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# ON A NEW INEQUALITY SIMILAR TO THE HARDY - HILBERT INTEGRAL INEQUALITY

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Abstract. A new inequality similar to the Hardy-Hilbert integral inequality is proved. Some special cases are also deduced.

## 1. Introduction

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \text{ and } 0 < \int_0^\infty g^2(t)\,dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(t)\,dt \int_0^\infty g^2(t)\,dt\right)^{1/2}, \tag{1}$$

where the constant factor $\pi$ is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality has been extended by Hardy [1] as follows

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \text{ and } \int_0^\infty g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin\left(\pi/p\right)}\left(\int_0^\infty f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty g^q(t)\,dt\right)^{1/q}, \tag{2}$$

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where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert integral inequality and is important in analysis and applications (cf. Mitrinovic et al. [3]).

B. Yang gave the following extensions of (2) as follows :

**Theorem 1.** [4] *If* $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$ *satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty \ \text{and} \ \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$$

*then*

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}\,dx\,dy < k_\lambda(p) \left( \int_0^\infty t^{1-\lambda} f^p(t)\,dt \right)^{1/p} \left( \int_0^\infty t^{1-\lambda} g^q(t)\,dt \right)^{1/q} \quad (3)$$

*where the constant factor* $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ *is the best possible,* $B$ *is the beta function.*

**Theorem 2.** [5] *If* $n \in N - \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > n - \min_{1\leq i\leq n} \{p_i\}$, $f_i \geq 0$, *satisfy*

$$0 < \int_0^\infty t^{n-1-\lambda} f_i^{p_i}(t)\,dt < \infty \quad (i = 1, 2, \ldots, n),$$

*then*

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1 \ldots dx_n$$
$$< \frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right) \left( \int_0^\infty t^{n-1-\lambda} f_i^{p_i}(t)\,dt \right)^{1/p_i} \quad (4)$$

*where the constant factor* $\frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{p_i+\lambda-n}{p_i}\right)$ *is the best possible.*

Inequality (4) is a multiple extension of inequalities (1), (2) and (3).

**Theorem 3.** [6] *If* $n \in N - \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$, $f_i \geq 0$ *satisfy*

$$0 < \int_0^\infty t^{p_i-1-\lambda} f_i^{p_i}(t)\,dt < \infty \quad (i = 1, 2, \ldots, n),$$

*then*

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)\, dx_1 \dots dx_n$$
$$< \frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \left(\int_0^\infty t^{p_i-1-\lambda} f_i^{p_i}(t)\, dt\right)^{1/p_i}, \quad (5)$$

*where the constant factor* $\frac{1}{\Gamma\lambda} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ *is the best possible.*

## 2. Main result

We state and prove the following

**Theorem.** *Let* $n \in N - \{1\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_i > 0$, $1 \le i \le n$, $\lambda > \sum_{i=r+1}^n a_i$, $1 \le r < n$, $\lambda_{r+1} = (a_{r+1} - 1)(1-q)$, $K_{r+1} = \left(\prod_{j=r+1}^n \Gamma a_j\right) \Gamma\left(\lambda - \sum_{i=r+1}^n a_i\right)/\Gamma\lambda$. *Then, we have*

$$\left(\frac{\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda}\, dx_1 \dots dx_n}{K_{r+1} \int_0^\infty \cdots \int_0^\infty f_1^p(x_1) \dots f_r^p(x_r)\, dx_1 \dots dx_r}\right)^q$$
$$\le \frac{\int_0^\infty \cdots \int_0^\infty \frac{(x_1+\cdots+x_r)^{\sum_{i=r+1}^n a_i - \lambda} x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1}) \dots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\cdots+x_n)^\lambda}\, dx_1 \dots dx_n}{K_{r+1} \int_0^\infty \cdots \int_0^\infty f_1^p(x_1) \dots f_r^p(x_r)\, dx_1 \dots dx_r} \quad (6)$$

*Proof.*

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda}\, dx_1 \dots dx_n = \int_0^\infty \cdots \int_0^\infty f_1(x_1)\dots f_r(x_r)$$
$$\times \left(\int_0^\infty \cdots \int_0^\infty \frac{f_{r+1}(x_{r+1})\dots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda}\, dx_{r+1} \dots dx_n\right) dx_1 \dots dx_r$$
$$\le \left(\int_0^\infty \cdots \int_0^\infty f_1^p(x_1) \dots f_r^p(x_r)\, dx_1 \dots dx_r\right)^{1/p}$$
$$\times \left(\int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty \cdots \int_0^\infty \frac{f_{r+1}(x_{r+1})\dots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda} dx_{r+1} \dots dx_n\right)^q dx_1 \dots dx_r\right)^{1/q}$$

$$\leq \left(\int_0^\infty \cdots \int_0^\infty f_1^p(x_1)\dots f_r^p(x_r)\,dx_1\dots dx_r\right)^{1/p}$$

$$\times \left[\int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty \cdots \int_0^\infty \frac{x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1})\dots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\cdots+x_n)^\lambda}\,dx_{r+1}\dots dx_n\right)\right.$$

$$\left.\times \left(\int_0^\infty \cdots \int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\dots x_n^{a_n-1}}{(x_1+\cdots+x_n)^\lambda}\,dx_{r+1}\dots dx_n\right)^{q-1} dx_1 ... dx_r\right].$$

Now, we consider

$$I = \int_0^\infty \cdots \int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\dots x_n^{a_n-1}}{(x_1+\cdots+x_n)^\lambda}\,dx_{r+1}\dots dx_n$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\dots x_{n-1}^{a_{n-1}-1}}{(x_1+\cdots+x_{n-1})^{\lambda-a_n}}\,dx_{r+1}\dots dx_{n-1}$$

$$\times \int_0^\infty \frac{\left(\frac{x_n}{x_1+\cdots+x_{n-1}}\right)^{a_n-1} \frac{dx_n}{x_1+\cdots+x_{n-1}}}{\left(1+\frac{x_n}{x_1+\cdots+x_{n-1}}\right)^\lambda}$$

$$= B\left(a_n, \lambda - a_n\right) \int_0^\infty \cdots \int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\dots x_{n-1}^{a_{n-1}-1}}{(x_1+\cdots+x_{n-1})^{\lambda-a_n}}\,dx_{r+1}\dots dx_{n-1}.$$

Proceeding in this manner, we obtain

$$I = \prod_{j=r+1}^{n} B\left(a_j, \lambda - \sum_{i=j}^{n} a_i\right) (x_1+\cdots+x_r)^{\sum_{i=r+1}^n a_i - \lambda}$$

$$= \left(\frac{\Gamma\left(\lambda - \sum_{i=r+1}^n a_i\right)}{\Gamma\lambda} \prod_{j=r+1}^{n} \Gamma a_j\right) (x_1+\cdots+x_r)^{\sum_{i=r+1}^n a - \lambda_i}.$$

Therefore, we have

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda}\,dx_1\dots dx_n$$

$$\leq K_{r+1}^{1/p} \left(\int_0^\infty \cdots \int_0^\infty f_1^p(x_1)\dots f_r^p(x_r)\,dx_1\dots dx_r\right)^{1/p}$$

$$\times\left(\int_0^\infty\cdots\int_0^\infty\frac{(x_1+\cdots+x_r)^{\sum_{i=r+1}^n a_i-\lambda}x_{r+1}^{\lambda_{r+1}}}{(x_1+\cdots+x_n)^\lambda}\right.$$

$$\left.\times f_{r+1}^q(x_{r+1})\dots x_n^{\lambda_n}f_n^q(x_n)\,dx_1\dots dx_n\right)^{1/q}.$$

This implies

$$\left(\frac{\int_0^\infty\cdots\int_0^\infty\frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\cdots+x_n)^\lambda}\,dx_1\dots dx_n}{K_{r+1}\int_0^\infty\dots\int_0^\infty f_1^p(x_1)\dots f_r^p(x_r)\,dx_1\dots dx_r}\right)^q$$

$$\le\frac{\int_0^\infty\dots\int_0^\infty\frac{(x_1+\cdots+x_r)^{\sum_{i=r+1}^n a_i-\lambda}x_{r+1}^{\lambda_{r+1}}f_{r+1}^q(x_{r+1})\dots x_n^{\lambda_n}f_n^q(x_n)}{(x_1+\dots x_n)^\lambda}\,dx_1\dots dx_n}{K_{r+1}\int_0^\infty\dots\int_0^\infty f_1^p(x_1)\dots f_r^p(x_r)\,dx_1\dots dx_r}.$$

□

## 3. Applications

1. Putting $n=2, r=1$ in (6), we obtain

$$\left(\frac{\int_0^\infty\int_0^\infty\frac{f_1(x_1)\,f_2(x_2)}{(x_1+x_2)^\lambda}\,dx_1\,dx_2}{B\left(a_2,\lambda-a_2\right)\int_0^\infty f_1^p(x_1)\,dx_1}\right)^q\le\frac{\int_0^\infty\int_0^\infty\frac{x_1^{a_2-\lambda}x_2^{(a_2-1)(1-q)}}{(x_1+x_2)^\lambda}\,dx_1\,dx_2}{B\left(a_2,\lambda-a_2\right)\int_0^\infty f_1^p(x_1)\,dx_1}. \quad (7)$$

2. Putting $n=3, r=1$ in (6), we obtain

$$\left(\frac{\int_0^\infty\int_0^\infty\int_0^\infty\frac{f_1(x_1)\,f_2(x_2)\,f_3(x_3)}{(x_1+x_2+x_3)^\lambda}\,dx_1\,dx_2\,dx_3}{B\left(a_2,\lambda-a_2\right)\int_0^\infty f_1^p(x_1)\,dx_1}\right)^q$$

$$\le\frac{\int_0^\infty\int_0^\infty\int_0^\infty\frac{x_1^{a_2+a_3-\lambda}\,x_2^{(a_2-1)(1-q)}x_3^{(a_3-1)(1-q)}f_2^q(x_2)\,f_3^q(x_3)}{(x_1+x_2+x_3)^\lambda}\,dx_1\,dx_2\,dx_3}{B\left(a_2,\lambda-a_2\right)\int_0^\infty f_1^p(x_1)\,dx_1}. \quad (8)$$

3. Putting $n=3, r=2$ in (6), we have

$$\left(\frac{\int_0^\infty\int_0^\infty\int_0^\infty \frac{f_1(x_1)\,f_2(x_2)\,f_3(x_3)}{(x_1+x_2+x_3)^\lambda}\,dx_1\,dx_2\,dx_3}{B\left(a_3,\lambda-a_3\right)\int_0^\infty\int_0^\infty f_1^p(x_1)\,f_2^p(x_2)\,dx_1\,dx_2}\right)^q$$

$$\leq \frac{\int_0^\infty\int_0^\infty\int_0^\infty (x_1+x_2)^{a_3-\lambda}\,x_3^{(a_3-1)(1-q)}\,f_3^q(x_3)\,dx_1dx_2dx_3}{B\left(a_3,\lambda-a_3\right)\int_0^\infty\int_0^\infty f_1^p(x_1)\,f_2^p(x_2)\,dx_1\,dx_2}. \quad (9)$$

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