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SOME NOISELESS CODING THEOREMS OF INACCURACY MEASURE OF ORDER α AND TYPE β

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Abstract. In this paper, we propose a parametric 'useful' mean code length which is weighted by utilities and generalizes some well known mean code lengths available in the literature. The object of this paper is to establish some results on noiseless coding theorems for the proposed parametric 'useful' mean code length in terms of generalized 'useful' inaccuracy measure of order α and type β.

1. Introduction

Consider the model given below for a finite random experiment scheme having $(x_1, x_2, \ldots, x_n)$ as the complete system of events, happening with respective probabilities $P = (p_1, p_2, \ldots, p_n)$ and credited with utilities $U = (u_1, u_2, \ldots, u_n)$, $u_i > 0$, $i = 1, 2, \ldots, n$. Denote

$$\chi = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ p_1 & p_2 & \ldots & p_n \\ u_1 & u_2 & \ldots & u_n \end{bmatrix}. \tag{1.1}$$

We call (1.1) the utility information scheme.

Let $Q = (q_1, q_2, \ldots, q_n)$ be the predicted distribution having the utility distribution $U = (u_1, u_2, \ldots, u_n)$, $u_i > 0$, $i = 1, 2, \ldots, n$. Taneja and Tuteja [19] have suggested and characterized the 'useful' inaccuracy measure

$$I(P, Q; U) = -\sum_{i=1}^{n} u_i p_i \log q_i. \tag{1.2}$$

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By considering the weighted mean code word length [6]

$$L(U) = \frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i} \tag{1.3}$$

where $l_1, l_2, \ldots, l_n$ are the code lengths of $x_1, x_2, \ldots, x_n$ respectively.

Taneja and Tuteja [19] derived the lower and upper bounds on $L(U)$ in terms of $I(P, Q; U)$.

Bhatia [3] defined the 'useful' average code length of order t as

$$L^t(U) = \frac{1}{t} \log \left[\frac{\sum_{i=1}^{n} u_i^{t+1} p_i D^{tl_i}}{\left(\sum_{i=1}^{n} u_i p_i \right)^{t+1}} \right], \quad -1 < t < \infty \tag{1.4}$$

where D is the size of the code alphabet. He also derived the bounds for the 'useful' average code length of order t in terms of generalized 'useful' inaccuracy measure, given by

$$I_\alpha(P, Q; U) = \frac{1}{1-\alpha} \log \left[\frac{\sum_{i=1}^{n} u_i p_i q_i^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i} \right], \quad \alpha > 0 (\neq 1) \tag{1.5}$$

under the condition

$$\sum_{i=1}^{n} p_i q_i^{-1} D^{-l_i} \leq 1 \tag{1.6}$$

where D is the size of the code alphabet. Inequality (1.6) is generalized Kraft's inequality [5]. A code satisfying the generalized Kraft's inequality would be termed as a personal probability code.

Longo [12], Gurdial and Pessoa [7], Autar and Khan [1], Jain and Tuteja [9], Taneja et al [20], Hooda and Bhaker [8], Bhatia [2] and Singh, Kumar and Tuteja [18] considered the problem of 'useful' information measures and used it studying the noiseless coding theorems for sources involving utilities.

In this paper, we study some coding theorems by considering a new function depending on the parameters α and β and a utility function. Our motivation for studying this function is that it generalizes some information measures already existing in the literature.

2. Coding theorems

Consider a function

$$I_{\alpha}^{\beta}(P,Q;U)=\frac{1}{1-\alpha}\log\left[\frac{\sum_{i=1}^{n}u_ip_i^{\beta}q_i^{\beta(\alpha-1)}}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right],\quad \alpha>0(\neq 1),\ \beta>0. \tag{2.1}$$

(i) When $\beta=1$, (2.1) reduces to a measure of 'useful' information measure of order α due to Bhatia [3].
(ii) When $\beta=1$, $u_i=1$, $\forall i=1,2,\dots,n$. (2.1) reduces to the inaccuracy measure given by Nath [13], further it reduces to Renyi's [14] entropy by taking $p_i=q_i$, $\forall i=1,2,\dots,n$.
(iii) When $\beta=1$, $u_i=1$, $\forall i=1,2,\dots,n$ and $\alpha\to 1$. (2.1) reduces to the measure due to Kerridge [10].
(iv) When $u_i=1$, $\forall i=1,2,\dots,n$ and $p_i=q_i$, $\forall i=1,2,\dots,n$ the measure (2.1) becomes the entropy for the β-power distribution derived from P studied by Roy [15]. We call $I_{\alpha}^{\beta}(P,Q;U)$ in (2.1) the generalized 'useful' inaccuracy measure of order α and type β.

Further consider

$$L_{\beta}^{t}(U)=\frac{1}{t}\log\left[\frac{\sum_{i=1}^{n}u_i^{t+1}p_i^{\beta}D^{tl_i}}{\left(\sum_{i=1}^{n}u_ip_i^{\beta}\right)^{t+1}}\right],\quad -1<t<\infty. \tag{2.2}$$

(i) For $\beta=1$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the 'useful' mean length $L^{t}(U)$ of the code given by Bhatia [3].
(ii) For $\beta=1$, $u_i=1$, $\forall i=1,2,\dots,n$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the mean length given by Campbell [4].
(iii) For $\beta=1$, $u_i=1$, $\forall i=1,2,\dots,n$ and $\alpha\to 1$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the optimal code length identical to Shannon [16].
(iv) For $u_i=1$, $\forall i=1,2,\dots,n$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the mean length given by Khan and Haseen [11].

Now we find the bounds for $L_{\beta}^{t}(U)$ in terms of $I_{\alpha}^{\beta}(P,Q;U)$ under the condition

$$\sum_{i=1}^{n}p_i^{\beta}q_i^{-\beta}D^{-l_i}\leq 1 \tag{2.3}$$

where D is the size of the code alphabet.

Theorem 2.1. *For every code whose lengths $l_1, l_2, \ldots, l_n$ satisfies (2.3), then the average code length satisfies*

$$L_\beta^t(U) \geq I_\alpha^\beta(P, Q; U) \tag{2.4}$$

where $\alpha = \frac{1}{1+t}$, the equality occurs if and only if

$$l_i = -\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\beta(\alpha-1)}}. \tag{2.5}$$

Proof. By Hölder's inequality [17]

$$\sum_{i=1}^{n} x_i y_i \geq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p < 1(\neq 0)$, $q < 0$ or $q < 1(\neq 0)$, $p < 0$ and $x_i, y_i > 0$, $i = 1, 2, \ldots, n$. We see the equality holds if and only if there exists a positive constant c such that

$$x_i^p = c y_i^q. \tag{2.7}$$

Making the substitutions $p = -t$, $q = \frac{t}{1+t}$

$$x_i = u_i^{-\left(\frac{t+1}{t}\right)} p_i^{-\frac{\beta}{t}} \left(\frac{1}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{-\left(\frac{1+t}{t}\right)} D^{-l_i}$$

$$y_i = u_i^{\frac{t+1}{t}} p_i^{\beta\left(\frac{1+t}{t}\right)} \left(\frac{1}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{\frac{1+t}{t}} q_i^{-\beta}$$

in (2.6) and using (2.3), we get

$$\left[\frac{\sum_{i=1}^{n} u_i^{t+1} p_i^\beta D^{l_i t}}{\left(\sum_{i=1}^{n} u_i p_i^\beta\right)^{1+t}}\right]^{\frac{1}{t}} \geq \left[\frac{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\beta(\alpha-1)}}{\sum_{i=1}^{n} u_i p_i^\beta}\right]^{\frac{1+t}{t}}.$$

Taking logarithms of both sides with base D, we obtain (2.4). □

Theorem 2.2. *For every code whose lengths $l_1, l_2, \ldots, l_n$ satisfies (2.3), L_β^t (U) can be made to satisfy the inequality*

$$L_\beta^t(U) < I_\alpha^\beta(P, Q; U) + 1. \tag{2.8}$$

Proof. Let l_i be the positive integer satisfying

$$-\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} \le l_i < -\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} + 1. \tag{2.9}$$

Consider the interval

$$\delta_i = \left[-\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}}, \; -\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} + 1 \right] \tag{2.10}$$

of length 1. In every δ_i, there is exactly one positive integer l_i such that

$$0 < -\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} \le l_i < -\log \frac{u_i q_i^{\alpha\beta}}{\sum u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} + 1. \tag{2.11}$$

We will first show that the sequences $\{l_1, l_2, \ldots, l_n\}$, thus defined satisfy (2.3). From (2.11) we have

$$-\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} \le l_i = -\log_D D^{-l_i}$$

i.e.

$$\frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} \ge D^{-l_i}$$

Multiplying both sides by $p_i^{\beta} q_i^{-\beta}$ and summing over $i = 1, 2, \ldots, n$, we get (2.3). The last inequality in (2.11) gives

$$l_i < -\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} + 1 = -\log \frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} + \log_D D$$

i.e,

$$D^{l_i t} < \left(\frac{u_i q_i^{\alpha\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta(\alpha-1)}} \right)^{-t} D^{t}$$

Multiplying both sides by $\frac{u_i^{t+1}p_i^\beta}{\left(\sum\limits_{i=1}^n u_i p_i^\beta\right)^{t+1}}$ and summing over $i = 1, 2, \ldots, n$, we get

$$\frac{\sum\limits_{i=1}^n u_i^{t+1} p_i^\beta D^{tl_i}}{\left(\sum\limits_{i=1}^n u_i p_i^\beta\right)^{t+1}} < \left[\frac{\sum\limits_{i=1}^n u_i p_i^\beta q_i^{\beta(\alpha-1)}}{\sum\limits_{i=1}^n u_i p_i^\beta}\right]^{t+1} D^t$$

Taking logarithms of both sides with base D and then dividing both sides by t, we obtain (2.8). □

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