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MAXIMAL CLIFFORD SEMIGROUPS OF MATRICES

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Abstract. All maximal Clifford semigroups of matrices are identified up to isomorphism. If the ground field of the matrices is finite, then there exists a unique Clifford semigroup of maximum order.

1. Introduction

A *Clifford semigroup* is a regular semigroup with central idempotents. Clifford semigroups are precisely the completely regular semigroups that are also inverse semigroups. The reader is referred to [2] for more information on Clifford semigroups.

The set $\mathrm{M}_n(\mathcal{F})$ of all $n \times n$ matrices over a field $\mathcal{F}$ is a semigroup under usual matrix multiplication. In this article, Clifford semigroups in $\mathrm{M}_n(\mathcal{F})$ are investigated. It is shown that up to isomorphism, the number of distinct maximal Clifford semigroups in $\mathrm{M}_n(\mathcal{F})$ is precisely the number of partitions of the integer n. Furthermore, if $\mathcal{F}$ is finite, then the semigroup $\mathrm{GL}_n(\mathcal{F})\cup\{\mathbf{0}\}$ (the general linear group with the $n\times n$ zero matrix $\mathbf{0}$ adjoined) is the unique Clifford semigroup in $\mathrm{M}_n(\mathcal{F})$ of maximum order.

The reader is referred to [2] for all undefined notation and terminology of semigroup theory. A property of $\mathrm{M}_n(\mathcal{F})$ that is important to the present investigation of semigroups in $\mathrm{M}_n(\mathcal{F})$ is:

Lemma 1. *All semilattices in* $\mathrm{M}_n(\mathcal{F})$ *are finite.*

Proof. This follows from well-known results. See [3, Lemma 2.1]. □

2. Maximal Clifford semigroups in $\mathrm{M}_n(\mathcal{F})$

We first recall the construction and some properties of Σ-semigroups introduced in [3]. Basically, these semigroups are unions of sums of completely simple (multiplicative) semigroups in rings containing no infinite semilattices. However, for this article, it suffices to consider Σ-semigroups in the special case of unions of sums of groups in $\mathrm{M}_n(\mathcal{F})$.

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Let $\mathcal{G} = \{G_i : i \in \Lambda\}$ be a collection of groups in $\mathrm{M}_n(\mathcal{F})$ such that $G_i \neq \{\mathbf{0}\}$ for all $i \in \Lambda$. Suppose further that $\mathcal{G}$ is *orthogonal*, that is, $G_iG_j = \{\mathbf{0}\}$ whenever $i \neq j$. Let e_i be the identity element of G_i. Since $\{\mathbf{0}\} \cup \{e_i : i \in \Lambda\}$ is a semilattice in $\mathrm{M}_n(\mathcal{F})$, the set Λ (and hence $\mathcal{G}$) must be finite by Lemma 1.

For any subset I of Λ, define

$$G_I = \begin{cases} \{\mathbf{0}\} & \text{if } I = \emptyset \\ \sum_{i \in I} G_i & \text{otherwise.} \end{cases}$$

Lemma 2. *Let* $S = \bigcup_{I \subseteq \Lambda} G_I$. *Then*:

(1) *The sets in* $\{G_I : I \subseteq \Lambda\}$ *are pairwise disjoint*;
(2) *If* $I \neq \emptyset$, *then* G_I *is a group isomorphic to the direct product* $\prod_{i \in I} G_i$;
(3) S *is a Clifford semigroup*;
(4) S *is completely determined by the groups in* $\mathcal{G}$.

Proof. Parts (1), (2), and (3) follow from Lemma 3.1, Lemma 3.2, and Theorem 3.3 in [3], respectively. Part (4) follows easily from parts (1) and (2) and the definition of S. □

The semigroup S in Lemma 2 is called the Σ-*semigroup with foundation* $\mathcal{G}$. More generally, by a Σ-*semigroup* we mean a semigroup S in $\mathrm{M}_n(\mathcal{F})$ for which there exists an orthogonal collection $\mathcal{G}$ of groups such that S is the Σ-semigroup with foundation $\mathcal{G}$.

We now identify all maximal Clifford semigroups in $\mathrm{M}_n(\mathcal{F})$ up to isomorphism. Suppose S is a maximal Clifford semigroup in $\mathrm{M}_n(\mathcal{F})$. Then S is a Σ-semigroup by [3, Corollary 6.2], whence we may assume $S = \bigcup_{I \subseteq \Lambda} G_I$ with foundation $\mathcal{G} = \{G_i : i \in \Lambda\}$. The idempotents of S commute so that they are simultaneously diagonalizable (by the matrix a, say). Since the conjugation map $x \mapsto axa^{-1}$ is an isomorphism, we may assume without loss of generality that the idempotents of S are diagonal matrices with entries from $\{0, 1\}$.

Since $G_iG_j = \{\mathbf{0}\}$ whenever $i \neq j$, the diagonal matrices e_i and e_j do not share any common nonzero diagonal entry. Suppose the (k, k)-entry of every e_i is 0. Let f denote the matrix in $\mathrm{M}_n(\mathcal{F})$ with 1 in its (k, k)-entry and 0 everywhere else. Then f is an idempotent that does not belong to S. Moreover, since $fG_i = G_if = \{\mathbf{0}\}$ for all $i \in \Lambda$ so that $fG_I = G_If = \{\mathbf{0}\}$ for all $I \subseteq \Lambda$, the set $S \cup \{f\}$ is a Clifford semigroup that strictly contains S, contradicting the maximality of S. Consequently, f does not exist, whence for each $k \in \{1, \ldots, n\}$, exactly one e_i has 1 in its (k, k)-entry. Equivalently, the sets

$$E_i = \{k : \text{the } (k, k)\text{-entry of } e_i \text{ is } 1\},$$

where $i \in \Lambda$, form a partition of the set $\{1, \ldots, n\}$. Note that $|E_i| = \mathrm{rank}(e_i)$. By conjugating S with an appropriate permutation matrix and relabelling of the indices i, we may assume that the integers in each E_i are consecutive, and that the integers $|E_i|$ $(i \in \Lambda)$ are in non-increasing order. For example, consider the idempotent matrices

$$e_1 = \begin{bmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix}$$

in $\mathrm{M}_5(\mathcal{F})$. Then after conjugation by the permutation matrix

$$\begin{bmatrix} 0&0&0&0&1\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\1&0&0&0&0 \end{bmatrix}$$

and relabelling of indices, we obtain

$$\begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix},\quad \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0 \end{bmatrix},\quad \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix},$$

where $E_1 = \{1, 2\}$, $E_2 = \{3, 4\}$, $E_3 = \{5\}$, and $|E_1| \geq |E_2| \geq |E_3|$.

It remains to determine which groups each G_i in $\mathcal{G}$ can possibly be, keeping in mind that $\mathcal{G}$ must be orthogonal. Let $\mathrm{M}_{E_i}(\mathcal{F})$ denote the set of all matrices in $\mathrm{M}_n(\mathcal{F})$ with 0 in their (s, t)-entries for all $(s, t) \notin E_i \times E_i$. Since $G_i = e_i G_i e_i \subseteq \mathrm{M}_{E_i}(\mathcal{F})$ and $\mathrm{M}_{E_j}(\mathcal{F})\,\mathrm{M}_{E_k}(\mathcal{F}) = \{\mathbf{0}\}$ whenever $j \neq k$, the required property of $\mathcal{G}$ being orthogonal will not be violated as long as G_i is chosen to be any group in $\mathrm{M}_{E_i}(\mathcal{F})$ (with identity element e_i). But by the maximality of S, the group G_i must contain all matrices in $M_{E_i}(\mathcal{F})$ of rank $|E_i|$ (so that $G_i \cong \mathrm{GL}_{|E_i|}(\mathcal{F})$).

We have thus shown:

Proposition 3. *Up to isomorphism, each maximal Clifford semigroup in* $\mathrm{M}_n(\mathcal{F})$ *is a* Σ*-semigroup* $\bigcup_{I \subseteq \Lambda} G_I$ *with foundation* $\mathcal{G} = \{G_i : i \in \Lambda\}$*, and there exists a partition* $\{E_i : i \in \Lambda\}$ *of the set* $\{1, \ldots, n\}$ *such that* $G_i \cong \mathrm{GL}_{|E_i|}(\mathcal{F})$ *for all* $i \in \Lambda$*. Consequently, the number of non-isomorphic maximal Clifford semigroups in* $\mathrm{M}_n(\mathcal{F})$ *is precisely the number of partitions of* n*.*

Let $P = (n_1, \ldots, n_r)$ be a partition of the integer n, that is, $n_1, \ldots, n_r$ are positive integers in non-increasing order such that $n_1 + \cdots + n_r = n$. In view of Proposition 3, up to isomorphism, P corresponds uniquely to the maximal Clifford semigroup $\bigcup\{G_I : I \subseteq \{1, \ldots, r\}\}$ with foundation $\mathcal{G} = \{G_1, \ldots, G_r\}$, where $G_i \cong \mathrm{GL}_{n_i}(\mathcal{F})$. This maximal Clifford semigroup is said to be *associated with* the partition P and is denoted by C^P. Note then that $G_I \cong \prod_{i\in I} \mathrm{GL}_{n_i}(\mathcal{F})$.

3. The Clifford semigroup in $\mathrm{M}_n(\mathcal{F})$ of maximum order

In this section, we assume $\mathcal{F}$ is a finite field with q elements. Since $\mathrm{M}_n(\mathcal{F})$ is already a Clifford semigroup (of maximum order q) when $n = 1$, we may also assume that $n \geq 2$, whence the order of $\mathrm{GL}_n(\mathcal{F})$ is

$$\gamma(n) = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}) = \textstyle\prod_{i=0}^{n-1}(q^n - q^i)$$

(see, for example, [1]). For any partition $P = (n_1, \ldots, n_r)$ of n, define

$$\begin{aligned}
\sigma_0(n,P) &= 1,\\
\sigma_1(n,P) &= \gamma(n_1) + \gamma(n_2) + \cdots + \gamma(n_r),\\
\sigma_2(n,P) &= \gamma(n_1)\,\gamma(n_2) + \gamma(n_1)\,\gamma(n_3) + \cdots + \gamma(n_{r-1})\,\gamma(n_r),\\
&\;\;\vdots\\
\sigma_k(n,P) &= \textstyle\sum\{\gamma(n_{i_1})\cdots\gamma(n_{i_k}) : 1 \leq i_1 < \cdots < i_k \leq r\},\\
&\;\;\vdots\\
\sigma_r(n,P) &= \gamma(n_1)\cdots\gamma(n_r).
\end{aligned}$$

By Lemma 2(2), the order of $\mathrm{GL}_{n_{i_1}}(\mathcal{F}) \times \cdots \times \mathrm{GL}_{n_{i_k}}(\mathcal{F})$ is $\gamma(n_{i_1})\cdots\gamma(n_{i_k})$. Therefore $\sigma_k(n,P)$ is the sum of all $|G_I|$ where $|I| = k$. Hence:

Theorem 4. *The order of the maximal Clifford semigroup C^P in $\mathrm{M}_n(\mathcal{F})$ is*

$$\sigma(n,P) = \sigma_0(n,P) + \cdots + \sigma_r(n,P) = \textstyle\sum_{k=0}^{r}\sigma_k(n,P).$$

Note that if T is the trivial partition (n) of n, then $C^T = \mathrm{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$ and $\sigma(n,T) = 1 + \gamma(n)$. For the rest of this article, we show that C^T is the unique Clifford semigroup in $\mathrm{M}_n(\mathcal{F})$ of maximum order.

Lemma 5. *Suppose $n = s + t$ where $s, t \geq 1$. Then $4\,\gamma(s)\,\gamma(t) \leq \gamma(n)$.*

Proof. Since

$$\begin{aligned}
2\,\gamma(s) &= 2(q^s - 1)(q^s - q)\cdots(q^s - q^{s-1})\\
&\leq (q^n - 1)(q^n - q)\cdots(q^n - q^{s-1})
\end{aligned}$$

and

$$\begin{aligned}
2\,\gamma(t) &= 2(q^t - 1)(q^t - q)\cdots(q^t - q^{t-1})\\
&= 2\cdot\frac{q^{s+t} - q^s}{q^s}\cdot\frac{q^{s+t} - q^{s+1}}{q^s}\cdots\frac{q^{s+t} - q^{s+t-1}}{q^s}\\
&\leq (q^n - q^s)(q^n - q^{s+1})\cdots(q^n - q^{n-1}),
\end{aligned}$$

we have $4\,\gamma(s)\,\gamma(t) \leq \prod_{i=0}^{s-1}(q^n - q^i)\prod_{j=s}^{n-1}(q^n - q^j) = \gamma(n)$. □

Lemma 6. *If P is any partition of n, then $\sigma(n, P) \leq 1 + \gamma(n)$.*

Proof. It suffices to assume that P is nontrivial. We proceed by induction on n. For $n = 2$, the only nontrivial partition is $P = (1, 1)$, whence

$$\begin{aligned}\sigma(2, P) &= 1 + 2(q-1) + (q-1)^2\\ &\leq 1 + (q^2-1)(q^2-q)\\ &= 1 + \gamma(2)\end{aligned}$$

for all $q \geq 2$. Suppose the inequality holds for all integers strictly less than n. Let $P = (n_1, \ldots, n_r)$ be a nontrivial partition of n. Note that for $1 \leq k \leq r$,

$$\begin{aligned}\sigma_k(n, P) &= \textstyle\sum\{\gamma(n_{i_1})\cdots\gamma(n_{i_k}) : 1 \leq i_1 < \cdots < i_k \leq r\}\\ &= \textstyle\sum\{\gamma(n_{i_1})\cdots\gamma(n_{i_k}) : 1 \leq i_1 < \cdots < i_k \leq r-1\}\\ &\quad + \textstyle\sum\{\gamma(n_{i_1})\cdots\gamma(n_{i_{k-1}})\,\gamma(n_r) : 1 \leq i_1 < \cdots < i_{k-1} \leq r-1\}\\ &= \sigma_k(n-n_r, P') + \sigma_{k-1}(n-n_r, P')\cdot\gamma(n_r),\end{aligned}$$

where P' is the partition $(n_1, \ldots, n_{r-1})$ of $n - n_r$. Hence

$$\begin{aligned}\sigma(n, P) &= 1 + \textstyle\sum_{k=1}^{r}\sigma_k(n, P)\\ &= (1 + \textstyle\sum_{k=1}^{r-1}\sigma_k(n-n_r, P')) + \sigma_r(n-n_r, P')\\ &\quad + \gamma(n_r)\textstyle\sum_{k=1}^{r-1}\sigma_{k-1}(n-n_r, P')\\ &= \sigma(n-n_r, P') + 0 + \gamma(n_r)\cdot\sigma(n-n_r, P')\\ &= (1+\gamma(n_r))\cdot\sigma(n-n_r, P').\end{aligned}$$

Since $\sigma(n - n_r, P') \leq 1 + \gamma(n - n_r)$ by induction hypothesis, we have

$$\begin{aligned}\sigma(n, P) &\leq (1+\gamma(n_r))\cdot(1+\gamma(n-n_r))\\ &\leq (2\,\gamma(n_r))\cdot(2\,\gamma(n-n_r))\\ &\leq 1 + \gamma(n),\end{aligned}$$

where the last inequality holds by Lemma 5. □

Theorem 7. *Let $\mathcal{F}$ be a finite field. Then $\mathrm{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$ is the unique Clifford semigroup in $\mathrm{M}_n(\mathcal{F})$ of maximum order.*

Proof. By Proposition 3 and Lemma 6, a Clifford semigroup S in $\mathrm{M}_n(\mathcal{F})$ of maximum order is isomorphic to $C^T = \mathrm{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$. Since $\mathrm{GL}_n(\mathcal{F})$ is the unique maximal group in $\mathrm{M}_n(\mathcal{F})$ of rank n, we have $S = \mathrm{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$. □

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