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RESULT ON VARIATIONAL INEQUALITY PROBLEM

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Abstract. The aim of this paper is to present an improved and extended version of the variational inequality problem of Vetrivel and Nanda [7] by using a weaker condition in the topological vector space and without using the result of Lassonde [3].

1. Introduction

An existence theorem dealing with the variational inequality problem was discussed by Gwinner [1], which is, an infinite dimensional version of the Walras excess demand theorem (see also Zeidler [8]), as follows:

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty compact convex subsets of Hausdorff locally convex topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $f : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ be continuous. Let $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a multifunction. Suppose that*

(i) *for each $y \in \mathcal{B}, \{x \in \mathcal{A} : f(x, y) < t\}$ is convex for all $t \in \mathbb{R}$,*
(ii) *$\mathcal{T}$ is an upper semicontinuous multifunction with nonempty compact convex values. Then there exists $x_0 \in \mathcal{A}$ and $y_0 \in \mathcal{T}(x_0)$ such that $f(x_0, y_0) \leq f(x, y_0)$ for all $x \in \mathcal{A}$.*

This result was improved by Vetrivel and Nanda [7] for multifunction with open inverse values in the same space in the line of Tarafdar and Yuan [6].

In this paper, our purpose is to improve and extend the result of Vetrivel and Nanda [7] by using a weaker assumption in the topological vector space instead of locally convex Hausdorff topological vector space. Since any convex or star-shaped set in a topological vector space is acyclic, we assume that the function involved is an upper semicontinuous multifunction with acyclic values instead of convex values. For this reason, we use the result of Shioji [4, Lemma 1] instead of Lassonde [3] along with the result of Horvath [2]. To prove the theorem, we follow the method of Tarafdar and Watson [5].

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2. Preliminaries

To prove our main result, Let us recall the following:

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty sets. The collection of all non-empty subsets of $\mathcal{X}$ is denoted by $2^{\mathcal{X}}$.

A multifunction or set-valued function from $\mathcal{X}$ to $\mathcal{Y}$ is defined to be a function that assigns to each elements of $\mathcal{X}$ a non-empty subset of $\mathcal{Y}$.

If $\mathcal{T}$ is a multifunction from $\mathcal{X}$ to $\mathcal{Y}$, then it is designated as $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$, and for every $x \in \mathcal{X}$, $\mathcal{T}x$ is called a value of $\mathcal{T}$.

For $\mathcal{A} \subseteq \mathcal{X}$, the image of $\mathcal{A}$ under $\mathcal{T}$, denoted by $\mathcal{T}(\mathcal{A})$, is defined as

$$\mathcal{T}(\mathcal{A}) = \bigcup_{x\in\mathcal{A}} \mathcal{T}x.$$

For $\mathcal{B} \subseteq \mathcal{Y}$, the preimage or inverse image of $\mathcal{B}$ under $\mathcal{T}$, denoted by $\mathcal{T}^{-1}(\mathcal{B})$, is defined as

$$\mathcal{T}^{-1}(\mathcal{B}) = \{x \in \mathcal{X} : \mathcal{T}x \cap \mathcal{B} \neq \phi\}.$$

If $y \in \mathcal{Y}$, then $\mathcal{T}^{-1}(y)$ is called a inverse value of $\mathcal{T}$. If it is open, then it called an open inverse value.

A multivalued function $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ is upper semicontinuous (usc)(lower semicontinuous(lsc)) if $\mathcal{T}^{-1}(\mathcal{B}) = \{x \in \mathcal{X} : \mathcal{T}x \cap \mathcal{B} \neq \phi\}$ is closed(open) for each closed (open) subset $\mathcal{B}$ of $\mathcal{Y}$. If $\mathcal{T}$ is both usc and lsc, then it is continuous.

A multifunction $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ is said to be a compact multifunction, if $\mathcal{T}(\mathcal{X})$ is contained in a compact subset of $\mathcal{Y}$.

It is known that if $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ is an upper semicontinuous multifunction with compact values, then $\mathcal{T}(\mathcal{K})$ is a compact in $\mathcal{Y}$ whenever $\mathcal{K}$ is compact subset of $\mathcal{X}$.

Let Δ_n be the standard $n-$dimensional simplex with vertices $e_0, e_1, e_2, \ldots, e_n$. If $J_n = \{0, 1, 2, \ldots, n\}$. We denote by $\Delta_J = Co\{e_j : j \in J\}$ for any non-empty subset J of J_n.

A topological space $\mathcal{X}$ is said to be contractible, if the identity mapping $\mathcal{I}_{\mathcal{X}}$ of $\mathcal{X}$ is homotopic to a constant function. A topological space is said to be an acyclic space if all of its reduced *C*ech homology groups over the rationals vanish. In particular, any contractible space is acyclic, and hence, any convex or star-shaped set in a topological vector space is acyclic. For a topological space $\mathcal{X}$, we shall denote by $ka(\mathcal{X})$, the family of all compact acyclic subsets of $\mathcal{X}$.

For our main result, we need following results due to Horvath [2] and Shioji [4, Lemma 1]:

Theorem 2.1. [2]. *Let $\mathcal{X}$ be a topological space. For any nonempty subset J of $\{0, 1, \ldots, n\}$, let Γ_J be a nonempty contractible subset of $\mathcal{X}$. If $\phi \neq J \subset J' \subset \{0, 1, \ldots, n\}$ implies $\Gamma_J \subset \Gamma_{J'}$, then there exists a single valued* *continuous function $f : \Delta_n \to \mathcal{X}$ such that $g[\Delta_J] \subseteq \Gamma_J$ for all non-empty subsets J of $\{0, 1, \ldots, n\}$.*

Theorem 2.2. [4]. *Let Δ_n be an $n-$dimensional simplex with the Euclidean topology and $\mathcal{X}$ a compact topological space. Let $h : \mathcal{X} \to \Delta_n$ be a single-valued continuous mapping and $\mathcal{T} : \Delta_n \to ka(\mathcal{X})$ be a upper semicontinuous set-valued mapping. Then there exists a point $x_0 \in \Delta_n$ such that $x_0 \in h(\mathcal{T}(x_0))$.*

3. Main result

Theorem 3.1. *Let $\mathcal{A}$ be as in Theorem 1.1 and $\mathcal{B}$ be an arbitrary subset of topological vector space $\mathcal{Y}$. Let $f : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ be continuous. Let $T : \mathcal{A} \to \mathcal{B}$ be a multifunction. Suppose that*

(i) *for each $y \in \mathcal{B}, \{x \in \mathcal{A} : f(x, y) < t\}$ is convex for all $t \in \mathbb{R}$;*
(ii) *$\mathcal{T}^{-1}(y)$ contains an open set $\mathcal{O}_y$(which may be empty) such that $\bigcup_{y \in \mathcal{T}(\mathcal{A})} \mathcal{O}_y = \mathcal{A}$;*
(iii) *for every open set $\mathcal{V}$ in $\mathcal{A}$, the set $\cap\{\mathcal{T}v : v \in \mathcal{V}\}$ is empty or contractible;*
(iv) *$\mathcal{T}(\mathcal{A})$ is compact and contractible;*

Then there exist $x_0 \in \mathcal{A}$ and $y_0 \in \mathcal{T}(x_0)$ such that $f(x_0, y_0) \leq f(x, y_0)$ for all $x \in \mathcal{A}$.

Proof. Since $\mathcal{A}$ is compact and $\bigcup_{y \in \mathcal{T}(\mathcal{A})} \mathcal{O}_y = \mathcal{A}$, there exists a finite subset $\{y_0, y_1, y_2, \ldots, y_n\} \subset \mathcal{T}(\mathcal{A})$ such that $\mathcal{A} = \bigcup_{i=0}^{n} \mathcal{O}_{y_i}$. Now, for each nonempty subset J of $N = \{0, 1, 2, \ldots, n\}$, define

$$\Gamma_J = \begin{cases} \cap\{\mathcal{T}(x) : x \in \bigcap_{j \in J} \mathcal{O}_{y_j}\}, & \text{if } \bigcap_{j \in J} \mathcal{O}_{y_j} \neq \phi, \\ \mathcal{T}(\mathcal{A}), & \text{otherwise} \end{cases} .$$

Evidently, if $x \in \bigcap_{j \in J} \mathcal{O}_{y_j}$, then $\{y_j : j \in J\} \subset \mathcal{T}(x)$. By (ii), each Γ_J is nonempty contractible and it is clear that $\Gamma_J \subseteq \Gamma_{J'}$, whenever $\phi \neq J \subset J' \subset N$.

By Theorem 2.1, there exists a single valued continuous function $f : \Delta_n \to \mathcal{T}(\mathcal{A})$ such that $f[\Delta_J] \subseteq \Gamma_J$, for all $\phi \neq J \subset N$. Let $\{h_0, h_1, \ldots, h_n\}$ be a continuous partition of unity subordinated to the open covering $\{\mathcal{O}_{y_i}\}_{i \in N}$ i.e., for each $i \in N$, $h_i : \mathcal{A} \to [0, 1]$ is continuous; $\{x \in \mathcal{A} : h_i(x) \neq 0\} \subset \mathcal{O}_{y_i}$ such that $\sum_{i=0}^{n} h_i(x) = 1$ for all $x \in \mathcal{A}$.

Define $h : \mathcal{A} \to \Delta_n$ by

$$h(x) = (h_0(x), h_1(x), h_2(x), \ldots, h_n(x)) \quad \text{for all } x \in \mathcal{A}.$$

Then, h is continuous. Then, $h(x) \subset \Delta_{J(x)}$ for all $x \in \mathcal{A}$, where $J(x)$: $\{J(x) := j \in N : h_j(x) \neq 0\}$. Therefore, we have

$$f(h(x)) \in f(\Delta_{J(x)}) \subseteq \Gamma_{J(x)} \subseteq \mathcal{T}(x), \;\; textforall \;\; x \in \mathcal{A}. \tag{3.1}$$

Consider $\mathcal{G} : \mathcal{T}(\mathcal{A}) \to \mathcal{A}$ defined by $\mathcal{G}(y) = \{z \in \mathcal{A} : f(z,y) \le f(w,y)$ for all $w \in \mathcal{A}\}$. For each $y \in \mathcal{T}(\mathcal{A})$, $\mathcal{G}(y)$ is nonempty since f assumes its minimum on the compact set $\mathcal{A}$. Also, it is closed and hence compact. Further, $\mathcal{G}(y)$ is acyclic. Indeed, let z_1 and $z_2 \in \mathcal{A}$ be such that $f(z_i, y) \le f(w, y)$ for all $w \in \mathcal{A}$ and $i = 1, 2$. Since any convex or star-shaped set in a topological vector space is acyclic. So, $\mathcal{G}(y)$ is acyclic. By the assumption on f, $f(\lambda z_1 + (1-\lambda)z_2, y) \le f(w,y)$ for all $w \in \mathcal{A}$. Since f is continuous, the graph of $\mathcal{G}$, $Gr(\mathcal{G}) = \{(y,z) : y \in \mathcal{T}(\mathcal{A}), z \in \mathcal{G}(y)\}$ is a closed subset of the compact set $\mathcal{T}(\mathcal{A}) \times \mathcal{A}$. Then it follows that $\mathcal{G}$ is upper semicontinuous.

Thus, by the above discussion $\mathcal{G}$ is upper semicontinuous with nonempty compact acyclic values and $f : \Delta_n \to \mathcal{T}(\mathcal{A})$ is continuous, it follows that the composition mapping $\mathcal{G} \circ f : \Delta \to \mathcal{A}$ is also upper semicontinuous with nonempty compact acyclic values. Since $h : \mathcal{A} \to \Delta_n$ is continuous and hence, Theorem 2.2 guarantees the existence of a point $x_0 \in \Delta_n$ such that $x_0 \in h(\mathcal{G} \circ f(x_0))$. Let $y_0 \in f(x_0)$, then we have

$$y_0 = f(x_0) \in f(h(\mathcal{A} \circ f(x_0))) = f(h(\mathcal{G}(y_0))),$$

so that there exists $x_0 \in \mathcal{G}(y_0)$ such that $y_0 = f(h(x_0)) \subset \mathcal{T}(x_0)$.

□

Remark 3.2. In the light of the fact that any convex or star-shaped set in a topological vector space is acyclic and condition (ii) of Theorem 3.1, our Theorem 3.1, it turn, improves and extends the Theorem 3.1 of Vetrivel and Nanda [7] in any topological vector space.

References

[1] J. Gwinner, *On fixed points and variational inequalities - a circular tour,* Nonlinear Anal., Theory Methods Appl., 5 (5) (1981), 565-583.

[2] C. D. Horvath, *Contractibility and generalized convexity,* J. Math. Anal. Appl., 156 (2) (1991), 341–357.

[3] M. Lassonde, *Fixed points for Kakutani factorizable multifunctions,* J. Math. Anal. Appl., 152 (1) (1990), 46–60.

[4] N. Shioji, *A further generalization of Knaster-Kuratoski-Masurkiewicz theorem,* Proc. Amer. Math. Soc., 111 (1991), 187–195.

[5] E. U. Tarafdar and P. J. Watson, *A coincidence point theorem and related results*, Appl. Math. Lett., 11 (1) (1998), 37–40.

[6] E. Tarafdar and X. Z. Yuan, *A remark on coincidence theorems,* Proc. Amer. Math. Soc., 122 (3) (1994), 957–859.

[7] V. Vetrivel and S. Nanda, *A remark on Gwinner's existence theorem on variational inequality problem,* Internat. J. Math. Math. Sci., 24 (8) (2000), 573–575.

[8] E. Zeidler, *Nonlinear Functional Analysis and its Applications III, Variational methods and optimization*, Translated from the German by Leo F. Boron, Springer-Verlag, New York, Berlin, 1985.

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