open set G in Y containing $f(x)$ such that $cl(G) \subset V$. Since f is $(\pi g, s)$-continuous, there exists $U \in \pi GO(X, x)$ such that $f(U) \subset cl(G)$. Then $f(U) \subset cl(G) \subset V$. Hence, f is πg-continuous. □

Theorem 25. *Let $f : X \to Y$ be a function from a πg-$T_{1/2}$ space (X,τ) to an extremely disconnected space (Y,υ). Then the following are equivalent:*

(1) *f is $(\pi g,s)$-continuous,*
(2) *f is (g,s)-continuous,*
(3) *f is (θ,s)-continuous,*
(4) *f is almost contra-super-continuous,*
(5) *f is contra R-map,*
(6) *f is regular set-connected,*
(7) *f is almost s-continuous.*

Proof. (7) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) : Follows from Remark 14.

(1) $\Rightarrow$ (7) : Let V be any semi-open semi-closed set of Y. Since V is semi-open, $cl(V) = cl(int(V))$ and hence $cl(V)$ is open in Y. Since V is semi-closed, $int(cl(V)) \subset V \subset cl(V)$ and hence $int(cl(V)) = V = cl(V)$. Therefore, V is clopen in Y and $V \in RO(Y) \cap RC(Y)$. Since f is $(\pi g,s)$-continuous, $f^{-1}(V)$ is πg-open and πg-closed in X. Since X is πg-$T_{1/2}$, then $\tau = \pi GO(X)$. Thus, $f^{-1}(V)$ is clopen in X and hence f is almost s-continuous [33, Theorem 3.1]. □

Definition 26. *A space is said to be weakly P_Σ* [31] *if for any $V \in RO(X)$ and each $x \in V$, there exists $F \in RC(X,x)$ such that $x \in F \subset V$.*

Theorem 27. *Let $f : (X,\tau) \to (Y,\sigma)$ be a $(\pi g,s)$-continuous function and let $\pi GC(X)$ be closed under arbitrary intersections. If Y is weakly P_Σ and X is πg-$T_{1/2}$, then f is regular set-connected.*

Proof. Let V be any regular open set of Y. Since Y is weakly P_Σ, there exists a subfamily Φ of $RC(Y)$ such that $V = \cup\{A : A \in \Phi\}$. Since f is $(\pi g,s)$-continuous, $f^{-1}(A)$ is πg-open in X for each $A \in \Phi$ and $f^{-1}(V)$ is πg-open in X. Also, $f^{-1}(V)$ is πg-closed in X since f is $(\pi g,s)$-continuous. Since X is πg-$T_{1/2}$, then $\tau = \pi GO(X)$. Hence, $f^{-1}(V)$ is clopen in X and then f is regular set-connected. □

Definition 28. *A function $f : X \to Y$ is said to be πg-irresolute* [17] *if $f^{-1}(V)$ is πg-open in X for every $V \in \pi GO(Y)$.*

Theorem 29. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold:*

(1) *If f is πg-irresolute and g is $(\pi g,s)$-continuous, then $g \circ f$ is $(\pi g,s)$-continuous.*
(2) *If f is $(\pi g,s)$-continuous and g is contra R-map, then $g \circ f$ is almost πg-continuous.*
(3) *If f is πg-continuous and g is (θ,s)-continuous, then $g \circ f$ is $(\pi g,s)$-continuous.*

(4) *If f is $(\pi g, s)$-continuous and g is RC-continuous, then $g \circ f$ is πg-continuous.*

5. Fundamental properties

Definition 30.

(1) *A subset S of a space X is said to be πg-compact relative to X if for every cover $\{A_i : i \in I\}$ of S by πg-open sets of X, there exists a finite subset I_0 of I such that $S \subset \cup\{A_i : i \in I_0\}$.*

(2) *A space X said to be πg-compact* [17] *if every πg-open cover of X has a finite subcover.*

(3) *A graph $G(f)$ of a function $f : X \to Y$ is said to be $(\pi g, s)$-graph if there exist a πg-open set A in X containing x and a semi-open set B in Y containing y such that $(A \times cl(B)) \cap G(f) = \emptyset$ for each $(x, y) \in (X \times Y) \backslash G(f)$.*

Theorem 31.

(1) *Every πg-closed subset of a πg-compact space X is πg-compact relative to X.*

(2) *The surjective $(\pi g, s)$-continuous image of a πg-compact space is S-closed.*

(3) *If $f : (X, \tau) \to (Y, \sigma)$ is πg-irresolute and a subset A of X is πg-compact relative to X, then its image $f(A)$ is πg-compact relative to Y.*

Proof. (1) : Let A be a πg-closed subset of a πg-compact space (X, τ). Let $\{U_i : i \in I\}$ be a cover of A by πg-open subsets of X. So $A \subset \underset{i \in I}{\cup} U_i$ and then $(X \backslash A) \cup (\underset{i \in I}{\cup} U_i) = X$. Since X is πg-compact, there exists a finite subset I_0 of I such that $(X \backslash A) \cup (\underset{i \in I_0}{\cup} U_i) = X$. Then $A \subset \underset{i \in I_0}{\cup} U_i$ and hence A is πg-compact relative to X.

(2) : Let (X, τ) be a πg-compact space and $f : (X, \tau) \to (Y, \sigma)$ be a surjective $(\pi g, s)$-continuous function. Let $\{U_i : i \in I\}$ be a cover of X by regular closed sets. Then $\{f^{-1}(U_i) : i \in I\}$ is a cover of X by πg-open sets, since f is $(\pi g, s)$-continuous. By πg-compactness of X, there exists a finite subset I_0 of I such that $X = \underset{i \in I_0}{\cup} f^{-1}(U_i)$. Since f is surjective, $Y = \underset{i \in I_0}{\cup} U_i$ and hence Y is S-closed.

(3) : Similar to that of (2). □

Proposition 32. *The following properties are equivalent for a function f:*

(1) *$G(f)$ is $(\pi g, s)$-graph,*

(2) *for each* $(x, y) \in (X \times Y)\backslash G(f)$, *there exist a* πg*-open set* A *in* X *containing* x *and a semi-open set* B *in* Y *containing* y *such that* $f(A) \cap cl(B) = \emptyset$,

(3) *for each* $(x, y) \in (X \times Y)\backslash G(f)$, *there exist a* πg*-open set* A *in* X *containing* x *and a regular closed set* K *in* Y *containing* y *such that* $f(A) \cap K = \emptyset$.

Definition 33. *A subset* S *of a space* X *is said to be S-closed relative to* X [29] *if for every cover* $\{A_i : i \in I\}$ *of* S *by semi-open sets of* X, *there exists a finite subset* I_0 *of* I *such that* $S \subset \cup\{cl(A_i) : i \in I_0\}$.

Theorem 34. *If a function* $f : X \to Y$ *has a* $(\pi g, s)$*-graph and the collection of* πg*-closed sets of* X *is closed under arbitrary intersections, then* $f^{-1}(A)$ *is* πg*-closed in* X *for every subset* A *which is S-closed relative to* Y.

Proof. Suppose that A is S-closed relative to Y and $x \notin f^{-1}(A)$. We have $(x, y) \in X \times Y\backslash G(f)$ for each $y \in A$ and there exist a πg-open set B_y containing x and a semi-open set C_y containing y such that $f(B_y) \cap cl(C_y) = \varnothing$. Since $\{C_y : y \in A\}$ is a cover by semi-open sets of Y, there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ of A such that $A \subset \cup\{cl(C_{y_i}) : i = 1, 2, 3, \ldots, n\}$. Take $B = \cap\{B_{y_i} : i = 1, 2, 3, \ldots, n\}$. Then B is a πg-open containing x and $f(B) \cap A = \varnothing$. Thus, $B \cap f^{-1}(A) = \varnothing$ and hence $f^{-1}(A)$ is πg-closed in X. □

Theorem 35. *Let* $f : X \to Y$ *be a* $(\pi g, s)$*-continuous functions. Then the following properties hold:*

(1) $G(f)$ *is a* $(\pi g, s)$*-graph if* Y *is an s-Urysohn.*

(2) f *is almost* πg*-continuous if* Y *is an s-Urysohn and* $\pi GC(X)$ *is closed under arbitrary intersections.*

Proof. (1) : Let Y be s-Urysohn and $(x, y) \in (X \times Y)\backslash G(f)$. Then $f(x) \neq y$. Since Y is s-Urysohn, there exist $M \in SO(Y, f(x))$ and $N \in SO(Y, y)$ such that $cl(M) \cap cl(N) = \emptyset$. Since f is $(\pi g, s)$-continuous, there exists a πg-open set A in X containing x such that $f(A) \subset cl(M)$. Hence, $f(A) \cap cl(N) = \emptyset$ and $G(f)$ is $(\pi g, s)$-graph in $X \times Y$.

(2) : Let F be a regular closed set in Y. By Theorem 3.3 and 3.4 [29], F is S-closed relative to Y. Hence, by Theorem 34 and (1), $f^{-1}(F)$ is πg-closed in X and hence f is almost πg-continuous. □

Definition 36. *A space* X *is said to be*

(1) πg-T_2 [18] *if for each pair of distinct points* x *and* y *in* X, *there exist* $U \in \pi GO(X, x)$ *and* $V \in \pi GO(X, y)$ *such that* $U \cap V = \emptyset$.

(2) πg-T_1 [18] *if for each pair of distinct points in* X, *there exist* πg*-open sets* U *and* V *containing* x *and* y, *respectively, such that* $y \notin U$ *and* $x \notin V$.

(3) r-T_1 [15] *if for each pair of distinct points in* X, *there exist regular open sets* U *and* V *containing* x *and* y, *respectively, such that* $y \notin U$ *and* $x \notin V$.

Theorem 37. *Let* f, $g : X \to Y$ *be functions and* πg-$cl(S)$ *be* πg-*closed for each* $S \subset X$. *If*

(1) f *and* g *are* $(\pi g, s)$-*continuous,*
(2) Y *is s-Urysohn,*

then $A = \{x \in X : f(x) = g(x)\}$ *is* πg-*closed in* X.

Proof. Let $x \in X \backslash A$, then it follows that $f(x) \neq g(x)$. Since Y is s-Urysohn, there exist $M \in SO(Y, f(x))$ and $N \in SO(Y, g(x))$ such that $cl(M) \cap cl(N) = \emptyset$. Since f and g are $(\pi g, s)$-continuous, there exist πg-open sets U and V containing x such that $f(U) \subset cl(M)$ and $g(V) \subset cl(N)$. Hence, $U \cap V = P \in \pi GO(X)$, $f(P) \cap g(P) = \emptyset$ and then $x \notin \pi g$-$cl(A)$. Thus, A is πg-closed in X. □

A subset A of a topological space X is said to be πg-dense in X if πg-$cl(A) = X$.

Theorem 38. *Let* f, $g : X \to Y$ *be functions and* πg-$cl(S)$ *be* πg-*closed for each* $S \subset X$. *If*

(1) Y *is s-Urysohn,*
(2) f *and* g *are* $(\pi g, s)$-*continuous,*
(3) $f = g$ *on* πg-*dense set* $A \subset X$,

then $f = g$ *on* X.

Proof. Since f and g are $(\pi g, s)$-continuous and Y is s-Urysohn, by Theorem 37, $B = \{x \in X : f(x) = g(x)\}$ is πg-closed in X. We have $f = g$ on πg-dense set $A \subset X$. Since $A \subset B$ and A is πg-dense set in X, then $X = \pi g$-$cl(A) \subset \pi g$-$cl(B) = B$. Hence, $f = g$ on X. □

Theorem 39. *The following properties hold for a function* $f : X \to Y$:

(1) *If* f *is a* $(\pi g, s)$-*continuous injection and* Y *is s-Urysohn, then* X *is* πg-T_2.
(2) *If* f *is a* $(\pi g, s)$-*continuous injection and* Y *is weakly Hausdorff, then* X *is* πg-T_1.
(3) *If* f *is surjective and it has a* $(\pi g, s)$-*graph, then* Y *is weakly* T_2.

Proof. (1) : Let Y be s-Urysohn. By the injectivity of f, $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is s-Urysohn, there exist $A \in SO(Y, f(x))$ and $B \in SO(Y, f(y))$ such that $cl(A) \cap cl(B) = \emptyset$. Since f is a $(\pi g, s)$-continuous, there exist πg-open sets C and D in X containing x and y, respectively, such that $f(C) \subset cl(A)$ and $f(D) \subset cl(B)$ such that $C \cap D = \emptyset$. Thus, X is πg-T_2.

(2) : Let Y be weakly Hausdorff. For $x \neq y$ in X, there exist A, $B \in RC(Y)$ such that $f(x) \in A$, $f(y) \notin A$, $f(x) \notin B$ and $f(y) \in B$. Since f is $(\pi g, s)$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are πg-open subsets of X such that $x \in f^{-1}(A)$, $y \notin f^{-1}(A)$, $x \notin f^{-1}(B)$ and $y \in f^{-1}(B)$. Hence, X is πg-T_1.

(3) : Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$, for some $x \in X$ and $(x, y_2) \in (X \times Y) \backslash G(f)$. Then, there exist a πg-open set A in X containing x and a regular closed set K in Y containing y_2 such that $f(A) \cap K = \emptyset$. Hence $y_1 \notin K$. This implies that Y is weakly T_2. □

Definition 40. *A space X said to be*

(1) *countably πg-compact* [18] *if every countable cover of X by πg-open sets has a finite subcover,*
(2) *πg-Lindelof* [18] *if every πg-open cover of X has a countable subcover.*

Theorem 41. *Let $f : X \to Y$ be a $(\pi g, s)$-continuous surjection. Then the following statements hold:*

(1) *if X is πg-Lindelof, then Y is S-Lindelof.*
(2) *if X is countably πg-compact, then Y is countably S-closed.*

Definition 42. *A space X is called*

(1) *πg-connected if X is not the union of two disjoint nonempty πg-open sets.*
(2) *πg-ultra-connected if every two non-void πg-closed subsets of X intersect,*
(3) *hyperconnected* [37] *if every open set is dense.*

Theorem 43. *Let $f : X \to Y$ be a $(\pi g, s)$-continuous surjection.*

(1) *If X is πg-connected, then Y is connected.*
(2) *If X is πg-ultra-connected, then Y is hyperconnected.*

Proof. (1) : Assume that Y is not connected space. Then there exist nonempty disjoint open sets A and B such that $Y = A \cup B$. Also, A and B are clopen in Y. Since f is $(\pi g, s)$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are πg-open in X. Moreover, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty disjoint and $X = f^{-1}(A) \cup f^{-1}(B)$. This shows that X is not πg-connected. This contradicts the assumption that Y is not connected. □

Definition 44. [18] *The πg-frontier of a subset A of a space X is given by πg-$fr(A) = \pi g$-$cl(A) \cap \pi g$-$cl(X \backslash A)$.*

Theorem 45. *Suppose that $\pi GC(X)$ is closed under arbitrary intersections. A function $f : X \to Y$ is not $(\pi g, s)$-continuous at x if and only if $x \in \pi g$-$fr(f^{-1}(F))$ for some $F \in RC(Y, f(x))$.*

Proof. $(\Rightarrow)$: Let f be not $(\pi g, s)$-continuous at x. Then there exists $F \in RC(Y, f(x))$ for which $f(U) \nsubseteq F$ for every $U \in \pi GO(X, x)$. Thus, $f(U) \cap (Y \backslash F) \neq \varnothing$ for every $U \in \pi GO(X, x)$ and hence $U \cap (X \backslash f^{-1}(F)) \neq \varnothing$ for every $U \in \pi GO(X, x)$. Thus, $x \in \pi g\text{-}cl(X \backslash f^{-1}(F))$. Since $x \in f^{-1}(F)$, $x \in \pi g\text{-}fr(f^{-1}(F))$.

$(\Leftarrow)$: Let $x \in X$ and suppose that there exists $F \in RC(Y, f(x))$ such that $x \in \pi g\text{-}fr(f^{-1}(F))$. Suppose f is $(\pi g, s)$-continuous at x. Then there exists a πg-open set U such that $x \in U$ and $U \subset f^{-1}(F)$. Hence, $x \notin \pi g\text{-}cl(X \backslash f^{-1}(F))$. This contradiction implies that f is not $(\pi g, s)$-continuous at x. □

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