SARAJEVO JOURNAL OF MATHEMATICS
Vol.3 (15) (2007), 123–130

A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN COMPACT METRIC SPACES SATISFYING AN IMPLICIT RELATION

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Abstract. We prove a common fixed point Theorem for four mappings in compact metric spaces satisfying an implicit relation using the concept of weak compatibility without decreasing assumption which generalizes Theorem 1 of V. Popa [9].

1. Introduction

Let S and T be self-mappings of a metric space (X,d). S and T are commuting if $STx = TSx$ for all $x \in X$. Sessa [10] defined S and T to be weakly commuting if for all $x \in X$

$$d(STx, TSx) \leq d(Tx, Sx) \tag{1.1}$$

Jungck [1] defined S and T to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper. See [1] and [10].

Jungck et al [2] defined S and T to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0. \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

2000 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. Weakly compatible mappings, common fixed point, compact metric space, implicit relation.

Clearly, weakly commuting implies compatible of type (A). By [2], the converse is not true. Examples are given to show that the two concepts of compatibility are independent. See [2].

Recently, Pathak and Khan [6] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, T^2x_n)\Big] \quad \text{and}$$
$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)\Big] \qquad (1.4)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true. See [6]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous. See [6].

Pathak et al [7] defined S and T to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0 \qquad (1.5)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous. See [7].

Pathak et al [8] defined S and T to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{3}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, S^2x_n) + \lim_{n\to\infty} d(Tt, T^2x_n)\Big] \quad \text{and}$$
$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{3}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, T^2x_n) + \lim_{n\to\infty} d(St, S^2x_n)\Big] \qquad (1.6)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous. See [8].

2. Preliminaries

Definition 1. [3]. *S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.*

Lemma 1. [1, 2, 6, 7, 8]. *If S and T are compatible, or compatible of type* (A)*, or compatible of type* (P)*, or compatible of type* (B)*, or compatible of type* (C)*, then they are weakly compatible.*

The following example shows that the converse is not true in general.

Example 1. Let $(X, d) = ([0, 10], |.|)$. Define S and T by:

$$Sx = \begin{cases} 3 & \text{if } x \in (0, 2] \\ 0 & \text{if } x \in \{0\} \cup (2, 10] \end{cases}, \quad Tx = \begin{cases} 0 & \text{if } x = 0 \\ x + 8 & \text{if } x \in (0, 2] \\ x - 2 & \text{if } x \in (2, 10] \end{cases}.$$

We have $Sx = Tx$ iff $x = 0$. $ST(0) = TS(0) = 0$. Then, (S, T) is weakly compatible. Let $\{x_n\}$ be a sequence in X defined by: $x_n = 2 + \frac{1}{n}$, $n \geq 1$.

$Sx_n = S(2 + \frac{1}{n}) = 0$, $Tx_n = T(2 + \frac{1}{n}) = \frac{1}{n}$. $Sx_n, Tx_n \to t = 0$ as $n \to \infty$. $STx_n = S(\frac{1}{n}) = 3$, $TSx_n = T(0) = 0$.

Since

$$\lim_{n\to\infty} |STx_n - TSx_n| = 3 \neq 0,$$

so (S, T) is not compatible.

$S^2x_n = S(0) = 0$, $T^2x_n = T(\frac{1}{n}) = 8 + \frac{1}{n}$. Therefore, $\left|TSx_n - S^2x_n\right| = 0$.

Since

$$\left|STx_n - T^2x_n\right| = 5 + \frac{1}{n} \to 5 \neq 0 \text{ as } n \to \infty,$$

then (S, T) is not compatible of type (A).

Since

$$\lim_{n\to\infty} \left|STx_n - T^2x_n\right| = 5 > \frac{1}{2}\Big[\lim_{n\to\infty} |STx_n - St| + \lim_{n\to\infty} \left|St - S^2x_n)\right|\Big] = \frac{3}{2},$$

hence (S, T) is not compatible of type (B).

Since

$$\lim_{n\to\infty} \left|S^2x_n - T^2x_n\right| = 8 \neq 0,$$

therefore, (S, T) is not compatible of type (P).

Since

$$\lim_{n\to\infty} \left|STx_n - T^2x_n\right| = 5 > \frac{1}{3}\Big[\lim_{n\to\infty} |STx_n - St| + \lim_{n\to\infty} \left|St - T^2x_n\right| + \lim_{n\to\infty} \left|St - S^2x_n\right|\Big] = \frac{11}{3},$$

then (S, T) is not compatible of type (C).

Definition 2. [4]. *S and T are said to be R–weakly commuting if there exists $R > 0$ such that*

$$d(STx, TSx) \leq Rd(Tx, Sx) \text{ for all } x \in X. \tag{2.1}$$

Definition 3. [5]. *S and T are said to be pointwise R–weakly commuting if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.*

It is proved in [5] that R–weak commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R– weakly commuting if and only if they are weakly compatible.

Let $\mathbb{R}_+$ be the set of all non-negative real numbers and F_6 the family of all continuous mappings $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is decreasing in variables t_5 and t_6.
(F_2) : for all $u, v \geq 0$ with
(F_a) : $F(u, v, v, u, u + v, 0) < 0$ or
(F_b) : $F(u, v, u, v, 0, u + v) < 0$
we have $u < v$.
(F_3) : $F(u, u, 0, 0, u, u) \geq 0$ for all $u > 0$.

The following Theorem was proved by Popa [9].

Theorem 1. *Let f, g, I and J be self-mappings of a compact metric space (X, d) such that*

(a) *$f(X) \subset J(X)$ and $g(X) \subset I(X)$.*

$$F(d(fx, gy), d(Ix, Jy), d(Ix, fx), d(Jy, gy), d(Ix, gy), d(fx, Jy)) < 0 \tag{2.2}$$

for all x, y in X and $F \in F_6$ for which one of $d(Ix, Jy), d(Ix, fx)$ and $d(Jy, gy)$ is positive.

(b) *The pair (f, I) is compatible or compatible of type* (A) *or compatible of type* (P) *and the pair (g, J) is weakly compatible.*

(c) *f and I are continuous.*

Then, f, g, I and J have a unique common fixed point z in X. Further, z is a common fixed point of f and I and of g and J.

Our purpose in this paper is to prove a common fixed point theorem for weakly compatible mappings satisfying an implicit relation in compact metric spaces without using decreasing assumption which generalizes Theorem 1 of [9].

3. Implicit relation

Let F_6 the family of all continuous mappings $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(C_1) : For all $u \geq 0, v > 0$ and $w \geq 0$ with
(C_a) : $F(u, v, v, u, w, 0) < 0$ or
(C_b) : $F(u, v, u, v, 0, w) < 0$
we have $u < v$.
(C_2) : For all $u > 0$, $F(u, 0, 0, u, u, 0) \geq 0$.
(C_3) : For all $u > 0$, $F(u, u, 0, 0, u, u) \geq 0$.

Remark 1. In the paper of Popa [9], the condition (C_2) should be added because the condition (F_a^*) in [9] implies if $v = 0$, $u < 0$ which is a contradiction since $u \geq 0$.

Example 2. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4\} + b(t_5 + t_6)$, $b > 0$.
(C_1) : Let $u, v > 0$ and $w \geq 0$. We have
$F(u, v, v, u, w, 0) = u - \max\{v, u\} + bw < 0$.
If $v \leq u$, then $u < u$ which is a contradiction. Therefore, $u < v$. Similarly, if $F(u, v, u, v, 0, w) < 0$ then $u < v$.
If $u = 0$, $v > 0$ and $w \geq 0$, then $u < v$.
(C_2) : For all $u > 0$, $F(u, 0, 0, u, u, 0) = bu > 0$.
(C_3) : $F(u, u, 0, 0, u, u) = 2bu > 0$ for all $u > 0$.

Example 3. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4\} + bt_5t_6$, $b > 0$.
$(C_1), (C_2)$ and (C_3) as in Example 2.

Example 4. $F(t_1, t_2, t_3, t_4, t_5, t_6) = (1 + pt_2)t_1 - pt_3t_4 - \max\{t_2, t_3, t_4\}$ $+b(t_5 + t_6)$, $b > 0$ and $p \geq 0$.
$(C_1), (C_2)$ and (C_3) as in Example 2.

Example 5. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_2^2 - b\frac{t_3^2 + t_4^2}{t_5 + t_6 + 1}$, $0 < a, b < 1$ and $a + 2b = 1$.
(C_1) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u^2 - av^2 - b\frac{(u^2 + v^2)}{w + 1} < 0$.
Then, $u^2 < \frac{a+b}{1-b}v^2 = v^2$. Hence, $u < v$. Similarly, if $F(u, v, u, v, 0, w) < 0$, then $u < v$.
If $u = 0$, $v > 0$ and $w \geq 0$ then $u < v$.
(C_2) : For all $u > 0$, $F(u, 0, 0, u, u, 0) = u^2 - b\frac{u^2}{u+1} > 0$.
(C_3) : For all $u > 0$, $F(u, u, 0, 0, u, u) = (1 - a)u^2 > 0$.

Example 6. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_2^2 - b\frac{t_3^2 + t_4^2}{t_5 t_6 + 1}$, $0 < a, b < 1$ and $a + 2b = 1$.
$(C_1), (C_2)$ and (C_3) as in Example 5.

Example 7. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \frac{t_3^2 t_4^2}{t_2 + t_5 + t_6 + 1}$.
(C_1) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u^3 - \frac{u^2 v^2}{v + w + 1} < 0$. Then $u < \frac{v^2}{v + w + 1} < v$. Similarly, if $F(u, v, u, v, 0, w) < 0$ then $u < v$.

If $u = 0$, $v > 0$ and $w \geq 0$ then $u < v$.
(C_2) : For all $u > 0$, $F(u, 0, 0, u, u, 0) = u^3 > 0$.
(C_3) : $F(u, u, 0, 0, u, u) = u^3 > 0$ for all $u > 0$.

Example 8. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \frac{t_3^2 t_4^2}{t_2 + t_5 t_6 + 1}$.
$(C_1), (C_2)$ and (C_3) as in Example 7.

Example 9. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_3 - c\frac{t_4 t_5}{t_5 + t_6 + 1}$, $0 < a, b, c < 1$ and $a + b + c = 1$.
(C_1) : Let $u, v > 0$, $w \geq 0$ and $F(u, v, v, u, w, 0) = u - av - bv - c\frac{uw}{w+1} < 0$. Then $u < \frac{a+b}{1-c}v = v$. Similarly, if $F(u, v, u, v, 0, w) < 0$ then $u < v$.
If $u = 0$, $v > 0$ and $w \geq 0$ then $u < v$.
(C_2) : For all $u > 0$, $F(u, 0, 0, u, u, 0) = u - c\frac{u^2}{u+1} > 0$
(C_3) : $F(u, u, 0, 0, u, u) = (1 - a)u > 0$ for all $u > 0$.

Example 10. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b\frac{t_3 t_6}{t_5 + t_6 + 1} - ct_4$, $0 < a, b, c < 1$ and $a + b + c = 1$.
$(C_1), (C_2)$ and (C_3) as in Example 9.

4. Main results

Theorem 2. *Let f, g, S and T be self-mappings of a compact metric space (X, d) satisfying the following conditions.*

$$S(X) \subset g(X) \text{ and } T(X) \subset f(X) \tag{4.1}$$

$$F(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) < 0 \tag{4.2}$$

for all $x, y \in X$ and $F \in F_6$ satisfies (C_1), (C_2) and (C_3) for which one of $d(fx, gy), d(fx, Sx)$ and $d(gy, Ty)$ is positive. Suppose that the pairs (S, f) and (T, g) are weakly compatible and S and f are continuous. Then, f, g, S and T have a unique common fixed point z in X. Further, z is a common fixed point of S and f and of T and g.

Proof. Let

$$m = \inf\{d(fx, Sx), x \in X\}.$$

Since X is a compact metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that

$$\lim_{n\to\infty} d(fx_n, Sx_n) = m.$$

Since

$$d(fx_0, Sx_0) \leq d(fx_0, fx_n) + d(fx_n, Sx_n) + d(Sx_n, Sx_0) \ ,$$

by the continuity of f and S and $\lim_{n\to\infty} x_n = x_0$ we get $d(fx_0, Sx_0) \leq m$ and therefore $d(fx_0, Sx_0) = m$.

Since $S(X) \subset g(X)$, then there exists $v \in X$ such that $Sx_0 = gv$ and $d(fx_0, gv) = m$.

Suppose that $m > 0$. Using (4.2) we have

$$F(d(Sx_0, Tv), d(fx_0, gv), d(fx_0, Sx_0), d(gv, Tv), d(fx_0, Tv), d(Sx_0, gv))$$
$$= F(d(gv, Tv), m, m, d(gv, Tv), d(fx_0, Tv), 0) < 0.$$

By (C_a) we get $d(gv, Tv) < m$.

Since $T(X) \subset f(X)$, then there exists $u \in X$ such that $fu = Tv$ and $d(fu, gv) < m$.

Since $d(fu, Su) \geq m > 0$. Using (4.2) we have

$$F(d(Su, Tv), d(fu, gv), d(fu, Su), d(gv, Tv), d(fu, Tv), d(Su, gv))$$
$$= F(d(fu, Su), d(gv, Tv), d(fu, Su), d(gv, Tv), 0, d(Su, gv)) < 0.$$

By (C_b) we get

$$m \leq d(fu, Su) < d(gv, Tv) < m$$

which is a contradiction. Then, $m = 0$ which implies that $fx_0 = Sx_0 = gv$.

If $d(gv, Tv) > 0$, using (4.2) we have

$$F(d(Sx_0, Tv), d(fx_0, gv), d(fx_0, Sx_0), d(gv, Tv), d(fx_0, Tv), d(Sx_0, gv))$$
$$= F(d(gv, Tv), 0, 0, d(gv, Tv), d(gv, Tv), 0) < 0$$

which is a contradiction of (C_2). Therefore, $z = fx_0 = Sx_0 = gv = Tv$.

Since the pair (S, f) is weakly compatible, we get $fz = Sz$.

If $z \neq Sz$, using (4.2) we have

$$F(d(Sz, Tv), d(fz, gv), d(fz, Sz), d(gv, Tv), d(fz, Tv), d(Sz, gv))$$
$$= F(d(Sz, z), d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)) < 0$$

which is a contradiction of (C_3). Therefore, $z = Sz = fz$.

Since the pair (g, T) is weakly compatible we get $Tz = gz$.

If $z \neq Tz$, using (4.2) we have

$$F(d(Sz, Tz), d(fz, gz), d(fz, Sz), d(gz, Tz), d(fz, Tz), d(Sz, gz))$$
$$= F(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) < 0$$

which is a contradiction of (C_3). Therefore, $z = gz = Tz$. Hence, z is a common fixed point of f, g, S and T.

The uniqueness of z follows from (4.2) and (C_3). □

Remark 2. Theorem 2 generalizes Theorem 1 of [9].

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(Received: July 25, 2006)
(Revised: November 29, 2006)

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