

## STABILITY CRITERIA FOR SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

GEVORG A. GRIGORIAN

**ABSTRACT.** We use some properties of solutions of Riccati equation for establishing boundedness and stability criteria for solutions of second order linear ordinary differential equations. We show that the conditions on coefficients of the equations, appearing in the proven criteria, do not follow from the conditions, which ensure the application of the WKB approximation to the second order linear equations. On these examples we compare the obtained results with the results obtained by the Liapunov and Bogdanov methods, by a method involving estimates of solutions in the Lozinski's logarithmic norms, and by the freezing method. We compare these results with the Wazevski's theorem as well.

### 1. INTRODUCTION

Let  $p(t)$  and  $q(t)$  be complex valued continuous functions on  $[t_0; +\infty)$ . Consider the equation

$$\phi''(t) + p(t)\phi'(t) + q(t)\phi(t) = 0, \quad t \geq t_0. \quad (1.1)$$

Study of the boundedness and stability behavior of solutions of Eq. (1.1) is an important problem of the qualitative theory of differential equations and many works are devoted to it (see e.g., the book [1] and cited works therein, [2 - 14]).

Let  $p(t)$  be continuously differentiable. In Eq. (1.1) make the substitution

$$\phi(t) = E(t)\psi(t), \quad t \geq t_0, \quad (1.2)$$

where  $E(t) \equiv \exp \left\{ -\frac{1}{2} \int_{t_0}^t p(\tau) d\tau \right\}$ . We get

$$\psi''(t) - \frac{D(t)}{4}\psi(t) = 0, \quad t \geq t_0, \quad (1.3)$$

where  $D(t) \equiv 2p'(t) + p^2(t) - 4q(t)$ ,  $t \geq t_0$ . One of important methods of studying the boundedness and stability problems of the solutions of Eq. (1.1) is the application of the Liouville's transformation (see [2], pp. 131, 132, 152, 153). In the book [5] on the basis of the Liouville's transformation a substantiation of asymptotic representation of the solutions of Eq. (1.3) and their derivatives is given (see. [5], pp.

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54 - 61, WKB approximation [Wentzel-Kramers-Brillouin]). It is assumed therein, that  $D(t)$  is twice continuously differentiable,  $D(t) \neq 0$ ,  $t \geq t_0$ ,  $\operatorname{Re} \sqrt{D(t)} \geq 0$  for  $t \gg 1$  and

$$\int_{t_0}^{+\infty} \left| 36 \frac{D''(\tau)}{D(\tau)^{3/2}} - 5 \frac{D'(\tau)^2}{D(\tau)^{5/2}} \right| d\tau < +\infty. \quad (1.4)$$

By virtue of (1.2) the WKB approximation gives possibility to describe wide classes of equations (1.1) with bounded and (or) unbounded solutions, classes of stable and (or) unstable equations (1.1) in terms of their coefficients.

Assume  $x(t)$  is a nonnegative continuous function on the half line  $[t_0; +\infty)$ . Consider the Riccati equation

$$y'(t) + y^2(t) = x(t), \quad t \geq t_0. \quad (1.5)$$

**Definition 1.1.** *The solution  $y(t)$  of Eq. (1.5) satisfying the initial condition  $y(t_0) = \sqrt{x(t_0)}$  is called differential root.*

For the study of the boundedness and stability problem of solutions of Eq. (1.1) in this work the Riccati equations method is applied, which (in this work) basically is an application of properties of the differential root of  $\frac{D(t)}{4}$ , corresponding to the solutions of Eq. (1.3). Unlike conditions on  $D(t)$ , providing of use WKB approximation, here other restrictions are imposed on  $D(t)$  assuming the condition A)  $D(t) > 0$ ,  $t \geq t_0$ , and  $p(t)$ ,  $D(t)$  are continuously differentiable functions; and other conditions, different from (1.4), be satisfied. Note that the case  $D(t) < 0$ ,  $t \geq t_0$ , is studied in [10]. Boundedness and stability tests for the solutions of Eq. (1.1) in terms of their coefficients are proved. Examples, to which the mentioned tests are applicable and which do not satisfy the condition (1.4), are represented.

## 2. MAIN RESULTS

For any positive and continuously differentiable on  $[t_0; +\infty)$  function  $x(t)$  denote

$$R_x(t_1; t) \equiv \frac{1 + \sqrt{x(t_0)}(t_1 - t_0)}{1 + \sqrt{x(t_0)}(t - t_0)} \exp \left\{ - \int_{t_1}^t \sqrt{x(s)} ds \right\} \sup_{\xi \in [t_0; t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} + \sup_{\xi \in [t_1; t]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}},$$

$$\rho_x(t) \equiv \inf_{t_1 \in [t_0; t]} R_x(t_1; t), \quad t_0 \leq t_1 \leq t.$$

In our main results the functions

$$\begin{aligned} \rho_{D/4}(t), \quad r_1(t) &\equiv \int_{t_0}^t [\sqrt{D(\tau)} - \operatorname{Re} p(\tau)] d\tau - \frac{1}{2} \ln D(t), \\ r_2(t) &\equiv \int_{t_0}^t [\sqrt{D(\tau)} - \operatorname{Re} p(\tau)] d\tau - \frac{1}{2} \ln D(t) + 2 \ln [1 + |p(t) - \sqrt{D(t)}|], \quad t \geq t_0, \end{aligned}$$

play a crucial role

**Theorem 2.1.** *Let the conditions*

A)  $D(t) > 0$ ,  $t \geq t_0$ , and  $p(t)$ ,  $D(t)$  are continuously differentiable functions; and one of the following groups of conditions

B)  $D(t)$  is a nondecreasing function; for some  $\varepsilon > 0$  the function  $\frac{D'(t)}{D(t)^{3/2-\varepsilon}}$  is bounded;

C)  $D(t) \geq \varepsilon > 0$ ,  $t \geq t_0$ , the function  $\frac{D'(t)}{D(t)}$  is bounded and  $\int_{t_0}^{+\infty} \rho_{D/4}(s) \frac{|D'(s)|}{D(s)^{3/2}} ds < +\infty$  be satisfied. Then all solutions of Eq. (1.1) are bounded (vanish on  $+\infty$ ) if and only if the function  $r_1(t)$  is bounded from above ( $\lim_{t \rightarrow +\infty} r_1(t) = -\infty$ ).

In many cases in applications of Eq. (1.1) its stability property plays an important role, and the property of boundedness of its solutions is a necessary condition for stability of Eq. (1.1). However this property (even the property of vanishing of all solutions to Eq. (1.1) in  $+\infty$ ) still does not guarantee the stability of Eq. (1.1). The next theorem indicates some conditions on the coefficients of Eq. (1.1) which guarantee Liapunov stability (asymptotically stability) of Eq. (1.1).

**Theorem 2.2.** *Let the condition A) and the group of conditions C) of Theorem 2.1 or the group of conditions*

D)  $D(t)$  is a nondecreasing function;  $\frac{D'(t)}{D(t)}$  is bounded, be satisfied. Then Eq. (1.1) is Liapunov stable (asymptotically) if and only if the function  $r_2(t)$  is bounded from above ( $\lim_{t \rightarrow +\infty} r_2(t) = -\infty$ ).

**Corollary 2.1.** *Let  $D(t) \geq \varepsilon > 0$ ,  $t \geq t_0$ ;  $\frac{|D'(t)|}{D(t)} \leq \frac{c}{(1+t-t_0)^\alpha}$ ,  $t \geq t_0$ ,  $c > 0$ ,  $\alpha > 0$ ;  $\int_{t_0}^{+\infty} \frac{d\tau}{\sqrt{D(\tau)(1+\tau-t_0)^{2\alpha}}} < +\infty$  and let the condition A) be satisfied. Then the following assertions are valid:*

A<sub>1</sub>) All solutions of Eq. (1.1) are bounded (vanish on  $+\infty$ ) if and only if the function  $r_1(t)$  is bounded above ( $\lim_{t \rightarrow +\infty} r_1(t) = -\infty$ ).

B<sub>1</sub>) Eq. (1.1) is Liapunov stable (asymptotically) if and only if the function  $r_2(t)$  is bounded above ( $\lim_{t \rightarrow +\infty} r_2(t) = -\infty$ ).

**Example 2.1.** *Consider the equation*

$$\phi''(t) + p_1(t)\phi'(t) + q_1(t)\phi(t) = 0, \quad t \geq 1, \quad (2.1)$$

where  $p_1(t) \equiv \lambda t$ ,  $q_1(t) \equiv \frac{\lambda}{2} + \frac{\lambda^2 t^2}{4} - \frac{t}{4} - \frac{1}{4} \int_1^t \sin^2 e^\tau d\tau$ ,  $t \geq 1$ ,  $\lambda = \text{const} \in \mathbf{C}$ . For

this equation we have  $D(t) = D_1(t) \equiv t + \int_1^t \sin^2 e^\tau d\tau$ ,  $t \geq 1$ , and  $D_1(t)$  is an increasing function on  $[1; +\infty)$ . Therefore for Eq. (2.1) the conditions A) and B) hold. For Eq. (2.1) we have

$$r_1(t) = \int_1^t \left[ \sqrt{\tau + \int_1^\tau \sin^2 e^s ds} - \text{Re} \lambda \tau \right] d\tau - \frac{1}{2} \ln \left[ t + \int_1^t \sin^2 e^\tau d\tau \right], \quad t \geq 1.$$

Hence  $\lim_{t \rightarrow +\infty} r_1(t) = \begin{cases} -\infty, & \text{if } \operatorname{Re} \lambda > 0; \\ +\infty, & \text{if } \operatorname{Re} \lambda \leq 0. \end{cases}$  Therefore by Theorem 2.1 if  $\operatorname{Re} \lambda > 0$  then all solutions of Eq. (2.1) vanish on  $+\infty$  and if  $\operatorname{Re} \lambda \leq 0$  then Eq. (2.1) has unbounded solution, i.e., Eq. (2.1) is unstable. It is not difficult to show, that for  $D(t) = D_1(t)$  the condition (1.4) does not hold. Therefore the WKB approximation is not applicable to Eq. (2.1). The substitution  $\phi'(t) = \psi(t)$ ,  $t \geq 1$ , in Eq. (2.1) reduces it to the system

$$\begin{cases} \phi'(t) = & \psi(t); \\ \psi'(t) = -q_1(t)\phi(t) - p_1(t)\psi(t), & t \geq 1. \end{cases}$$

It is not difficult to verify that the application of estimates of Liapunov ([4], p. 132) and Bogdanov ([4], p. 133), the estimate by Lozinski's logarithmic norms ([4], p. 137), as well as the estimation by freezing method ([4], p. 139) to the last system give no result. The application of Wazevski's theorem to the last system also gives no result. Hence these estimates and the Wazvski's theorem give no result for Eq. (2.1).

**Example 2.2.** Consider the equation

$$\phi''(t) + p_2(t)\phi'(t) + q_2(t)\phi(t) = 0, \quad t \geq 1, \quad (2.2)$$

where  $p_2(t) \equiv \lambda t^2$ ,  $q_2(t) \equiv \lambda t + \frac{\lambda^2 t^4}{4} - \frac{t^2}{4} - \frac{1}{4} \left( \int_1^t \sin e^\tau d\tau \right)^2$ ,  $t \geq 1$ ,  $\lambda = \text{const} \in \mathbf{C}$ .

For this equation we have  $D(t) = D_2(t) = t^2 + \left( \int_1^t \sin e^\tau d\tau \right)^2$ ,  $r_2(t) = \int_1^t \left[ \sqrt{\tau^2 + \left( \int_1^\tau \sin e^s ds \right)^2} - \operatorname{Re} \lambda \tau^2 \right] d\tau - \frac{1}{2} \ln D_2(t) + 2 \ln [1 + |p_2(t) - \sqrt{D_2(t)}|]$ ,  $t \geq 1$ .

It is not difficult to check that the conditions A) and C) for Eq. (2.2) hold and for  $D(t) = D_2(t)$  the condition (1.4) does not fulfill. Therefore Theorem 2.2 is applicable to Eq. (2.2) and the WKB approximation is not applicable to Eq. (2.2). We have

$$\lim_{t \rightarrow +\infty} r_2(t) = \begin{cases} -\infty, & \text{if } \operatorname{Re} \lambda > 0; \\ +\infty, & \text{if } \operatorname{Re} \lambda \leq 0. \end{cases}$$

By Theorem 2.2 from here it follows that for  $\operatorname{Re} \lambda > 0$  Eq. (2.2) is asymptotically stable and for  $\operatorname{Re} \lambda \leq 0$  Eq. (2.2) is unstable. Moreover we also can use Theorem 2.1 to Eq. (2.2) and show that for  $\operatorname{Re} \lambda \leq 0$  it has an unbounded solution. It is not difficult to verify that the application of the mentioned above estimates and the Wazevski's theorem to Eq. (2.2) gives no result.

**Example 2.3.** Consider the equation

$$\phi''(t) + p_3(t)\phi'(t) + q_3(t)\phi(t) = 0, \quad t \geq 1, \quad (2.3)$$

where  $p_3(t) \equiv \lambda + \mu \sin t$ ,  $q_3(t) \equiv \frac{\mu \cos t}{2} + \frac{(\lambda + \mu \sin t)^2}{4} - \frac{1}{4} \left( \alpha + \beta \cos \ln t + \gamma \int_1^t \frac{\sin^2 \tau}{\tau} d\tau \right)$ ,  $\lambda = \text{const} \in \mathbf{C}$ ,  $\mu = \text{const} \in \mathbf{C}$ ,  $\alpha = \text{const} \geq \beta = \text{const} > 0$ ,  $\gamma = \text{const} > 0$ .

For this equation we have  $D(t) = D_3(t) = \alpha + \beta \cos \ln t + \gamma \int_1^t \frac{\sin^2 \tau}{\tau} d\tau$ ,  $r_1(t) = \int_1^t \left[ \sqrt{\alpha + \beta \cos \ln \tau + \gamma \int_1^\tau \frac{\sin^2 s}{s} ds} - \operatorname{Re} \lambda - \operatorname{Re} \mu \sin \tau \right] d\tau - \frac{1}{2} \ln \left( \alpha + \beta \cos \ln t + \gamma \int_1^t \frac{\sin^2 \tau}{\tau} d\tau \right)$ ,  
 $r_2(t) = r_1(t) + 2 \ln \left[ 1 + \left| \lambda + \mu \sin t - \alpha - \beta \cos \ln t - \gamma \int_1^t \frac{\sin^2 \tau}{\tau} d\tau \right| \right]$ ,  $t \geq 1$ . Hence

$$\lim_{t \rightarrow +\infty} r_1(t) = \lim_{t \rightarrow +\infty} r_2(t) = \begin{cases} -\infty, & \text{if } \operatorname{Re} \lambda > \sqrt{\alpha}; \\ +\infty, & \text{if } \operatorname{Re} \lambda \leq \sqrt{\alpha}. \end{cases} \quad (2.4)$$

We can easily check that for  $D(t) = D_3(t)$  condition (1.4) does not hold. Therefore the WKB approximation is not applicable to Eq. (2.3). It is not difficult to verify that for Eq. (2.3) all conditions of Corollary 2,1 are fulfilled. Therefore taking into account (2.4) we get:

for  $\operatorname{Re} \lambda > \sqrt{\alpha}$  Eq. (2.3) is asymptotically stable;

for  $\operatorname{Re} \lambda \leq \sqrt{\alpha}$  Eq. (2.3) has unbounded solution.

It is not difficult to verify that the application of the mentioned above estimates and the Wazewski's theorem to Eq. (2.3) gives no result. Note that the results of work [11] concern to the case  $D(t) < 0$ ,  $t \geq t_0$ , and the results of work [12] concern to the case of periodic functions  $p(t)$  and  $q(t)$ . Therefore the results of these works cannot be applicable to the equations (2.1) - (2.3).

### 3. PROOF OF THE MAIN RESULTS

To prove the main results at fist we shall formulate and prove some preliminary propositions. Let  $x_1(t)$  be a real valued continuous function on  $[t_0; +\infty)$ . Along with Eq. (1.5) consider the Riccaty equation

$$y'(t) + y^2(t) = x_1(t), \quad t \geq t_0. \quad (3.1)$$

The following assertion is valid (see [15]).

**Theorem 3.1.** *Let Eq. (1.5) have a real valued solution  $y_0(t)$  on  $[t_0; +\infty)$ , and let  $x_1(t) \geq x(t)$ ,  $t \geq t_0$ . Then for each  $y_{(0)} \geq y_0(t_0)$  Eq. (3.1) has a solution  $y_1(t)$  on  $[t_0; +\infty)$ , satisfying the initial condition  $y_1(t_0) = y_{(0)}$ , moreover  $y_1(t) \geq y_0(t)$ ,  $t \geq t_0$ .*

The proof of a more general theorem is presented in [16].

Since  $y_0(t) \equiv 0$  is a solution of the equation

$$y'(t) + y^2(t) = 0, \quad t \geq t_0,$$

from Theorem 3.1 we immediately get:

**Corollary 3.1.** *Let  $x(t) \geq 0$ ,  $t \geq t_0$ . Then for any  $y_{(0)} \geq 0$  Eq. (1.5) has a solution  $y_1(t)$  on  $[t_0; +\infty)$ , satisfying the initial condition  $y_1(t_0) = y_{(0)}$ , moreover  $y_1(t) \geq 0$ ,  $t \geq t_0$ .*

From Corollary 3.1 it follows, that the differential root is defined on  $[t_0; +\infty)$  and is nonnegative.

*Remark 3.1.* A more detailed study of the properties of the differential root is presented in [15].

In the sequel the differential root of  $x(t)$  we shall denote by  $y_x(t)$ .

Let  $x(t)$  be continuously differentiable and  $x(t) > 0$ ,  $t \geq t_0$ . Then

$$[y_x(t) - \sqrt{x(t)}]' + (y_x(t) + \sqrt{x(t)})[y_x(t) - \sqrt{x(t)}] = -(\sqrt{x(t)})', \quad t \geq t_0.$$

It follows from here, that  $u_0(t) \equiv y_x(t) - \sqrt{x(t)}$  ( $t \geq t_0$ ) is a solution of the first order linear equation:

$$u'(t) + F(t)u(t) = -(\sqrt{x(t)})', \quad t \geq t_0,$$

where  $F(t) \equiv y_x(t) + \sqrt{x(t)}$ ,  $t \geq t_0$ . Therefore by Cauchy formula

$$y_x(t) - \sqrt{x(t)} = \exp\left\{-\int_{t_1}^t F(\tau) d\tau\right\} \left[ y_x(t_1) - \sqrt{x(t_1)} - \int_{t_1}^t \exp\left\{\int_{t_1}^{\tau} F(s) ds\right\} (\sqrt{x(\tau)})' d\tau \right], \quad t, t_1 \geq t_0, \quad (3.2)$$

in particular,

$$y_x(t) - \sqrt{x(t)} = -\int_{t_0}^t \exp\left\{-\int_{\tau}^t F(s) ds\right\} (\sqrt{x(\tau)})' d\tau, \quad t \geq t_0.$$

Hence

$$\begin{aligned} |y_x(t_1) - \sqrt{x(t_1)}| &= \int_{t_0}^{t_1} F(\tau) \exp\left\{-\int_{\tau}^{t_1} F(s) ds\right\} \frac{(\sqrt{x(\tau)})'}{F(\tau)} d\tau \\ &\leq \sup_{\xi \in [t_0, t_1]} \frac{|(\sqrt{x(\xi)})'|}{F(\xi)} \int_{t_0}^{t_1} d \left[ \exp\left\{-\int_{\tau}^{t_1} F(s) ds\right\} \right] \\ &= \sup_{\xi \in [t_0, t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} \left[ 1 - \exp\left\{-\int_{t_0}^{t_1} F(s) ds\right\} \right] \leq \sup_{\xi \in [t_0, t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}}. \end{aligned}$$

Then from (3.2) we get:

$$\begin{aligned} |y_x(t) - \sqrt{x(t)}| &\leq \\ &\leq \exp\left\{-\int_{t_1}^t F(\tau) d\tau\right\} \sup_{\xi \in [t_0, t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} + \int_{t_1}^t \exp\left\{-\int_{\tau}^t F(s) ds\right\} |(\sqrt{x(\tau)})'| d\tau \\ &\leq \exp\left\{-\int_{t_1}^t F(\tau) d\tau\right\} \sup_{\xi \in [t_0, t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} + \sup_{\xi \in [t_1, t]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}}, \quad t_0 \leq t_1 \leq t, \quad (3.3) \end{aligned}$$

as far as

$$\int_{t_1}^t \exp\left\{-\int_{\tau}^t F(s) ds\right\} |(\sqrt{x(\tau)})'| d\tau = \int_{t_1}^t F(\tau) \exp\left\{-\int_{\tau}^t F(s) ds\right\} \frac{|(\sqrt{x(\tau)})'|}{F(\tau)} d\tau \leq$$

$$\begin{aligned}
 &\leq \sup_{\xi \in [t_0; t_1]} \frac{|(\sqrt{x(\xi)})'|}{F(\xi)} \int_{t_1}^t F(\tau) \exp \left\{ - \int_{\tau}^t F(s) ds \right\} d\tau \\
 &= \sup_{\xi \in [t_0; t_1]} \frac{|(\sqrt{x(\xi)})'|}{F(\xi)} \int_{t_1}^t d \left[ \exp \left\{ - \int_{\tau}^t F(s) ds \right\} \right] \\
 &= \sup_{\xi \in [t_0; t_1]} \frac{|(\sqrt{x(\xi)})'|}{F(\xi)} \left[ 1 - \exp \left\{ - \int_{t_1}^t F(s) ds \right\} \right] \leq \sup_{\xi \in [t_1; t]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}}.
 \end{aligned}$$

**Lemma 3.1.** For every  $s \geq t_0$  the inequality

$$y_x(s) \geq \frac{y_x(t_0)}{1 + y_x(t_0)(s - t_0)}$$

is valid.

See the proof in [15].

By virtue of this lemma we have:

$$\int_{t_1}^t y_x(s) ds \geq \int_{t_1}^t \frac{\sqrt{x(t_0)} ds}{1 + \sqrt{x(t_0)}(s - t_0)} = \ln \frac{1 + \sqrt{x(t_0)}(t - t_0)}{1 + \sqrt{x(t_0)}(t_1 - t_0)}, \quad t, t_1 \geq t_0, \quad t_1 \leq t.$$

From here and from (3.3) it follows:

$$\begin{aligned}
 |y_x(t) - \sqrt{x(t)}| &\leq \left[ \frac{1 + \sqrt{x(t_0)}(t_1 - t_0)}{1 + \sqrt{x(t_0)}(t - t_0)} \exp \left\{ - \int_{t_1}^t \sqrt{x(s)} ds \right\} \sup_{\xi \in [t_0; t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} + \right. \\
 &\quad \left. + \sup_{\xi \in [t_1; t]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} \right] = R_x(t_1; t), \quad t_0 \leq t_1 \leq t.
 \end{aligned}$$

It means that,

$$|y_x(t) - \sqrt{x(t)}| \leq \inf_{t_1 \in [t_0; t]} R_x(t_1; t) = \rho_x(t), \quad t \geq t_0. \quad (3.4)$$

If  $\frac{1}{2} \frac{|x'(t)|}{x(t)} \leq c$ ,  $t \geq t_0$ , then it is evident, that

$$\rho_x(t) \leq R_x(t_1; t) \leq c, \quad t \geq t_0. \quad (3.5)$$

Let

$$x(t) \geq \varepsilon > 0, \quad \frac{|x'(t)|}{x(t)} \leq \frac{c}{[1 + \sqrt{x(t_0)}(t - t_0)]^\alpha}, \quad t \geq t_0, \quad c > 0, \quad \alpha > 0. \quad (3.6)$$

Let us define  $t_1 = t_1(t)$  by relation

$$t - t_1 = \frac{\alpha}{\sqrt{\varepsilon}} \ln[1 + \sqrt{x(t_0)}(t - t_0)], \quad t > \bar{t},$$

where  $\bar{t}$  ( $< +\infty$ ) satisfies the condition:

$\ln[1 + \sqrt{x(t_0)}(t - t_0)] < \frac{1}{2}(t - t_0)$ ,  $t > \bar{t}$  (since  $\frac{\ln[1 + \sqrt{x(t_0)}(t - t_0)]}{t - t_0} \rightarrow 0$  for  $t \rightarrow +\infty$ , the number  $\bar{t}$  always exists). Since  $x(t) \geq \varepsilon > 0$ ,  $t \geq t_0$ , we have

$$\int_{t_1}^t \sqrt{x(s)} ds \geq \sqrt{\varepsilon}(t - t_1) = \ln[1 + \sqrt{x(t_0)(t - t_0)}]^\alpha, \quad t_0 \leq t_1 \leq t.$$

Therefore, taking into account (3.6) we get:

$$\begin{aligned} R_x(t_1(t); t) &\leq \frac{1}{2} \frac{1 + \sqrt{x(t_0)(t_1 - t_0)}}{[1 + \sqrt{x(t_0)(t - t_0)}]^{1+\alpha}} \sup_{\xi \in [t_0; t_1]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} + \sup_{\xi \in [t_1; t]} \frac{|(\sqrt{x(\xi)})'|}{\sqrt{x(\xi)}} \\ &\leq \frac{c[1 + \sqrt{x(t_0)(t_1 - t_0)}]}{2[1 + \sqrt{x(t_0)(t - t_0)}]^{1+2\alpha}} + \frac{c}{2[1 + \sqrt{x(t_0)(t_1 - t_0)}]^\alpha}, \quad t > \bar{t}. \end{aligned} \quad (3.7)$$

From definition of  $t_1(t)$  it follows, that  $t - t_1 < \frac{1}{2}(t - t_0)$  for  $t > \bar{t}$ . Then  $t_1 - t_0 > \frac{1}{2}(t - t_0)$  for  $t > \bar{t}$ , and therefore from (3.7) we obtain:

$$R_x(t_1(t); t) \leq \frac{c}{2} \left\{ \frac{1}{[1 + \sqrt{x(t_0)(t - t_0)}]^\alpha} + \frac{1}{[1 + \frac{\sqrt{x(t_0)}{2}(t - t_0)]^\alpha} \right\} \leq \frac{2^{\alpha-1}c}{[1 + \sqrt{x(t_0)(t - t_0)}]^\alpha},$$

$t > \bar{t}$ . From here we immediately get:

**Lemma 3.2.** *Let  $x(t)$  satisfy the conditions (3.6). Then*

$$\rho_x(t) \leq \frac{2^{\alpha-1}c}{[1 + \sqrt{x(t_0)(t - t_0)}]^\alpha}, \quad t > \bar{t}.$$

Consider the sets

$$A_t = A_t(x) \equiv \{s \in [t_0; t] : y'_x(s) \geq 0\}, \quad B_t = B_t(x) \equiv \{s \in [t_0; t] : y'_x(s) < 0\}.$$

It is evident, that  $A_t$  and  $B_t$  are measurable and

$$A_t \cup B_t = [t_0; t], \quad A_t \cap B_t = \emptyset. \quad (3.8)$$

Suppose  $s \in A_t$ . Then  $y'_x(s) \geq 0$ ,  $y_x(s) \leq \sqrt{x(s)}$ , and therefore

$$\int_{A_t} \frac{y'_x(s)}{2\sqrt{x(s)}} ds \leq \int_{A_t} \frac{y'_x(s)}{y_x(s) + \sqrt{x(s)}} ds \leq \int_{A_t} \frac{y'_x(s)}{2y_x(s)} ds. \quad (3.9)$$

For  $s \in B_t$  we have:  $y'_x(s) < 0$ ,  $y_x(s) > \sqrt{x(s)}$ . Then

$$\int_{B_t} \frac{y'_x(s)}{2\sqrt{x(s)}} ds \leq \int_{B_t} \frac{y'_x(s)}{y_x(s) + \sqrt{x(s)}} ds \leq \int_{B_t} \frac{y'_x(s)}{2y_x(s)} ds.$$

Summarizing each part of these inequalities with the corresponding parts of (3.9) and taking into account (3.8) we get:

$$\int_{t_0}^t \frac{y'_x(s)}{2\sqrt{x(s)}} ds \leq \int_{t_0}^t \frac{y'_x(s)}{y_x(s) + \sqrt{x(s)}} ds \leq \int_{t_0}^t \frac{y'_x(s)}{2y_x(s)} ds, \quad t \geq t_0.$$

Due to the equality  $y'_x(s) = (\sqrt{x(s)} - y_x(s))(\sqrt{x(s)} - y_x(s))$ ,  $s \geq t_0$ , from here we obtain the inequality

$$-\frac{1}{2} \ln \frac{y_x(t)}{y_x(t_0)} \leq \int_{t_0}^t [y_x(s) - \sqrt{x(s)}] ds \leq - \int_{t_0}^t \frac{y'_x(s)}{2\sqrt{x(s)}} ds, \quad t \geq t_0.$$

Therefore,

$$\frac{1}{4} \ln \left[ \frac{x(t)}{y_x^2(t)} \right] \leq \int_{t_0}^t [y_x(s) - \sqrt{x(s)}] ds + \frac{1}{4} \ln \left[ \frac{x(t)}{x(t_0)} \right] \leq \int_{t_0}^t \frac{(\sqrt{x(s)} - y_x(s))'}{2\sqrt{x(s)}} ds, \quad t \geq t_0.$$

Then integrating the last integral by parts we obtain:

$$\begin{aligned} \frac{1}{4} \ln \left[ \frac{x(t)}{y_x^2(t)} \right] &\leq \int_{t_0}^t [y_x(s) - \sqrt{x(s)}] ds + \frac{1}{4} \ln \left[ \frac{x(t)}{x(t_0)} \right] \\ &\leq \frac{1}{2} - \frac{y_x(t)}{2\sqrt{x(t)}} + \int_{t_0}^t \frac{[\sqrt{x(s)} - y_x(s)]x'(s)}{4x(s)^{3/2}} ds. \quad t \geq t_0, \end{aligned} \quad (3.10)$$

or

$$\begin{aligned} \frac{1}{4} \ln \left[ \frac{x(t)}{y_x^2(t)} \right] &\leq \int_{t_0}^t [y_x(s) - \sqrt{x(s)}] ds + \frac{1}{4} \ln \left[ \frac{x(t)}{x(t_0)} \right] \\ &\leq \frac{1}{2} - \frac{y_x(t)}{2\sqrt{x(t)}} + \int_{t_0}^t \frac{y'_x(s)x'(s)}{4[\sqrt{x(s)} + y_x(s)]x(s)^{3/2}} ds, \quad t \geq t_0. \end{aligned} \quad (3.11)$$

Consider the function

$$Q_x(t) \equiv \int_{t_0}^t [y_x(s) - \sqrt{x(s)}] ds + \frac{1}{4} \ln x(t), \quad t \geq t_0.$$

**Lemma 3.3.** *Let  $x(t)$  be a monotone nondecreasing function, and let for some  $\varepsilon > 0$  the function  $\frac{x'(t)}{x(t)^{3/2-\varepsilon}}$  be bounded. Then  $Q_x(t)$  is bounded.*

*Proof.* Since  $x(t)$  is a monotone nondecreasing function, then (see [15])

$y_x(t) \leq \sqrt{x(t)}$  from here and from the first inequality of (3.11) it follows, that  $Q_x(t) \geq \frac{1}{4} \ln x(t_0) > -\infty$ ,  $t \geq t_0$ . Therefore,  $Q_x(t)$  is bounded from below. Suppose  $\frac{|x'(t)|}{x(t)^{3/2-\varepsilon}} \leq c$ ,  $t \geq t_0$ , for some  $\varepsilon > 0$ ,  $c > 0$ . Then taking into account the inequality  $y_x(t) \leq \sqrt{x(t)}$ ,  $t \geq t_0$ , we will have:

$$\begin{aligned} \int_{t_0}^t \frac{y'_x(s)x'(s)}{4[\sqrt{x(s)} + y_x(s)]x(s)^{3/2}} ds &\leq \int_{t_0}^t \frac{y'_x(s)x'(s)}{8y_x(s)^{1+2\varepsilon}x(s)^{3/2-\varepsilon}} ds \leq \frac{c}{8} \int_{t_0}^{+\infty} \frac{d(y_x(s))^{1+2\varepsilon}}{y_x(s)} \\ &\stackrel{def}{=} d_0 < +\infty, \quad t \geq t_0. \end{aligned}$$

From here and from the second inequality of (3.11) it follows, that  $Q_x(t) \leq d_0 + \frac{1}{2} + \frac{1}{4} \ln x(t_0) < +\infty, t \geq t_0$ . Therefore  $Q_x(t)$  is bounded above. The lemma is proved.  $\square$

**Lemma 3.4.** *Let  $x(t) \geq \varepsilon > 0, t \geq t_0, \frac{x'(t)}{x(t)}$  be bounded, and let  $\int_{t_0}^{+\infty} \rho_x(s) \frac{|x'(s)|}{x(s)^{3/2}} ds < +\infty$ . Then  $Q_x(t)$  is bounded.*

*Proof.* By virtue of mean value theorem

$$\ln \left[ \frac{x(t)}{y_x(t)} \right] = 2[\ln \sqrt{x(t)} - \ln y_x(t)] = 2 \frac{\sqrt{x(t)} - y_x(t)}{\xi(t)}, \quad (3.12)$$

where  $\xi(t) \in [\min\{\sqrt{x(t)}, y_x(t)\}; \max\{\sqrt{x(t)}, y_x(t)\}]$ ,  $t \geq t_0$ . Since  $x(t) \geq \varepsilon, t \geq t_0$ , we have  $\xi(t) \geq \min\{\sqrt{x(t)}, y_x(t)\} \geq \sqrt{\varepsilon}, t \geq t_0$ . From here, from the boundedness of  $\frac{x'(t)}{x(t)}$  and from (3.4), (3.5), (3.12) it follows:

$$\left| \ln \left[ \frac{x(t)}{y_x^2(t)} \right] \right| \leq \frac{2}{\sqrt{\varepsilon}} \rho_x(t) \leq \frac{c}{\sqrt{\varepsilon}} < +\infty, \text{ where } \frac{|x'(t)|}{x(t)} \leq 2c, \quad t \geq t_0.$$

From here and from the first inequality of (3.10) it follows, that  $Q_x(t) \geq \frac{1}{4} \ln x(t_0) - \frac{c}{\varepsilon} > -\infty, t \geq t_0$ . Therefore,  $Q_x(t)$  is bounded below. From (3.4) it follows:

$$\int_{t_0}^t \frac{[\sqrt{x(s)} - y_x(s)]x'(s)}{4x(s)^{3/2}} ds \leq \int_{t_0}^{+\infty} \frac{\rho_x(s)|x'(s)|}{4x(s)^{3/2}} ds \stackrel{\text{def}}{=} d_1 < +\infty.$$

Then taking into account the second inequality of (3.11) we will have:  $Q_x(t) \leq \frac{1}{2} + d_1 + \frac{1}{4} \ln x(t_0) < +\infty$ . Therefore,  $Q_x(t)$  is bounded above. The lemma is proved.  $\square$

Consider the Riccati equation

$$y'(t) + y^2(t) = \frac{D(t)}{4}, \quad t \geq t_0. \quad (3.13)$$

In the sequel we shall assume, that the conditions A) are satisfied. Each solution of Eq. (2.14), existing on  $[t_0; +\infty)$ , is connected with some solution  $\psi(t)$  of Eq. (1.3) by the equality (see [3], pp. 391, 392)

$$\psi(t) = \psi(t_0) \exp \left\{ \int_{t_0}^t y(\tau) d\tau \right\}, \quad t \geq t_0, \quad \psi(t_0) \neq 0.$$

By (1.2) from here it follows, that

$$\phi_0(t) \equiv \exp \left\{ \int_{t_0}^t [y_{D/4}(\tau) - \frac{1}{2}p(\tau)] d\tau \right\}, \quad t \geq t_0,$$

is a solution of Eq. (1.1). Since  $\phi_0'(t) = [y_{D/4}(t) - \frac{1}{2}p(t)]\phi_0(t), t \geq t_0$ , we have

$$|\phi_0'(t)| \leq |y_{D/4}(t) - \frac{1}{2}\sqrt{D(t)}| |\phi_0(t)| + \frac{1}{2} |\sqrt{D(t)} - p(t)| |\phi_0(t)|. \quad t \geq t_0.$$

By (3.4) from here it follows:

$$|\phi_0'(t)| \leq \rho_{D/4}(t)|\phi_0(t)| + \frac{1}{2} \left[ 1 + |p(t) - \sqrt{D(t)}| \right] |\phi_0(t)|, \quad t \geq t_0. \quad (3.14)$$

Since

$$\frac{1}{2} [\sqrt{D(t)} - p(t)] \phi_0(t) = \phi_0'(t) + \left[ \frac{1}{2} \sqrt{D(t)} - y_{D/4}(t) \right] \phi_0(t), \quad t \geq t_0,$$

we have

$$\frac{1}{2} [1 + |p(t) - \sqrt{D(t)}|] |\phi_0(t)| \leq |\phi_0'(t)| + [1 + |\sqrt{D(t)} - y_{D/4}(t)|] |\phi_0(t)|, \quad t \geq t_0.$$

By virtue of (3.4) it follows from here, that

$$[1 + |p(t) - \sqrt{D(t)}|] |\phi_0(t)| \leq 2|\phi_0'(t)| + [1 + \rho_{D/4}(t)] |\phi_0(t)|, \quad t \geq t_0. \quad (3.15)$$

It is not difficult to see, that

$$|\phi_0(t)| = 2 \exp\{Q_{D/4}(t) + \frac{1}{2}r_1(t)\}, \quad t \geq t_0. \quad (3.16)$$

From here and from (3.14) it follows:

$$|\phi_0'(t)| \leq \rho_{D/4}(t)|\phi_0(t)| + \exp\{Q_{D/4}(t) + \frac{1}{2}r_2(t)\}, \quad t \geq t_0. \quad (3.17)$$

It follows from (3.15), that  $\exp\{Q_{D/4}(t) + \frac{1}{2}r_2(t)\} = \exp\left\{ \int_{t_0}^t [y_{D/4}(\tau) - \frac{1}{2}\operatorname{Re} p(\tau)] d\tau + \ln[1 + |p(t) - \sqrt{D(t)}|] - \frac{1}{4}\ln 4 \right\} \leq [1 + |p(t) - \sqrt{D(t)}|] |\phi_0(t)| \leq 2|\phi_0'(t)| + [1 + \rho_{D/4}(t)] |\phi_0(t)|, t \geq t_0$ . Therefore,

$$\exp\{r_2(t)\} \leq \exp\{-2Q_{D/4}(t)\} [2|\phi_0'(t)| + [1 + \rho_{D/4}(t)] |\phi_0(t)|], \quad t \geq t_0. \quad (3.18)$$

**Lemma 3.5.** *All solutions of Eq. (1.1) are bounded (vanish on  $+\infty$ ) if and only if the function  $\phi_0(t)$  is bounded (vanishes on  $+\infty$ ).*

See the proof in [15].

**Lemma 3.6.** *Eq. (1.1) is Liapunov stable (asymptotically) if and only if  $\phi_0(t)$  and  $\phi_0'(t)$  are bounded (vanish on  $+\infty$ ).*

See the proof in [15].

*Proof of Theorem 2.1.* Since the conditions A) hold by virtue of Lemma 3.3 if the conditions B) are satisfied, then the function  $Q_{D/4}(t)$  is bounded. If the conditions C) are satisfied, then the boundedness of  $Q_{D/4}(t)$  follows from Lemma 3.4. Thus the satisfiability of either B) or C) ensures the boundedness of  $Q_{D/4}(t)$ . Then from (3.15) it follows, that the function  $\phi_0(t)$  is bounded ( $\lim_{t \rightarrow +\infty} \phi_0(t) = 0$ ), if and only if the function  $r_1(t)$  is bounded above ( $\lim_{t \rightarrow +\infty} r_1(t) = -\infty$ ). By virtue of Lemma 3.5

from here it follows, that all solutions of Eq. (1.1) are bounded (vanish on  $+\infty$ ) if and only if the function  $r_1(t)$  is bounded above ( $\lim_{t \rightarrow +\infty} r_1(t) = -\infty$ ).

The theorem is proved.  $\square$

*Proof of Theorem 2.2.* From D) follows B). Therefore, by already proven the conditions A), C) and D) provide the boundedness of the function  $Q_{D/4}(t)$ . Then from (3.16) - (3.18) it follows, that  $\phi_0(t)$  and  $\phi'_0(t)$  are bounded (vanish on  $+\infty$ ) if and only if  $r_1(t)$  and  $r_2(t)$  are bounded above ( $\lim_{t \rightarrow +\infty} r_j(t) = -\infty$ ,  $j = 1, 2$ ). Since  $r_1(t) \leq r_2(t)$ ,  $t \geq t_0$ , from the boundedness above of  $r_2(t)$  (from the equality  $\lim_{t \rightarrow +\infty} r_2(t) = -\infty$ ) it follows the boundedness above of  $r_1(t)$  (the equality  $\lim_{t \rightarrow +\infty} r_1(t) = -\infty$ ). By virtue of Lemma 3.6 from here it follows that Eq. (1.1) is Liapunov stable (asymptotically) if and only if the function  $r_2(t)$  is bounded above ( $\lim_{t \rightarrow +\infty} r_2(t) = -\infty$ ).

The theorem is proved.  $\square$

*Proof of Corollary 2.1.* By virtue of Lemma 3.2 from the first two conditions of corollary it follows

$$\rho_{D/4}(t) \leq \frac{c_1}{(1+t-t_0)^\alpha}, \quad t \geq t_0, \quad c_1 = \text{const.}$$

Then

$$\int_{t_0}^{+\infty} \rho_{D/4}(\tau) \frac{|D'(\tau)|}{D(\tau)^{3/2}} d\tau \leq c_1 \int_{t_0}^{+\infty} \frac{d\tau}{\sqrt{D(s)}(1+\tau-t_0)^{2\alpha}} < +\infty.$$

Thus the group of conditions C) of Theorem 2.1 is satisfied. Then A<sub>1</sub>) follows from Theorem 3.1, and B<sub>1</sub>) follows from Theorem 2.2.

The corollary is proved.  $\square$

## REFERENCES

- [1] L. Cesary. *Asymptotic behavior and stability problems in ordinary differential equations*, Moscow, "Mir", 1964.
- [2] R. Bellman. *Stability theory of differential equations*, Moscow, Izdatelstvo inostranoj literatury, 1954.
- [3] Ph. Hartman. *Ordinary differential equations*, Moscow, "Mir", 1970.
- [4] L. Ya. Adrianova. *Introduction in the theory of linear systems of differential equations*, St. Peterburg, Izdatelstvo St. Peterburgskogo universiteta, 1992.
- [5] M. V. Fedoriuk. *Asymptotic methods for linear ordinary differential equations*, Moscow, "Nauka", 1983.
- [6] N V. McLachlan. *Theory and application of Mathieu functions*, Moscow, "Mir", 1953.
- [7] I. M. Sobol. *Study of the asymptotic behaviour of the solutions of the linear second order differential equations wit the aid of polar coordinates*, "Matematicheskij sbornik", vol. 28 (70), N° 3, 1951, pp. 707 - 714.
- [8] L. A. Gusarov. *Convergence to zero of solutions of linear second order differential equations*, DAN SSSR, vol. LXXI, N° 1, 1950, pp. 9 - 12.

- [9] G. A. Grigorian. *Some properties of the solutions of linear second order ordinary differential equations*, Trudy UrO RAN, vol. 19, N<sup>o</sup> 19, 2013, pp. 69 - 90.
- [10] G. A. Grigorian. *Boundedness and stability criteria for linear ordinary differential equations of the second order*, "Izvestia vuzov, Matematika, N<sup>o</sup> 12, 2013, pp. 11 - 18.
- [11] I. Knowles. *On stability Conditions for Second Order Linear Differential Equations*, Journal of Differential Equations 34, 179 - 203 (1979).
- [12] L. H. Erbe. *Stability results for Periodic second Order Linear Differential Equations*, Proc. Amer. Math. Soc., Vol. 93, Num. 2, 1985. pp. 272 - 276.
- [13] M Bohner and T. Li, *Kamenev - type criteria for nonlinear damped dynamic equations*, Sci. China Math. 58(2015), 1445 - 1452.
- [14] T. Li, Yu. V. Rogovchenko, and S. Tang, *Oscillation of second - order nonlinear differential equations with damping*, Math. Slovaca 64(2014), 1227 - 1236.
- [15] G. A. Grigorian. *Some properties of differential root and theirs applications*, Acta linebreak Math. Univ. Comenianae, Vol. LXXXV, 2 (2016), pp. 205 - 212.
- [16] G. A. Grigorian. *On two comparison tests for second-order linear ordinary differential equations*, (Russian) Differ. Uravn. 47 (2011), no. 9, 1225 - 1240; translation in Differ. Equ. 47 (2011), no. 9 1237 - 1252, 34C10.

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Gevorg A. Grigorian  
Institute of Mathematics of NAS of Armenia  
ul. Marshala Bagramyana 24/5  
Yerevan, 0019  
Republic of Armenia  
e-mail: [mathphys2@instmath.sci.am](mailto:mathphys2@instmath.sci.am)

