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(L^p, L^q) PROPERTIES OF THE POTENTIAL-TYPE INTEGRALS ASSOCIATED TO NON-DOUBLING MEASURES

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ABSTRACT. The (L^p, L^q) boundedness of the potential-type integrals associated to non-doubling measures are investigated.

1. INTRODUCTION

Let (X, μ) is a space with positive measure μ. By $L^p(X, d\mu)$ denote a class of all μ-measurable functions $f : X \to (-\infty, +\infty)$ with $\|f\|_{p,\mu} = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} < \infty$.

Let μ and ν are two positive measures on X and T is a linear operator from $L^p(X, d\mu)$ to $L^q(X, d\nu)$, where $p, q \in (0, \infty)$. T is said to be an operator of strong type $(L^p(X, d\mu), L^q(X, d\nu))$, if there exists a positive constant C such that

$$\|Tf\|_{q,\nu} \leq C \|f\|_{p,\mu}, \quad \text{for} \quad f \in L^p(X, d\mu).$$

If for arbitrary $\beta > 0$ and $f \in L^p(X, d\mu)$

$$\nu\{x : |Tf(x)| > \beta\} \leq \left(\frac{C\|f\|_{p,\mu}}{\beta}\right)^q,$$

then T is called an operator of weak type $(L^p(X, d\mu), L^q(X, d\nu))$.

If $X = R^n$, μ and ν are Lebesgue measures on R^n, then in the above conditions we simply say that T is an operator of strong (weak) type (p, q).

For $0 < \alpha < n$, the operator

$$I_\alpha f(x) = \int_{R^n} |x - y|^{\alpha - n} f(y) \, dy$$

is called a classical Riesz potential, where $|\cdot|$ denotes the Euclidean norm and dy is an element of the Lebesgue measure.

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By the classical Hardy-Littlewood-Sobolev theorem, if $1 < p < \infty$ and $\alpha p < n$, then $I_\alpha f$ is an operator of strong type (p, q), where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $p = 1$, then $I_\alpha f$ is an operator of weak type $(1, q)$, where $\frac{1}{q} = 1 - \frac{\alpha}{n}$ (see [7]).

The Hardy-Littlewood-Sobolev theorem is an important result in fractional integral theory and potential theory. There are a lot of generalizations of this theorem. In [5] and [6], the Hardy-Littlewood-Sobolev theorem is extended to Orlicz spaces for generalized Riesz potentials. In [2] and [4], generalized potential-type integral operators are considered and (L^p, L^q) properties of these operators are proved. The Hardy-Littlewood-Sobolev theorem is proved for Riesz potentials associated to non-doubling measures in [3].

In [2], the following integral operator is considered

$$\Lambda f(x) = \int_{R^n} \mathcal{K}(|x-y|) f(y)\, dy, \tag{1}$$

where $\mathcal{K}(\cdot)$ is a kernel satisfying the following three conditions.

($\mathcal{K}_1$) $\mathcal{K}(\cdot)$ is a nonnegative decreasing function on $(0, \infty)$ and $\lim_{t\to 0} \mathcal{K}(t) = \infty$;

($\mathcal{K}_2$) there exist positive constants A_1 and σ such that for any $0 < h < \infty$

$$\int_0^h \mathcal{K}(t)\, t^{n-1} dt \le A_1 h^\sigma;$$

($\mathcal{K}_3$) there exist a positive constant A_2 and positive $\gamma(p) = \gamma(p, n)$ such that for any $0 < h < \infty$

$$\left(\int_h^\infty \mathcal{K}^{p'}(t)\, t^{n-1} dt\right)^{\frac{1}{p'}} \le A_2 h^{-\gamma(p)},\ if\ 1 < p < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

If $p = 1$, then $\mathcal{K}(h) \le A_2 h^{-\gamma(p)}$.

The following theorem is proved in [2].

Theorem A. *Let $1 \le p < \infty$. Then*

1) *if $f \in L^p(R^n, dx)$, then integral (1) converges for almost every x;*
2) *if $l = \left(1 + \frac{\sigma}{\gamma(p)}\right)p$, then Λ is the operator of weak type (p, l);*
3) *if $1 < p < r$ and*

$$\frac{1}{q} = \frac{1}{p}\left[\frac{r-p}{r-1}\frac{\gamma(1)}{\sigma+\gamma(1)} + \frac{p-1}{r-1}\frac{\gamma(r)}{\sigma+\gamma(r)}\right],$$

then Λ is the operator of strong type (p, q).

Let μ and ν be two positive measures on X and λ be a nonnegative, symmetric function on $X \times X$. Suppose that there exist positive constants M_1, M_2, d and m such that

$$\mu\left(B\left(x,r\right)\right) \leq M_1 r^d \tag{2}$$

and

$$\nu\left(B\left(x,r\right)\right) \leq M_2 r^m, \tag{3}$$

where $B\left(x,r\right)$ is an open ball with center x and λ-radius r, i.e.,

$$B\left(x,r\right) = \left\{y \in X,\ \lambda\left(x,y\right) < r\right\}.$$

Consider the generalized potential-type integral

$$Lf\left(x\right) = \int_X K\left(\lambda\left(x,y\right)\right) f\left(y\right) d\mu\left(y\right), \tag{4}$$

where $K\left(\cdot\right)$ is a kernel.

In this work we have found the sufficient conditions on the kernel $K\left(\cdot\right)$ for the boundedness of operator (4) from $L^p\left(X, d\mu\right)$ to $L^q\left(X, d\nu\right)$.

2. Main result

Theorem 1. *Let $1 \leq p < \infty$, μ and ν be two positive measures on X, the conditions* (2), (3) *be satisfied and the function $K\left(\cdot\right)$ satisfy the following conditions:*

(K_1) $K : (0, \infty) \to (0, \infty)$ *is a decreasing, bijective function and for any* $0 < h < \infty$

$$\int_0^h K\left(t\right) t^{d-1} dt < \infty;$$

(K_2) *there exist positive constants A_1 and σ such that for any* $0 < h < \infty$

$$-\int_0^h \frac{dK\left(t\right)}{dt} t^{d+\frac{m-d}{p}} dt \leq A_1 h^{\sigma};$$

(K_3) *there exist a positive constant A_2 and positive $\gamma\left(p\right) = \gamma\left(p,d\right)$ such that for any* $0 < h < \infty$

$$-\int_h^{\infty} \frac{dK\left(t\right)}{dt} t^{\frac{d}{p'}} dt \leq A_2 h^{-\gamma(p)},\ \ if\, p > 1$$

and

$$K\left(h\right) \leq A_2 h^{-\gamma(1)},\ \ if\, p = 1.$$

Then

i) *if $f \in L^p\left(X, d\mu\right)$, then integral* (4) *converges for ν -almost every x.*

ii) *if $l = \left(1 + \frac{\sigma}{\gamma(p)}\right)p$, then L is the operator of weak type $\left(L^p(X, d\mu),\ L^l (X, d\nu)\right)$*

iii) *if* $1 < p < r$ *and*

$$\frac{1}{q} = \frac{1}{p}\left[\frac{r-p}{r-1}\,\frac{\gamma(1)}{\sigma+\gamma(1)} + \frac{p-1}{r-1}\,\frac{\gamma(r)}{\sigma+\gamma(r)}\right]$$

then L *is an operator of strong type* $(L^p(X,d\mu),\, L^q(X,d\nu))$.

Proof. Take arbitrary $x_0 \in X$. Let $s > 0$, $i \in N$, $H_{s,i} = \{y : y \in B(x_0,i), L|f|(y) > s\}$ and $\nu_{s,i} = \nu|_{H_{s,i}}$. Then by Fubini's theorem

$$\begin{aligned} s\nu(H_{s,i}) &\le \int_{H_{s,i}} L|f|(y)\,d\nu(y) = \int_X L|f|(y)\,d\nu_{s,i}(y) \\ &= \int_X \left[\int_X K(\lambda(x,y))\,d\nu_{s,i}(y)\right]|f(x)|\,d\mu(x) \\ &= \int_X L\nu_{s,i}(x)\,|f(x)|\,d\mu(x). \end{aligned} \tag{5}$$

Consider $L\nu_{s,i}(x) = \int_X K(\lambda(x,y))\,d\nu_{s,i}(y)$. Applying Fubini's theorem

$$\begin{aligned} L\nu_{s,i}(x) &= \int\limits_X \left(\int\limits_0^{K(\lambda(x,y))} dt\right) d\nu_{s,i}(y) = \int_X \left(\int_0^\infty \chi_{\{t<K(\lambda(x,y))\}}(t)\,dt\right) d\nu_{s,i}(y) \\ &= \int_0^\infty \left(\int_{\lambda(x,y)<K^{-1}(t)} d\nu_{s,i}(y)\right) dt = \int_0^\infty \nu_{s,i}\left(B\left(x, K^{-1}(t)\right)\right) dt \\ &= -\int_0^\infty \nu_{s,i}(B(x,t))\,\frac{dK(t)}{dt}\,dt. \end{aligned}$$

For any $h > 0$ we have by (5)

$$\begin{aligned} s\nu(H_{s,i}) &\le -\int_X \left(\int_0^\infty \nu_{s,i}(B(x,t))\,\frac{dK(t)}{dt}\,dt\right)|f(x)|\,d\mu(x) \\ &= -\int\limits_0^h \left(\int_X |f(x)|\,\nu_{s,i}(B(x,t))\,d\mu(x)\right)\frac{dK(t)}{dt}\,dt \\ &\quad -\int_h^\infty \left(\int_X |f(x)|\,\nu_{s,i}(B(x,t))\,d\mu(x)\right)\frac{dK(t)}{dt}\,dt = J_1 + J_2. \end{aligned} \tag{6}$$

If $p > 1$, then by (3) one can write

$$\begin{aligned} \nu_{s,i}(B(x,t)) &= \nu_{s,i}(B(x,t))^{\frac{1}{p'}}\,\nu_{s,i}(B(x,t))^{\frac{1}{p}} \\ &\le M_2^{\frac{1}{p}}\,\nu_{s,i}(B(x,t))^{\frac{1}{p'}}\,t^{\frac{m}{p}}. \end{aligned}$$

Using Holder's inequality we have

$$J_1 \leq -M_2^{\frac{1}{p}} \|f\|_{p,\mu} \int_0^h \left(\int_X \nu_{s,\,i} \left(B\left(x,t\right)\right) d\mu\left(x\right) \right)^{\frac{1}{p'}} t^{\frac{m}{p}} \frac{dK\left(t\right)}{dt} dt. \tag{7}$$

Estimate $\int_X \nu_{s,\,i}\left(B\left(x,t\right)\right) d\mu\left(x\right)$. For $t > 0$ define

$$D_t = \left\{ (x,y) \in X \times X : \lambda\left(x,y\right) < t \right\}.$$

Let $G\left(x,y\right)$ be the characteristic function of D_t. Then by Fubini's theorem

$$\int_X \nu_{s,\,i}\left(B\left(x,t\right)\right) d\mu\left(x\right) = \int_X \int_{B(x,t)} d\nu_{s,\,i}\left(y\right) d\mu\left(x\right)$$

$$= \int_X \int_X G\left(x,y\right) d\nu_{s,\,i}\left(y\right) d\mu\left(x\right) = \int_X \int_X G\left(x,y\right) d\mu\left(x\right) d\nu_{s,\,i}\left(y\right)$$

$$= \int_X \mu\left(B(y,t\right) d\nu_{s,\,i}\left(y\right) \leq M_1 t^d \int_X d\nu_{s,\,i}\left(y\right) = M_1 t^d \nu\left(H_{s,\,i}\right).$$

From (7) and (K_2) we have

$$J_1 \leq -M_2^{\frac{1}{p}} M_1^{\frac{1}{p'}} \|f\|_{p,\mu} \nu\left(H_{s,\,i}\right)^{\frac{1}{p'}} \int_0^h t^{d+\frac{m-d}{p}} \frac{dK\left(t\right)}{dt} dt$$

$$\leq M_2^{\frac{1}{p}} M_1^{\frac{1}{p'}} A_1 \|f\|_{p,\mu} \nu\left(H_{s,\,i}\right)^{\frac{1}{p'}} h^{\sigma}. \tag{8}$$

It is clear that

$$\nu_{s,\,i}\left(B\left(x,t\right)\right) \leq \nu_{s,\,i}\left(B\left(x,t\right)\right)^{\frac{1}{p'}} \nu\left(H_{s,\,i}\right)^{\frac{1}{p}}.$$

Then by Holder's inequality and by (K_3)

$$J_2 \leq -\|f\|_{p,\mu} \nu\left(H_{s,\,i}\right)^{\frac{1}{p}} \int_h^\infty \left(\int_X \nu_{s,\,i}\left(B\left(x,t\right)\right) d\mu\left(x\right) \right)^{\frac{1}{p'}} \frac{dK\left(t\right)}{dt} dt$$

$$\leq -M_1^{\frac{1}{p'}} \|f\|_{p,\mu} \nu\left(H_{s,\,i}\right) \int_h^\infty t^{\frac{d}{p'}} \frac{dK\left(t\right)}{dt} dt$$

$$\leq M_1^{\frac{1}{p'}} A_2 \|f\|_{p,\mu} \nu\left(H_{s,\,i}\right) h^{-\gamma(p)}. \tag{9}$$

At last from (6), (8) and (9)

$$s\nu\left(H_{s,\,i}\right) \leq M_1^{\frac{1}{p'}} \|f\|_{p,\mu} \Big(M_2^{\frac{1}{p}} A_1 \nu\left(H_{s,\,i}\right)^{\frac{1}{p'}} h^{\sigma} + A_2 \nu\left(H_{s,\,i}\right) h^{-\gamma(p)} \Big). \tag{10}$$

If $s > M_1^{\frac{1}{p'}} A_2 \|f\|_{p,\mu} h^{-\gamma(p)}$, then

$$\nu\left(H_{s,\,i}\right) \leq M_2 A_1^p \left(\frac{s}{M_1^{\frac{1}{p'}} \|f\|_{p,\mu}} - A_2 h^{-\gamma(p)} \right)^{-p} h^{\sigma p}.$$

Let $H_s = \{y :\ L|f|(y) > s\}$. Since

$$\bigcup_i H_{s,\,i} = H_s$$

and

$$H_{s,\,1} \subset H_{s,\,2} \subset H_{s,\,3} \subset \dots$$

we have

$$\nu\{y :\ L|f|(y) = \infty\} \le \nu(H_s) = \lim_{i\to\infty} \nu(H_{s,\,i})$$
$$\le M_2 A_1^p \left(\frac{s}{M_1^{\frac{1}{p'}} \|f\|_{p,\mu}} - A_2 h^{-\gamma(p)} \right)^{-p} h^{\sigma p}.$$

and by arbitrariness s one get the proof of i) in the case $p > 1$.

If $p = 1$, then

$$J_1 \le -M_2 \|f\|_{1,\mu} \int_0^h t^m \frac{dK(t)}{dt} dt \le M_2 A_1 \|f\|_{1,\mu} h^\sigma,$$
$$J_2 \le -\|f\|_{1,\mu} \nu(H_{s,\,i}) \int_h^\infty \frac{dK(t)}{dt} dt \le A_2 \|f\|_{1,\mu} \nu(H_{s,\,i}) h^{-\gamma(1)}$$

and just as above one can get the proof of i) in the case $p = 1$.

Let us prove ii). If $p > 1$, then by (10) we have

$$s\nu(H_s) \le M_1^{\frac{1}{p'}} \|f\|_{p,\mu} \Big(M_2^{\frac{1}{p}} A_1 \nu(H_s)^{\frac{1}{p'}} h^\sigma + A_2 \nu(H_s) h^{-\gamma(p)} \Big).$$

Now let $h = \nu(H_s)^{\frac{1}{(\sigma+\gamma(p))p}}$. Then

$$s\nu(H_s) \le M_3 \|f\|_{p,\mu} \nu(H_s)^{\frac{p\sigma+p\gamma(p)-\gamma(p)}{(\sigma+\gamma(p))p}},$$

where $M_3 = M_1^{\frac{1}{p'}} \big(M_2^{\frac{1}{p}} A_1 + A_2 \big)$.

Hence

$$s\nu(H_s)^{\frac{1}{l}} = s\nu(H_s)\nu(H_s)^{\frac{1}{l}-1} \le M_3 \|f\|_{p,\mu} \nu(H_s)^{\frac{p\sigma+p\gamma(p)-\gamma(p)}{(\sigma+\gamma(p))p}+\frac{1}{l}-1}$$
$$= M_3 \|f\|_{p,\mu}.$$

So

$$\nu\{y : L|f|(y) > s\} \le \left(\frac{M_3 \|f\|_{p,\mu}}{s} \right)^l$$

and from this it follows that L is the operator of weak type $\big(L^p(X, d\mu), L^l(X, d\nu)\big)$ in the case $p > 1$. Taking $h = \nu(H_s)^{\frac{1}{\sigma+\gamma(1)}}$ in a similar way we can prove *ii*) in case $p = 1$.

Now prove iii). If take $p = 1$, then from *ii*) it is seen that L is an operator of weak type $\left(L^1(X, d\mu), L^{1+\frac{\sigma}{\gamma(1)}}(X, d\nu) \right)$. If we take $p = r$, then L is an operator of weak type $\left(L^r(X, d\mu), L^{\left(1+\frac{\sigma}{\gamma(r)}\right)r}(X, d\nu)\right)$. Applying the Marcinkiewic interpolation theorem (see [1]) with $p_0 = 1$, $q_0 = 1 + \frac{\sigma}{\gamma(1)}$ and $p_1 = r$, $q_1 = \left(1 + \frac{\sigma}{\gamma(r)}\right)r$ we obtain iii).

The theorem is proved. □

Examples.

1. Let $1 \le p < \infty$, $\alpha > 0$ and $d - m < \alpha p < d$. Let also $K(t) = t^{\alpha-d}$. If we take $\sigma = \alpha + \frac{m-d}{p}$ and $\gamma(p) = \frac{d}{p} - \alpha$, then one can see that the function $K(\cdot)$ satisfies the conditions (K_1), (K_2), (K_3). If also λ is quasi-metric and the measures μ and ν are equal, then by simple calculations $l = \frac{dp}{d-\alpha p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and we have the Hardy-Littlewood-Sobolev theorem for Riesz potentials associated to non-doubling measures (see [3, Theorem 3.2 and Theorem 3.4] and [4, Theorem 2.1]).

2. Let $1 \le p < \infty$, $\alpha > 0$ and assume there exists $\theta > 0$ such that $d - m < \alpha p < d - \theta p$. Let also $K(t) = t^{\alpha-d}\log(1+t)$. If take $\sigma = \alpha + \frac{m-d}{p}$ and $\gamma(p) = \frac{d}{p} - \alpha - \theta$, then the function $K(\cdot)$ satisfies the conditions (K_1), (K_2), (K_3).

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