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MEASURABILITY – PRESERVING WEAKLY MIXING TRANSFORMATIONS

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Abstract. In this paper we investigate measure-theoretic properties of the class of all weakly mixing transformations on a finite measure space which preserve measurability. The main result in this paper is the following theorem: If ϕ is a weakly mixing transformation on a finite measure space $(S, \mathcal{A}, \mu)$ with the property that $\phi(\mathcal{A}) \subseteq \mathcal{A}$, then for every A, B in $\mathcal{A}$ there is a subset $J(A, B)$ of the set of non-negative integers of density zero such that $\lim_{m\to\infty, m\notin J(A,B)} \mu(A \cap \phi^m(B)) = (\mu(A)/\mu(S)) \lim_{n\to\infty} \mu(\phi^n(B))$. Furthermore, we show that for most useful measure spaces we can strengthen the preceding statement to obtain a set of density zero that works for all pairs of measurable sets A and B. As corollaries we obtain a number of inclusion theorems. The results presented here extend the well-known classical results (for invertible weakly mixing transformations), results of R. E. Rice [17] (for strongly mixing), a result of C. Sempi [19] (for weakly mixing) and previous results of the author [8, 10] (for weakly mixing and ergodicity).

1. Introduction

In the broadest sense ergodic theory is the study of the qualitative properties of actions of groups on spaces (e.g. measure spaces, or topological spaces, or smooth manifolds) (cf., e.g., [20, §0.1] and [21]). In this paper we shall study actions of the group $\mathbf{Z}$ of integers on a measure space S , i.e., we study a transformation $\phi : S \to S$ and its iterates $\phi^n, n \in \mathbf{Z}$. It is customary in ergodic theory to assume that the underlying space is either a finite or σ- finite measure space. We shall assume that the measure is finite. It is commonly further assumed that the measure space is separable. However, we shall not make this assumption, principally because it would

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rule out some of our principal structure theorems. We shall generally refer to Billingsley [1], Brown [2], Choe [3], Kingman and Taylor [13] and Walters [20].

We shall use $\mathbf{Z}$ to denote the set of integers, $\mathbf{N}$ to denote the set of natural numbers, $\mathbf{N_0}$ to denote the set of nonnegative integers, and $\mathbf{R}$ to denote the set of real numbers. The empty set will be denoted by $\emptyset$, and the symmetric difference of sets A, B, i.e., the set $(A \setminus B) \cup (B \setminus A)$ will be denoted by $A \,\Delta\, B$.

Suppose $(S, \mathcal{A}, \mu)$ is a finite measure space. As usual, a transformation $\phi : S \to S$ is called: (i) *measurable* (μ - measurable) if, for any A in $\mathcal{A}$, the inverse image $\phi^{-1}(A)$ is in $\mathcal{A}$; (ii) *measure - preserving* (or μ is ϕ - *invariant*) if ϕ is measurable and $\mu(\phi^{-1}(A)) = \mu(A)$ for any A in $\mathcal{A}$; (iii) *ergodic* if the only members A of $\mathcal{A}$ with $\phi^{-1}(A) = A$ satisfy $\mu(A) = 0$ or $\mu(S \setminus A) = 0$; (iv) *weakly mixing (with respect to μ)* if ϕ is μ - measurable and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| \mu\left(\phi^{-i}(A) \cap B\right) - \frac{\mu(A)\mu(B)}{\mu(S)} \right| = 0 \tag{1.1}$$

for any two μ - measurable subsets A, B of S ; (v) (*strongly*) *mixing* (*with respect to* μ) if ϕ is μ - measurable and

$$\lim_{n\to\infty} \mu\left(\phi^{-n}(A) \cap B\right) = \frac{\mu(A)\mu(B)}{\mu(S)} \tag{1.2}$$

for any two μ - measurable subsets A, B of S. We say that the transformation $\phi : S \to S$ is *invertible* if ϕ is one-to-one (monic) and such that $\phi(A)$ is μ - measurable whenever A is μ - measurable subset of S.

A transformation ϕ on a finite measure space $(S, \mathcal{A}, \mu)$ is said to be *measurability - preserving* if $\phi(\mathcal{A}) \subseteq \mathcal{A}$ ([10, Definition 1]). In this case we also say that the transformation ϕ *preserves* μ *- measurability.*

The objects of interest are not really measure-preserving transformations, but equivalence classes of such transformations; two transformations are equivalent if they differ only on a set of measure zero. Measure-preserving transformations arise, e.g., in the investigation of classical dynamical systems. In this case ϕ is first obtained as a continuous transformation of some compact topological space, and the existence of an invariant measure μ is proved. The system $(S, \mathcal{A}, \mu, \phi)$ is then abstracted from the topological setting. Therefore, if $(S, \mathcal{A}, \mu)$ is a finite measure space, and $\phi : S \to S$ is a measure-preserving transformation (with respect to μ), then we say that $\Phi := (S, \mathcal{A}, \mu, \phi)$ is an *abstract dynamical system.* An abstract dynamical system is often called a *dynamical system with discrete time* or a *measure-theoretic dynamical system* or an *endomorphism* (see [2, pp. 1 - 7] and [4, pp. 6 - 26]; but see also [12] and [14] – [16]). We shall say that the abstract dynamical system Φ is: (i) *invertible* if ϕ is invertible; (ii) *ergodic* if ϕ is ergodic; (iii) *weakly* (resp. *strongly*) *mixing* if ϕ is weakly (resp. strongly) mixing.

If ϕ is a measure-preserving transformation on a finite measure space $(S, \mathcal{A}, \mu)$, then from the Birkhoff ergodic theorem (cf. [20, Theorem 1.14]), arguing as in the proof of Proposition 1.4 in [2] (see also the proof of Theorem 2 in [10]), one can deduce that ϕ is ergodic if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(\phi^{-i}(A) \cap B\right) = \frac{\mu(A)\mu(B)}{\mu(S)} \tag{1.3}$$

for each A, B in $\mathcal{A}$. Now, from (1.1) - (1.3) we readily obtain that every strongly mixing transformation is weakly mixing and every weakly mixing transformation is ergodic. Furthermore, if $\phi : S \to S$, in addition to being strongly mixing on S with respect to μ, is invertible, then (1.2) is equivalent to (the well-known result):

$$\lim_{n\to\infty} \mu\left(\phi^{n}(A) \cap B\right) = \frac{\mu(A)\mu(B)}{\mu(S)} \tag{1.4}$$

for any μ - measurable subsets A, B of S.

Investigations have shown, however, that many important consequences of (1.4) persist in the absence of invertibility (see [5], [17], [18, pp. 181-186, 190], [7] and [9, pp. 95 - 112, 114 - 119]) and/or the strongly mixing property (see [19], [8], [9, pp. 59-104, 112-121] and [10]). The following result (the most useful result of these investigations for the goals of this paper) is due to R.E. Rice [17, Theorem 1]:

Theorem 1.A. *Let ϕ be a strongly mixing transformation on the normalized measure space (probability space) $(S, \mathcal{A}, \mu)$. If ϕ is forward measurable, i.e., if $\phi(A)$ is μ - measurable whenever A is μ - measurable subset of S, then for any μ - measurable subsets A, B of S,*

$$\lim_{n\to\infty} \mu(\phi^{n}(A) \cap B) = \mu(B) \lim_{n\to\infty} \mu(\phi^{n}(A)). \tag{1.5}$$

Theorem 1.A has many consequences which are of interest because of the extreme simplicity of both their mathematical and physical realizations. These consequences have great relevance in the discussion of the recurrence paradox of Statistical Mechanics (see [5], [6] and [17]-[19]). It is therefore interesting to investigate how the conclusions of Theorem 1.A must be modified when the forward measurable transformation ϕ (i.e., the transformation ϕ which preserves μ - measurability) is assumed to have properties weaker then strongly mixing. In this direction we consider a case when the forward measurable transformation ϕ is assumed to have weakly mixing property.

Such transformations we will call *measurability - preserving weakly mixing transformations.*

We shall say that a subset J of $\mathbf{N_0}$ has *density zero* (or *null density*) if the number of elements in $J \cap \{0, 1, \ldots, n-1\}$ divided by n tends to 0 as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \frac{\text{cardinality}\,(J \cap \{0, 1, \ldots, n-1\})}{n} = 0.$$

Observe that the sets of density zero form a ring of sets: If both J_1 and J_2 have density zero then so does $J_1 \cup J_2$.

An example of a infinite subset J of $\mathbf{N_0}$ of density zero is given by $J := \{\lfloor n \log n \rfloor : n \in \mathbf{N}\}$, where $\lfloor x \rfloor$ denotes the integral part of $x \in \mathbf{R}$.

It is well known that union of a finite number of sets of density zero has density zero, and that this is false for a countable number (see, e.g., [2, p. 38]). Furthermore, a subset of $\mathbf{N_0}$ of density zero has an infinite complement in $\mathbf{N_0}$ (cf. [19, p. 5]).

The following well-known fact (see, e.g., [20, Theorem 1.20]) will be useful in the proof of the main results in this paper:

Theorem 1.B. *If* (a_n) *is a bounded sequence of real numbers, then the following are equivalent:*

(i) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=0}^{n-1} |a_i| = 0.$

(ii) *There exists a subset* J *of* $\mathbf{N_0}$ *of density zero such that* $\lim\limits_{n\to\infty} a_n = 0$ *provided* $n \notin J$.

2. Main results

The main objective of this section is to extend the previous work of R. E. Rice [17] and H. Fatkić [10] to weakly mixing transformations of a finite measure space.

First we make the following definitions.

Definition 2.1. A weakly mixing transformation on a finite measure space $(S, \mathcal{A}, \mu)$ with the property $\phi(\mathcal{A}) \subseteq \mathcal{A}$ is called a *measurability-preserving weakly mixing transformation* (*with respect to* μ).

Note that measurability-preserving weakly mixing transformations on a finite measure space are generalizations of the invertible weakly mixing transformations (they are not necessarily one-to-one).

Definition 2.2. Suppose that $(S, \mathcal{A}, \mu)$ is a finite measure space and $\Phi := (S, \mathcal{A}, \mu, \phi)$ is an abstract dynamical system. Then Φ is called:

(i) *measurability-preserving* if $\phi(\mathcal{A}) \subseteq \mathcal{A}$;

(ii) *measurability-preserving weakly mixing* if Φ is weakly mixing and $\phi(\mathcal{A}) \subseteq \mathcal{A}$.

Example 2.1. Let $\mathcal{A}$ consist of the Borel subsets of the half-open unit interval $S := [0, 1)$, with Lebesgue measure for μ, and let $\phi\,(x) = 2x \pmod 1$ (i.e., $\phi\,(x)$ is $2x$ on $[0, \frac{1}{2})$ and $2x-1$ on $[\frac{1}{2}, 1)$). If $f(x)$ is 0 for $x < \frac{1}{2}$ and 1 for $x \geq \frac{1}{2}$, then $f(\phi^{n-1}(x))$ is nth digit of the dyadic (base - 2) expansion of $x := \sum_{n=1}^{\infty} f\left(\phi^{n-1}(x)\right)/2^n$. Hence, if x has the expansion $x = 0, x_1x_2x_3\ldots$, then $\phi(x) = 0, x_2x_3\ldots$ (see [1, pp. 7, 11-13]). For every subinterval $I := [a, b]$ of $[0, 1)$ we have $\phi^{-1}(I) = [\frac{a}{2}, \frac{b}{2}] \cup [\frac{a}{2} + \frac{1}{2}, \frac{b}{2} + \frac{1}{2}]$, whence $\mu(\phi^{-1}(I)) = \mu(I)$. By considering the semi-algebra of all subintervals of $[0, 1)$ of the forms $[a, b]$, with $0 \leq a < b < 1$, (A collection $\mathcal{S}$ of subsets of S is called a *semi - algebra* if the following three conditions hold: (i) $\phi \in \mathcal{S}$; (ii) if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$; (iii) if $A \in \mathcal{S}$, then $S \setminus A = \cup_{i=1}^{n} E_i$ where each $E_i \in \mathcal{S}$ and $E_1, \ldots, E_n$ are pairwise disjoint subsets of S.) and using the well-known result which is desirable in checking whether transformations are measure-preserving or not (see [20, Theorem 1.1, p. 20]) we see that ϕ is measure-preserving. This transformation is called the *dyadic transformation*. It is not an invertible measure-preserving transformation. An application of the well-known criterion which is useful when checking whether or not examples have the weakly mixing properties (see, e.g., [20, Theorem 1.17]) shows that the dyadic transformation ϕ in addition to being ergodic, is weakly mixing. Furthermore, one readily gets $\phi(\mathcal{A}) \subseteq \mathcal{A}$. Therefore ϕ is an example of a measurability-preserving weakly mixing transformation which is not invertible.

As an application of the techniques developed in [10], we obtain the principal result of this section, which is an extension of Theorem 1 in [17] from strongly mixing to weakly mixing transformations and is an affirmative answer in one way to the question of H. Fatkić [10, Problem 2] wheather the relation (2.1) is a necessary and sufficient condition for the weakly mixing property of a measure-preserving transformation ϕ on a finite measure space $(S, \mathcal{A}, \mu)$ with the property that $\phi(\mathcal{A}) \subseteq \mathcal{A}$.

Theorem 2.1. *Let $(S, \mathcal{A}, \mu)$ be a finite measure space and let $\phi : S \to S$ be a measurability-preserving weakly mixing transformation with respect to μ. Then for every pair of elements A, B of $\mathcal{A}$ there is a subset $J(A, B)$ of $\mathbf{N_0}$ of density zero such that*

$$\lim_{m\to\infty, m\notin J(A,B)} \mu(A \cap \phi^m(B)) = \frac{\mu(A)}{\mu(S)} \lim_{n\to\infty} \mu(\phi^n(B)). \tag{2.1}$$

Proof. We first note that for any $B \subseteq S$ and for each n in $\mathbf{N}$,

$$\phi^{-1}\left(\phi^n(B)\right) = \phi^{-1}\left[\phi\left(\phi^{n-1}(B)\right)\right] \supseteq \phi^{n-1}(B). \tag{2.2}$$

Next, since ϕ preserves the measure μ and μ - measurability, from (2.2) we have, for every B in $\mathcal{A}$ and every n in $\mathbf{N}$,

$$\mu(S) \geq \mu(\phi^n(B)) = \mu(\phi^{-1}(\phi^n(B)) \geq \mu(\phi^{n-1}(B)),$$

whence we obtain that the sequence $(\mu(\phi^n(B))_{n=1}^{\infty}$ is bounded and non-decreasing. Therefore for every B in $\mathcal{A}$ the (finite) limit $L(B) := \lim_{n\to\infty} \mu(\phi^n(B))$ exists.

Since

$$\phi^{-n}(A_1 \cap \phi^n(A_2)) = \phi^{-n}(A_1) \cap \phi^{-n}(\phi^n(A_2)) \supseteq \phi^{-n}(A_1) \cap A_2 \tag{2.3}$$

for all subsets A_1, A_2 of S and every n in $\mathbf{N}$, it follows that (because ϕ preserves the measure μ and μ - measurability), for all A_1, A_2 in $\mathcal{A}$ and every n in $\mathbf{N}$,

$$\mu(A_1 \cap \phi^n(A_2)) = \mu(\phi^{-n}(A_1 \cap \phi^n(A_2))) \geq \mu(\phi^{-n}(A_1) \cap A_2). \tag{2.4}$$

Let A, $B \in \mathcal{A}$, $a := \frac{\mu(A)\mu(B)}{\mu(S)}$ and, for any k in $\mathbf{N_0}$, $a_k := \mu(\phi^{-k}(A) \cap B)$. Then, since ϕ is weakly mixing, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |a_i - a| = 0, \tag{2.5}$$

whence by Theorem 1.B, there exists a subset J of $\mathbf{N_0}$ of density zero such that $\lim_{n\to\infty, n\notin J} a_n = a$, i.e. for every pair of elements A, B of $\mathcal{A}$ there is a subset $J(= J(A, B))$ of $\mathbf{N_0}$ of density zero such that

$$\lim_{n\to\infty, n\notin J} \mu(\phi^{-n}(A) \cap B) = \frac{\mu(A)\mu(B)}{\mu(S)}. \tag{2.6}$$

Next, by (2.4) and (2.6), for every $n \in \mathbf{N_0}$ we have

$$\begin{aligned}
&\liminf_{m\to\infty, m\notin J} \mu(A \cap \phi^m(B)) = \liminf_{k\to\infty, n+k\notin J} \mu(A \cap \phi^{n+k}(B)) \\
&= \liminf_{k\to\infty, n+k\notin J} \mu(A \cap \phi^k(\phi^n(B))) \geq \lim_{k\to\infty, n+k\notin J} \mu(\phi^{-k}(A) \cap \phi^n(B)) \\
&= \frac{\mu(A)\,\mu(\phi^n(B))}{\mu(S)}.
\end{aligned} \tag{2.7}$$

Therefore

$$\liminf_{m\to\infty, m\notin J} \mu(A \cap \phi^m(B)) \geq \frac{\mu(A)}{\mu(S)} \lim_{n\to\infty} \mu(\phi^n(B)). \tag{2.8}$$

Since A was an arbitrary μ- measurable subset of S, (2.8) must hold with A replaced by its complement, $A^c(= S \setminus A)$. Hence

$$\liminf_{m\to\infty, m\notin J} \mu(A^c \cap \phi^m(B)) \geq \frac{\mu(A^c)}{\mu(S)} \lim_{n\to\infty} \mu(\phi^n(B)).$$

Consequently, since, for each $i \in \mathbf{N_0}$,

$$\mu\left(A \cap \phi^{\,i}\,(B)\right) = \mu\left(\phi^{\,i}\,(B)\right) - \mu\left(A^c \cap \phi^{\,i}\,(B)\right),$$

we have

$$\begin{aligned}\limsup_{m\to\infty, m\notin J} \mu\left(A \cap \phi^{\,m}\,(B)\right) &= \lim_{n\to\infty} \mu\left(\phi^{\,n}\,(B)\right) - \liminf_{m\to\infty, m\notin J} \mu\left(A^C \cap \phi^{\,m}\,(B)\right) \\ &\le \lim_{n\to\infty} \mu\left(\phi^{\,n}\,(B)\right) - \frac{\mu\left(A^c\right)}{\mu\left(S\right)} \lim_{n\to\infty} \mu\left(\phi^{\,n}\,(B)\right) \\ &= \left(1 - \frac{\mu\left(A^c\right)}{\mu\left(S\right)}\right) \lim_{n\to\infty} \mu\left(\phi^{\,n}\,(B)\right) \\ &= \frac{\mu\left(A\right)}{\mu\left(S\right)} \lim_{n\to\infty} \mu\left(\phi^{\,n}\,(B)\right).\end{aligned}$$

This taken together with (2.8) gives (2.1) and Theorem 2.1 is proved. □

As is seen from the following corollary in the case of invertible weakly mixing transformations, our above result coincides with the well-known (classical) result.

Corollary 2.1. *Let $(S, \mathcal{A}, \mu)$ be a finite measure space and let $\phi : S \to S$ be an invertible weakly mixing transformation with respect to μ. Then for every pair of elements A, B of $\mathcal{A}$ there is a subset $J(A,B)$ of $\mathbf{N_0}$ of density zero such that*

$$\lim_{m\to\infty, m\notin J(A,B)} \mu(A \cap \phi^m(B)) = \frac{\mu(A)\mu(B)}{\mu(S)}. \tag{2.9}$$

Proof. If ϕ is invertible, then $\mu\left(\phi^{\,n}\,(B)\right) = \mu\left(B\right)$ for any μ- measurable subset B of S and any $n \in \mathbf{N_0}$. Now the convergence property (2.9) follows from (2.1) and the corollary is proved. □

Remark 2.1. It follows at once from the proof of our Theorem 2.1 that, for every measurability-preserving weakly mixing transformation ϕ of the finite measure space $(S, \mathcal{A}, \mu)$ (not necessarily invertible), the following inequality holds for all A, $B \in \mathcal{A}$:

$$\liminf_{m\to\infty, m\notin J(A,B)} \mu(A \cap \phi^m(B)) \ge \frac{\mu(A)\mu(B)}{\mu(S)}, \tag{2.10}$$

where $J(A,B)$ is the subset of $\mathbf{N_0}$ of density zero. This result extends the result of Rice [17, Lemma 1.(ii)] from (strongly) mixing transformations on a probability space to weakly mixing transformations of a finite measure space, and also improves a result of C. Sempi [19, Lemma 2] and a previous result of the author (i.e., the fact that *for any $k \in \mathbf{N}$ there is a positive integer N such that $\mu\left(\phi^n(A) \cap B_i\right) > 0$ for $i = 1, \ldots, k$ and all n in the set $(\mathbf{N_0} \setminus J) \cap \{j \in \mathbf{N_0} | j \ge N\}$ where J is a subset of $\mathbf{N_0}$ of density zero,* *and* $A, B_1, \ldots, B_k \in \mathcal{A}$ *with* $\mu(A) > 0,\ \mu(B_1) > 0, \ldots,\ \mu(B_k) > 0)$ which is established and used in the proofs of Theorems 4 and 5 in [8]. Indeed, putting $n = 0$ in (2.7) gives (2.10).

Investigations have shown that many important consequences of (1.4) persist in the absence of invertibility (see [5], [17], [18, pp. 181-186, 190], [7] and [9, pp. 95-112, 114-119]) and/or the strongly mixing property (see [19], [8], [9, pp. 59-104, 112-121] and [10]). In this direction we now shall use our Theorem 2.1 to find several weakly mixing properties of abstract dynamical systems that are measurability-preserving, which are analogues of the classical weakly mixing properties of invertible dynamical systems.

Theorem 2.2. *Let* $(S, \mathcal{A}, \mu, \phi)$ *be a measurability-preserving weakly mixing dynamical system. For each* B *in* $\mathcal{A}$, *let* $L(B)$ *denote the limit of the sequence* $(\mu(\phi^n(B)))$, *i.e.,* $L(B) := \lim_{n\to\infty} \mu(\phi^n(B))$. *Then, for any two* μ *- measurable subsets* A, B *of* S, *the following three conditions hold and each of them is equivalent to the measurability-preserving weakly mixing property* (2.1):

(i)

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \left| \mu\left(A \cap \phi^i(B)\right) - \frac{\mu(A)\, L(B)}{\mu(S)} \right| = 0; \tag{2.11}$$

(ii)

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \left[\mu\left(A \cap \phi^i(B)\right) - \frac{\mu(A)\, L(B)}{\mu(S)} \right]^2 = 0; \tag{2.12}$$

(iii)

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \left[\mu\left(A \cap \phi^i(B)\right)\right]^2 = \frac{\mu(A)^2\, L(B)^2}{\mu(S)^2}. \tag{2.13}$$

Proof. Let A, B be μ - measurable subsets of S. Then by Theorem 1.2 there is a subset $J(= J(A, B))$ of $\mathbf{N_0}$ of density zero such that the measurability - preserving weakly mixing property (2.1) holds, i.e.,

$$\lim_{m\to\infty, m\notin J} \mu(A \cap \phi^m(B)) = \frac{\mu(A)\, L(B)}{\mu(S)}. \tag{2.14}$$

Now, applying Theorem 1.B with $a_m = \mu(A \cap \phi^m(B)) - \mu(A)\, L(B) / \mu(S)$, by (2.14) we conclude that the condition (2.11) holds and that the conditions (2.1) and (2.11) are equivalent for any two μ - measurable subsets A, B of S.

Also, it is clear that the condition (2.14) holds if and only if

$$\lim_{m\to\infty, m\notin J} \left| \mu(A \cap \phi^m(B)) - \frac{\mu(A)\, L(B)}{\mu(S)} \right| = 0,$$

i.e., if and only if

$$\lim_{m\to\infty, m\notin J}\left[\mu\left(A\cap\phi^{m}(B)\right)-\frac{\mu\left(A\right)L(B)}{\mu\left(S\right)}\right]^{2}=0$$

and this last condition is equivalent to the condition

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\left[\mu\left(A\cap\phi^{i}(B)\right)-\frac{\mu\left(A\right)L(B)}{\mu\left(S\right)}\right]^{2}=0.$$

This shows that the conditions (2.11) and (2.12) are equivalent.

Next, since ϕ is measurability-preserving ergodic (because every weakly mixing transformation is ergodic), it follows from Theorem 2 in [10] that for every A, B in $\mathcal{A}$ we have that the sequence $(\frac{1}{m}\sum_{i=0}^{m-1}\mu\left(A\cap\phi^{m}(B)\right))$ converges to $\frac{\mu(A)L(B)}{\mu(S)}$. This implies that

$$\begin{aligned}
&\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\left[\mu\left(A\cap\phi^{i}(B)\right)-\frac{\mu\left(A\right)L(B)}{\mu\left(S\right)}\right]^{2}\\
&\quad=\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\left[\mu\left(A\cap\phi^{i}(B)\right)^{2}-2\frac{\mu\left(A\right)L(B)}{\mu\left(S\right)}\mu\left(A\cap\phi^{i}(B)\right)\right]\\
&\quad+\frac{\mu\left(A\right)^{2}L(B)^{2}}{\mu\left(S\right)^{2}}=\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\left[\mu\left(A\cap\phi^{i}(B)\right)\right]^{2}-\frac{\mu\left(A\right)^{2}L(B)^{2}}{\mu\left(S\right)^{2}}.
\end{aligned}$$

Therefore the conditions (2.12) and (2.13) are equivalent. This completes the proof of Theorem 2.2. □

We now show that for most useful measure spaces we can strengthen our Theorem 2.1 to obtain a set of density zero that works for all pairs of μ - measurable subsets A and B of S. For this purpose, we begin with the following.

As before, let $(S,\mathcal{A},\mu)$ be a finite measure space. If there is a sequence $(E_k)_{k=1}^{\infty}$ of members of $\mathcal{A}$ such that for each $\varepsilon>0$ and each A in $\mathcal{A}$ there is some k with $\mu\left(A\,\Delta\,E_k\right)<\varepsilon$, we say that $(S,\mathcal{A},\mu)$ has a *countable basis.* This condition is equivalent to the condition that the Hilbert space $L^2(S,\mathcal{A},\mu)$ has a countable dense subset (i.e., that $L^2(S,\mathcal{A},\mu)$ is separable). If S is a metric space with a countable topological base and $\mathcal{A}$ is the σ - algebra of Borel subsets of S (the σ - algebra generated by the open sets), then $(S,\mathcal{A},\mu)$ has a countable basis for any normalized measure μ on $(S,\mathcal{A})$. This is also true if $\mathcal{A}$ is the completion, under μ, of the σ - algebra of Borel subsets of S (see [20, p. 10]). Any topological space with a countable (topological) base is separable, and any separable metric space has a countable topological base (see, e.g., [11; Theorems (6.22) and (6.23)]). Therefore most of the spaces we shall deal with have $L^2(S, \mathcal{A}, \mu)$ separable.

We then have the following result, which is a substantial strengthening of our Theorem 2.1 for the class of finite measure spaces having a countable basis and is the sequence of iterates of ϕ analogue of the classical result in terms of sequences of iterates of ϕ^{-1} (e.g., of Theorem 1.22 in [20, p. 45]).

Theorem 2.3. *Let $(S, \mathcal{A}, \mu)$ be a finite measure space having a countable basis and let $\phi : S \to S$ be a measurability-preserving weakly mixing transformation. Then there is a subset J of $\mathbf{N_0}$ of density zero such that, for all μ - measurable subsets A, B of S,*

$$\lim_{m\to\infty, m\notin J} \mu\left(A \cap \phi^m\left(B\right)\right) = \frac{\mu(A)}{\mu(S)} \lim_{n\to\infty} \mu(\phi^n(B)). \tag{2.15}$$

Proof. Let $(E_k)_{k=1}^{\infty}$ be a countable basis for $(S, \mathcal{A}, \mu)$. For any $m \in \mathbf{N_0}$, put

$$\alpha_m := \sum_{i,j=1}^{\infty} \frac{\left|\mu\left(\phi^{-m}\left(E_i\right) \cap E_j\right) - \mu\left(E_i\right)\mu\left(E_j\right)/\mu\left(S\right)\right|}{2^{i+j}}.$$

Since ϕ is weakly mixing we have $\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \alpha_i = 0$ so there is a subset J of $\mathbf{N_0}$ having density zero such that $\lim_{m\to\infty, m\notin J} \alpha_m = 0$. Therefore

$$\lim_{m\to\infty, m\notin J} \mu\left(\phi^{-m}\left(E_i\right) \cap E_j\right) = \frac{\mu\left(E_i\right)\mu\left(E_j\right)}{\mu\left(S\right)} \tag{2.16}$$

for all i, j in $\mathbf{N}$, and, by a usual approximation argument, it follows that the weakly mixing property (2.6) holds for all members A, B of $\mathcal{A}$. Indeed, let $\varepsilon > 0$ be given and let A, B in $\mathcal{A}$. Choose E_{i_0}, E_{j_0} in $\{E_k : k \in \mathbf{N}\}$ with $\mu\left(A \,\Delta\, E_{i_0}\right) < \frac{\varepsilon}{8}$ and $\mu\left(B \,\Delta\, E_{j_0}\right) < \frac{\varepsilon}{8}$. Then

$$\begin{aligned}|\mu(A) - \mu(E_{i_0})| &= |\mu(A \cap E_{i_0}) + \mu(A \setminus E_{i_0}) - \mu(E_{i_0} \cap A) - \mu(E_{i_0} \setminus A)| \\ &\le \mu(A \setminus E_{i_0}) + \mu(E_{i_0} \setminus A),\end{aligned}$$

i.e.,

$$|\mu\left(A\right) - \mu(E_{i_0})| \le \mu(A \,\Delta\, E_{i_0}).$$

Hence $|\mu\left(A\right) - \mu(E_{i_0})| < \frac{\varepsilon}{8}$, and similarly,

$$\begin{aligned}|\mu\left(B\right) - \mu(E_{j_0})| < \frac{\varepsilon}{8},\ &\left|\mu\left(\phi^{-m}(A) \cap B - \mu\left(\phi^{-m}(E_{i_0}) \cap E_{j_0}\right)\right)\right| \\ &\le \mu\left(\left(\phi^{-m}(A) \cap B\right) \,\Delta\, \left(\phi^{-m}(E_{i_0}) \cap E_{j_0}\right)\right).\end{aligned}$$

For any $m \in \mathbf{N_0}$ we have

$$\left(\phi^{-m}(A) \cap B\right) \,\Delta\, \left(\phi^{-m}(E_{i_0}) \cap E_{j_0}\right) \subseteq \left(\phi^{-m}(A) \,\Delta\, \phi^{-m}(E_{i_0})\right) \cup \left(B \,\Delta\, E_{j_0}\right),$$

and therefore

$$\left|\mu\left(\phi^{-m}(A)\cap B\right)-\mu\left(\phi^{-m}(E_{i_0})\cap E_{j_0}\right)\right|$$
$$\leq \mu\left(\phi^{-m}(A)\,\Delta\,\phi^{-m}(E_{i_0})\right)+\mu\left(B\,\Delta\,E_{j_0}\right)<\frac{\varepsilon}{4}.$$

Next, it follows from (2.16) that there is a positive integer $m_0(=m_0(\varepsilon))$ such that

$$\left|\mu\left(\phi^{-m}(E_{i_0})\cap E_{j_0}\right)-\frac{\mu(E_{i_0})\mu(E_{j_0})}{\mu(S)}\right|<\frac{\varepsilon}{2}$$

for all $m \geq m_0$, $m\notin J$. Therefore

$$\left|\mu\left(\phi^{-m}(A)\cap B\right)-\frac{\mu(A)\mu(B)}{\mu(S)}\right|\leq\left|\mu\left(\phi^{-m}(A)\cap B\right)-\mu\left(\phi^{-m}\left(E_{i_0}\right)\cap E_{j_0}\right)\right|$$
$$+\left|\mu\left(\phi^{-m}(E_{i_0})\cap E_{j_0}\right)-\frac{\mu(E_{i_0})\mu(E_{j_0})}{\mu(S)}\right|+\left|\frac{\mu(E_{i_0})\mu(E_{j_0})}{\mu(S)}-\frac{\mu(A)\mu(E_{j_0})}{\mu(S)}\right|$$
$$+\left|\frac{\mu(A)\mu(E_{j_0})}{\mu(S)}-\frac{\mu(A)\mu(B)}{\mu(S)}\right|<\frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}=\varepsilon$$

for all $m\geq m_0$, $m\notin J$. This proves that (if ϕ is a weakly mixing transformation on a finite measure space $(S,\mathcal{A},\mu)$ with a countable basis then) there is a subset J of $\mathbf{N_0}$ of density zero such that property (2.6) holds for all A, B in $\mathcal{A}$.

Now, since ϕ, in addition (to being weakly mixing on S with respect to μ), is measurability-preserving, arguing as in the proof of our Theorem 2.1, from (2.4) and (2.6) we obtain

$$\limsup_{m\to\infty,m\notin J}\mu\left(A\cap\phi^m(B)\right)\leq\frac{\mu\left(A\right)}{\mu\left(S\right)}\lim_{n\to\infty}\mu\left(\phi^n(B)\right)\leq\liminf_{m\to\infty,m\notin J}\mu\left(A\cap\phi^m(B)\right),$$

whence $\lim_{m\to\infty,m\notin J}\mu\left(A\cap\phi^m(B)\right)=\frac{\mu(A)}{\mu(S)}\lim_{n\to\infty}\mu\left(\phi^n(B)\right)$ for all A, B in $\mathcal{A}$, which proves that the stated condition (2.15) holds and completes the proof of Theorem 2.3. □

3. Concluding remarks

Remark 3.1. Note that each of the conditions (2.1), (2.11), (2.12) and (2.13) implies the relation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu\left(A\cap\phi^i(B)\right)=\frac{\mu\left(A\right)}{\mu\left(S\right)}\lim_{n\to\infty}\mu\left(\phi^n\left(B\right)\right)\tag{3.1}$$

for all A, B in $\mathcal{A}$. This is because if (a_n) is a sequence of real numbers then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|a_i-a|=0$$

implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i = a.$$

However, as was shown in [10], the relation (3.1) is a necessary and sufficient condition for a measure-preserving transformation ϕ on a finite measure space $(S, \mathcal{A}, \mu)$ with the property that $\phi(\mathcal{A}) \subseteq \mathcal{A}$ to be ergodic with respect to μ. Therefore each of the conditions (2.1) and (2.11) - (2.13) is the sufficient condition for a measure-preserving transformation ϕ on a finite measure space $(S, \mathcal{A}, \mu)$ with the property that $\phi(\mathcal{A}) \subseteq \mathcal{A}$ to be ergodic with respect to μ.

Remark 3.2. A classical characterization of weakly mixing in terms of sets of density zero (see, e.g., [20], Theorem 1.21) is that a measure-preserving transformation ϕ on a probability space $(S, \mathcal{A}, \mu)$ is weakly mixing if and only if for every pair of elements A, B of $\mathcal{A}$ there is a subset $J(A,\, B)$ of $\mathbf{N_0}$ of density zero such that

$$\lim_{n\to\infty, n\notin J(A,B)} \mu(\phi^{\,-n}(A) \cap B) = \mu(A)\mu(B). \tag{3.2}$$

However, in the case of invertible measure-preserving transformations, we also have another form of weakly mixing in terms of sets of density zero. Namely, if $(S, \mathcal{A}, \mu, \phi)$ is an invertible dynamical system, then

$$\mu\left(A \cap \phi^n\left(B\right)\right) = \mu\left(\phi^{-n}\left(A \cap \phi^n\left(B\right)\right)\right) = \mu\left(\phi^{-n}(A) \cap \phi^{-n}\left(\phi^n\left(B\right)\right)\right),$$

i.e., for any $A, B \in \mathcal{A}$ and any $n \in \mathbf{N_0}$, we have

$$\mu\left(A \cap \phi^n\left(B\right)\right) = \mu\left(\phi^{-n}(A) \cap B\right),$$

and therefore the condition (2.9) implies the condition (2.6). Thus, the next characterization of weakly mixing is that an invertible dynamical system $(S, \mathcal{A}, \mu, \phi)$ is weakly mixing if and only if for every A, B in $\mathcal{A}$ the condition (2.9) holds. If, in addition, $(S, \mathcal{A}, \mu)$ is a probability space, then the condition (2.9) reduces to the condition

$$\lim_{n\to\infty, n\notin J(A,B)} \mu\left(A \cap \phi^n\left(B\right)\right) = \mu(A)\mu(B). \tag{3.3}$$

Remark 3.3. We can use the condition (3.3) to give an intuitive description of weakly mixing. To say an invertible measure-preserving transformation ϕ on a probability space $(S, \mathcal{A}, \mu)$ is weakly mixing, means that for each set $A \in \mathcal{A}$ the sequence of sets $\phi^n(A)$ (and also the sequence of sets $\phi^{-n}(A)$) becomes, asymptotically, independent of any other set $B \in \mathcal{A}$ provided we neglect a few instants of time.

Note that the results (in measure set - theoretical form) presented here can be expressed in functional form (giving weakly mixing properties in terms of a unitary operator on the Hilbert space $L^2(S, \mathcal{A}, \mu)$ or in terms of the induced operator on the Banach space $L^p(S, \mathcal{A}, \mu)$, $(p \geq 1)$) (see [2, pp. 14, 19, 30-38] and [21, pp. 19-40]).

There is considerable evidence (see the proof of Theorem 2.1) to support a conjecture that our results (for weakly mixing dynamical systems with discrete time) which are contained in Theorems 2.1, 2.2 and 2.3 can be extended to weakly mixing dynamical systems with continuous time (see [4, pp. 6-26]).

Finally, we do not know if the converse of our Theorem 2.1 (and, also, of Theorem 2.3) is true; however, we show in Remark 3.2 that the converse is true for invertible measure-preserving transformations. Thus, it is not known if the condition (2.1) is always sufficient for a measure-preserving transformation ϕ on a finite measure space $(S, \mathcal{A}, \mu)$ with the property that $\phi(\mathcal{A}) \subseteq \mathcal{A}$ to be weakly mixing with respect to μ. Therefore, the question of the author [10, Problem 2] reduces to the following conjecture:

Conjecture 3.1. *Let ϕ be a noninvertible measure-preserving transformation on a finite measure space $(S, \mathcal{A}, \mu)$ with the property that $\phi(\mathcal{A}) \subseteq \mathcal{A}$. If for every pair of elements A, B of $\mathcal{A}$ there is a subset $J(A, B)$ of $\mathbf{N_0}$ having density zero such that*

$$\lim_{m\to\infty, m\notin J(A,B)} \mu(A \cap \phi^m(B)) = \frac{\mu(A)}{\mu(S)} \lim_{n\to\infty} \mu(\phi^n(B)),$$

then ϕ is weakly mixing.

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