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STRONG TRUNCATED MATRIX MOMENT PROBLEM OF HAMBURGER

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ABSTRACT. In this paper we consider the strong truncated matrix moment problem on the real line. We describe all the solutions of the problem in the form of a Nevanlinna type formula. We use M. G. Kreĭn's theory of representations for Hermitian operators and the technique of boundary triplets and the corresponding Weyl functions.

1. INTRODUCTION

In this paper we consider the following problem: Given a finite sequence of self-adjoint $N\times N$-matrices $\{S_k\}_{-2m}^{2m}$, find all self-adjoint nonnegative Borel $N\times N$-matrix measures $d\Sigma$ on $\mathbb{R}$ obeying the identities

$$\int_{-\infty}^{+\infty} t^k \, d\Sigma(t) = S_k \qquad (k = 0, \pm 1, \dots, \pm 2m). \tag{1}$$

This problem is called *the strong truncated matrix moment problem of Hamburger*. The matrices $\{S_k\}_{-2m}^{2m}$ are called *moments* and the measure $d\Sigma$ is called *a solution of the moment problem* (1).

Let us recall that for the classical truncated moment problem one is given a sequence $\{S_k\}_0^{2m}$ and seeks a measure $d\Sigma$ such that (1) holds only for nonnegative numbers k.

The classical matrix moment problem was investigated by M. G. Kreĭn (see [17, 18]). In [18], M. G. Kreĭn has described all the solutions of the full classical matrix moment problem for the completely indeterminate case. A description of all the solutions for the truncated classical matrix moment problem was originally obtained in [16] using the method of matrix inequalities developed by V. Potapov. Other approaches to the truncated classical matrix moment problem were presented in [8, 7, 1]. We follow [7] in our treatment of the strong moment problem.

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Investigations of the scalar strong moment problem and orthogonal Laurent polynomials originated in the papers of W. B. Jones, W. J. Thron, H. Waadeland, and O. Njåstad (see [14, 11, 13]). It is worth noting that a necessary and sufficient condition for the solvability of the strong moment problem was originally obtained by Yu. M. Berezanskiĭ (see [3]). A description for the solutions of the full scalar strong moment problem was obtained in [22, 23] for the Hamburger problem and in [15] for the Stieltjes problem. A detailed bibliography can be found in the survey [12].

To solve the moment problem means to answer the following questions:

(1) Under what conditions is the moment problem solvable?
(2) If the moment problem is solvable, how to determine whether it has a unique solution?
(3) How to describe all the solutions of the moment problem?

In this paper we give a necessary and sufficient condition for (1) to be solvable and describe all the solutions of (1) in terms of self-adjoint extensions of a certain linear operator. We also describe the solutions of (1) via a linear transformation of the Nevanlinna type under the assumption that the given sequence $\{S_k\}_{-2m}^{2m}$ is strictly positive and normalized.

Let us briefly outline the contents of the paper. In Section 2 we recall basic concepts of M. G. Kreĭn's theory of representations for Hermitian operators and some methods of the boundary triplets technique.

In Section 3 we establish a solvability criterion for the moment problem (1) (see Theorem 3.3). We also consider the space of Laurent polynomials of the form

$$\sum_{k=-m}^{m} \xi_k z^k \quad \left(\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N\right)$$

with the inner product generated by the Hankel quadratic form

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i \quad \left(\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N\right).$$

In this space, we introduce the multiplication operator A and determine a one-to-one correspondence between the set of minimal self-adjoint extensions of A and the set of all the solutions of (1) (see Theorem 3.4).

In Section 4 we recall some earlier results from [24] on orthogonal matrix Laurent polynomials of the first and the second kind.

In Section 5 we construct a boundary triplet (see Theorem 5.4) and the corresponding resolvent matrix (see Theorem 5.5) of the operator A and describe the set of all the solutions of (1) in the form of a Nevanlinna type formula (see Theorem 5.6).

In Section 6 we illustrate our approach with a simple example.

2. Representations of Hermitian operators

Let us recall basic concepts and statements of M. G. Kreĭn's theory of representations for Hermitian operators (see [17, 9]) and some methods of the boundary triplets technique (see [10, 5, 7, 6]).

A linear relation in a Hilbert space $\mathfrak{H}$ is a linear subspace in $\mathfrak{H} \oplus \mathfrak{H}$. Since any linear operator S in $\mathfrak{H}$ can be identified with its graph

$$\{\{f, Sf\} \in \mathfrak{H} \oplus \mathfrak{H} : \ f \in \operatorname{dom} S\},$$

we can regard any linear operator as a linear relation.

For arbitrary linear relations $\widetilde{S}, \widetilde{T}$ in $\mathfrak{H}$ and $\lambda \in \mathbb{C}$, we put

$$\operatorname{dom} \widetilde{S} = \left\{f : \ \{f, g\} \in \widetilde{S}\right\}, \qquad \operatorname{ran} \widetilde{S} = \left\{g : \ \{f, g\} \in \widetilde{S}\right\},$$
$$\ker \widetilde{S} = \left\{f : \ \{f, 0\} \in \widetilde{S}\right\}, \qquad \operatorname{mul} \widetilde{S} = \left\{g : \ \{0, g\} \in \widetilde{S}\right\},$$

$$\widetilde{S}^{-1} = \left\{\{g, f\} \in \mathfrak{H} \oplus \mathfrak{H} : \ \{f, g\} \in \widetilde{S}\right\},$$
$$\widetilde{S}^{*} = \left\{\{f', g'\} \in \mathfrak{H} \oplus \mathfrak{H} : \ (g, f') = (f, g') \text{ for all } \{f, g\} \in \widetilde{S}\right\},$$
$$\lambda\widetilde{S} = \left\{\{f, \lambda g\} \in \mathfrak{H} \oplus \mathfrak{H} : \ \{f, g\} \in \widetilde{S}\right\},$$
$$\widetilde{S} + \widetilde{T} = \left\{\{f, g + g'\} \in \mathfrak{H} \oplus \mathfrak{H} : \ \{f, g\} \in \widetilde{S}, \ \{f, g'\} \in \widetilde{T}\right\},$$
$$\widetilde{S}\widetilde{T} = \left\{\{f, h\} \in \mathfrak{H} \oplus \mathfrak{H} : \ \{f, g\} \in \widetilde{T}, \ \{g, h\} \in \widetilde{S}\right\}.$$

We define *the resolvent set* $\rho(\widetilde{S})$ of a linear relation $\widetilde{S}$ in $\mathfrak{H}$ by

$$\rho(\widetilde{S}) = \left\{\lambda \in \mathbb{C} : \ \ker(\widetilde{S} - \lambda) = 0, \ \operatorname{ran}(\widetilde{S} - \lambda) = \mathfrak{H}\right\}.$$

A linear relation is called *closed* if it is indeed closed as a subspace in $\mathfrak{H} \oplus \mathfrak{H}$. A linear relation $\widetilde{S}$ in $\mathfrak{H}$ is called *Hermitian* (*dissipative*) if $(f', f) \in \mathbb{R}$ ($\Im(f', f) \geq 0$) for any pair $\{f, f'\} \in \widetilde{S}$. A Hermitian (dissipative) relation $\widetilde{S}$ is called *self-adjoint* (*maximal dissipative*) if $\rho(\widetilde{S}) \neq \emptyset$.

Any maximal dissipative linear relation $\widetilde{S}$ in $\mathfrak{H}$ can be uniquely represented in the form

$$\widetilde{S} = S \oplus \widehat{\operatorname{mul}}\, \widetilde{S},$$

where

$$S = \left\{\{f, f'\} \in \widetilde{S} : \ f' \perp \operatorname{mul} \widetilde{S}\right\}, \quad \widehat{\operatorname{mul}}\, \widetilde{S} = \left\{\{0, f'\} \in \widetilde{S}\right\}.$$

S is an operator, which is called *the operator part* of $\widetilde{S}$. The relation $\widehat{\operatorname{mul}}\, \widetilde{S}$ is called *the multivalued part* of $\widetilde{S}$.

In this section, we consider a simple closed Hermitian operator A with finite deficiency indices (N, N) in a Hilbert space $\mathfrak{H}$. We assume that the domain of A is *not dense* in $\mathfrak{H}$ and $\dim(\mathfrak{H} \ominus \operatorname{dom} A) = N$. Let us put

$$\mathfrak{M}_\lambda = \operatorname{ran}(A-\lambda), \quad \mathfrak{N}_\lambda = \mathfrak{H} \ominus \mathfrak{M}_{\overline{\lambda}}, \quad \widehat{\mathfrak{N}}_\lambda = \{\{f_\lambda, \lambda f_\lambda\} \in \mathfrak{H} \oplus \mathfrak{H} : \ f_\lambda \in \mathfrak{N}_\lambda\},$$

$$\mathfrak{N}_\infty = \operatorname{mul} A^* = \mathfrak{H} \ominus \operatorname{dom} A, \quad \widehat{\mathfrak{N}}_\infty = \widehat{\operatorname{mul}} A^* = 0 \oplus \mathfrak{N}_\infty.$$

Let $\mathfrak{L}$ be a subspace in $\mathfrak{H}$ of dimension N. If there exist at least two points $\lambda_+ \in \mathbb{C}_+$ and $\lambda_- \in \mathbb{C}_-$ such that the decomposition

$$\mathfrak{H} = \mathfrak{L} \dotplus \mathfrak{M}_\lambda \tag{2}$$

holds for $\lambda = \lambda_\pm$, then $\mathfrak{L}$ is called *the module of a representation* of the operator A.

A point $\lambda \in \mathbb{C}$ is called *an $\mathfrak{L}$-regular point* of A if λ is a point of regular type for A and the decomposition (2) holds. Denote by $\rho(A; \mathfrak{L})$ the set of all $\mathfrak{L}$-regular points of A and put

$$\rho_s(A; \mathfrak{L}) = \left\{\lambda \in \mathbb{C} : \ \lambda, \overline{\lambda} \in \rho(A; \mathfrak{L})\right\}.$$

Let us define two holomorphic operator-valued functions

$$\mathcal{P}(\lambda), \mathcal{Q}(\lambda) : \ \mathfrak{H} \to \mathfrak{L} \qquad (\lambda \in \rho(A; \mathfrak{L}))$$

on the set $\rho(A; \mathfrak{L})$. Let $\mathcal{P}(\lambda)$ be the skew projection onto the subspace $\mathfrak{L}$ parallel to $\mathfrak{M}_\lambda$. In other words, $\mathcal{P}(\lambda)$ obeys

$$\mathcal{P}(\lambda) f \in \mathfrak{L}, \quad (I - \mathcal{P}(\lambda)) f \in \mathfrak{M}_\lambda \qquad (f \in \mathfrak{H}).$$

Define $\mathcal{Q}(\lambda)$ by the equality

$$\mathcal{Q}(\lambda) = P_{\mathfrak{L}} (A - \lambda)^{-1} (I - \mathcal{P}(\lambda)).$$

Henceforth, by P_H we denote the orthogonal projection onto a subspace H.

The function $\mathcal{P}$ establishes an isomorphism between the Hilbert space $\mathfrak{H}$ and the space of holomorphic functions

$$\mathfrak{H}_{\mathfrak{L}} = \{f_{\mathfrak{L}}(\lambda) = \mathcal{P}(\lambda) f : \ f \in \mathfrak{H}, \ \lambda \in \rho(A; \mathfrak{L})\}.$$

This isomorphism takes the operator A to the multiplication operator

$$\mathcal{P}(\lambda) A f = \lambda f_{\mathfrak{L}}(\lambda) \qquad (f \in \operatorname{dom} A).$$

It is easy to check the following properties of the functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$:

$$\mathcal{P}(\lambda) A f = \lambda \mathcal{P}(\lambda) f, \quad \mathcal{Q}(\lambda) A f = \lambda \mathcal{Q}(\lambda) f + P_{\mathfrak{L}} f \qquad (f \in \operatorname{dom} A),$$

$$\left.\begin{aligned} \widehat{\mathcal{P}}(\lambda)^* \phi &= \left\{\mathcal{P}(\lambda)^* \phi, \overline{\lambda} \mathcal{P}(\lambda)^* \phi\right\} \in A^*, \\ \widehat{\mathcal{Q}}(\lambda)^* \phi &= \left\{\mathcal{Q}(\lambda)^* \phi, \overline{\lambda} \mathcal{Q}(\lambda)^* \phi + \phi\right\} \in A^* \end{aligned}\right. \qquad (\phi \in \mathfrak{L}),$$

$$\begin{aligned} &\mathcal{P}(\lambda)\phi = \phi, && \mathcal{Q}(\lambda)\phi = 0 \qquad (\phi \in \mathfrak{L}),\\ &P_{\mathfrak{L}}\mathcal{P}(\lambda)^* = I_{\mathfrak{L}}, && P_{\mathfrak{L}}\mathcal{Q}(\lambda)^* = 0_{\mathfrak{L}},\\ &\mathcal{P}(\lambda)^*P_{\mathfrak{L}} = \mathcal{P}(\lambda)^*, && \mathcal{Q}(\lambda)^*P_{\mathfrak{L}} = \mathcal{Q}(\lambda)^*. \end{aligned}$$

It follows from the above that

$$\mathfrak{N}_\lambda = \ker(A^* - \lambda) = \mathcal{P}(\overline{\lambda})^*\mathfrak{L} \qquad (\lambda \in \rho_s(A;\mathfrak{L})). \tag{3}$$

Proposition 2.1 (see [6, 7])**.** *The following decomposition holds:*

$$A^* = A \dotplus \widehat{\mathcal{P}}(\lambda)^*\mathfrak{L} \dotplus \widehat{\mathcal{Q}}(\lambda)^*\mathfrak{L} \qquad (\lambda \in \rho_s(A;\mathfrak{L})).$$

Definition 2.1. Let $\widetilde{A}$ be a self-adjoint extension of the operator A, possibly in a larger Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. The extension $\widetilde{A}$ is called $\mathfrak{L}$-*minimal* if

$$\widetilde{\mathfrak{H}} = \overline{\operatorname{span}}\left\{\mathfrak{L},\ (\widetilde{A} - \lambda)^{-1}\mathfrak{L} :\ \lambda \in \rho(\widetilde{A})\right\}.$$

Definition 2.2. Let $\widetilde{A}$ be an $\mathfrak{L}$-minimal self-adjoint extension of the operator A. Then the operator-valued function

$$P_{\mathfrak{L}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{L}} \qquad (\lambda \in \rho(\widetilde{A}))$$

is called the $\mathfrak{L}$-*resolvent* of the operator A corresponding to the extension $\widetilde{A}$.

Definition 2.3 (see [10])**.** A triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$, where $\Gamma = \{\Gamma_0, \Gamma_1\}$ is a linear operator from A^* to $\mathfrak{L} \oplus \mathfrak{L}$, is called *a boundary triplet* for the linear relation A^* if the mapping Γ is surjective and obeys *the abstract Green identity*

$$(f', g) - (f, g') = (\Gamma_1\hat{f}, \Gamma_0\hat{g})_{\mathfrak{L}} - (\Gamma_0\hat{f}, \Gamma_1\hat{g})_{\mathfrak{L}} \quad (\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in A^*). \tag{4}$$

Proposition 2.2 (see [10])**.** *A boundary triplet* $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ *defines a one-to-one correspondence between the set of proper extensions* $\widetilde{A}$ *of the operator* A ($A \subset \widetilde{A} \subset A^*$) *and the set of linear relations* $\theta \subset \mathfrak{L} \oplus \mathfrak{L}$. *This correspondence is given by*

$$\widetilde{A} = \widetilde{A}_\theta \longleftrightarrow \theta = \Gamma \operatorname{dom} \widetilde{A} = \left\{\{\Gamma_0 f, \Gamma_1 f\} :\ f \in \operatorname{dom} \widetilde{A}\right\}.$$

The extension $\widetilde{A}_\theta$ *is Hermitian (self-adjoint) if and only if the relation* θ *has the same property.*

In particular, the operators Γ_0 and Γ_1 define two self-adjoint extensions $\widetilde{A}_0$ and $\widetilde{A}_1$ of the operator A:

$$\widetilde{A}_0 = \ker\Gamma_0, \quad \widetilde{A}_1 = \ker\Gamma_1.$$

The equality

$$\hat{\gamma}(\lambda) = \{\gamma(\lambda), \lambda\gamma(\lambda)\} = \left(\Gamma_0|_{\widehat{\mathfrak{N}}_\lambda}\right)^{-1} \qquad (\lambda \in \rho(\widetilde{A}_0))$$

defines two operator-valued functions $\hat{\gamma}(\lambda) : \mathfrak{L} \to \widehat{\mathfrak{N}}_\lambda$ and $\gamma(\lambda) : \mathfrak{L} \to \mathfrak{N}_\lambda$ holomorphic on $\rho(\widetilde{A}_0)$.

Definition 2.4 (see [6]). The operator-valued function $M(\lambda) : \mathfrak{L} \to \mathfrak{L}$ defined by the equality

$$M(\lambda)\Gamma_0\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda \qquad (\hat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda,\ \lambda \in \rho(\widetilde{A}_0))$$

is called *the Weyl function* of the operator A corresponding to the boundary triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$.

Proposition 2.3 (see [6]). *The functions $M(\lambda)$ and $\gamma(\lambda)$ obey the identities*

$$\gamma(\lambda) - \gamma(\mu) = (\lambda - \mu)(\widetilde{A}_0 - \lambda)^{-1}\gamma(\mu) \qquad (\lambda, \mu \in \rho(\widetilde{A}_0)), \tag{5}$$

$$M(\lambda) - M(\mu) = (\lambda - \mu)\gamma(\overline{\mu})^*\gamma(\lambda) \qquad (\lambda, \mu \in \rho(\widetilde{A}_0)). \tag{6}$$

Definition 2.5 (see [7]). It is said that a holomorphic function $\tau : \mathbb{C}_+ \to \mathfrak{L} \oplus \mathfrak{L}$ belongs to the class $\widetilde{\mathcal{N}}_{\mathfrak{L}}$ if $\tau(\lambda)$ is a maximal dissipative relation in $\mathfrak{L}$ for any $\lambda \in \mathbb{C}_+$.

It is said that τ belongs to the class $\mathcal{N}_{\mathfrak{L}}$ if $\tau(\lambda)$ is a maximal dissipative *operator* for each $\lambda \in \mathbb{C}_+$.

One can extend a function $\tau \in \widetilde{\mathcal{N}}_{\mathfrak{L}}$ to the domain $\mathbb{C}_-$ by the formula

$$\tau(\lambda) = \tau(\overline{\lambda})^* \qquad (\lambda \in \mathbb{C}_-).$$

By identities (5) and (6), it follows that $M(\lambda)$ belongs to the class $\mathcal{N}_{\mathfrak{L}}$. Moreover, identity (6) means that $M(\lambda)$ is a Q-function of the operator A corresponding to the extension $\widetilde{A}_0$ in the sense of [19, 20].

Definition 2.6 (see [21]). A $2N \times 2N$-matrix $W(\lambda) = (w_{ij}(\lambda))_1^2$ holomorphic on $\rho(A; \mathfrak{L})$ is called *an $\mathfrak{L}$-resolvent matrix* of the operator A if it obeys the identity

$$W(\lambda)JW(\mu)^* = J + i(\lambda - \overline{\mu})G(\lambda)G(\mu)^* \qquad (\lambda, \mu \in \rho(A; \mathfrak{L})),$$

where

$$J = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad G(\lambda) = \begin{pmatrix} -\mathcal{Q}(\lambda) \\ \mathcal{P}(\lambda) \end{pmatrix}.$$

An $\mathfrak{L}$-resolvent matrix is not unique. If $W_1(\lambda)$ and $W_2(\lambda)$ are two different $\mathfrak{L}$-resolvent matrices of A, then there exists a J-unitary matrix U such that

$$W_1(\lambda) = W_2(\lambda)U \qquad (\lambda \in \rho(A; \mathfrak{L})).$$

There exists a natural one-to-one correspondence between the set of $\mathfrak{L}$-resolvent matrices and the set of boundary triplets. The following theorem shows how to construct the $\mathfrak{L}$-resolvent matrix corresponding to a boundary triplet.

Theorem 2.4 (see [7]). *Let $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ be a boundary triplet of the operator A. Then the matrix function*

$$W_\Pi(\lambda) = \left(\Gamma \widehat{G}(\lambda)^*\right)^* = \begin{pmatrix} -\Gamma_0 \widehat{\mathcal{Q}}(\lambda)^* & \Gamma_0 \widehat{\mathcal{P}}(\lambda)^* \\ -\Gamma_1 \widehat{\mathcal{Q}}(\lambda)^* & \Gamma_1 \widehat{\mathcal{P}}(\lambda)^* \end{pmatrix}^*, \tag{7}$$

where

$$\widehat{G}(\lambda)^* = \begin{pmatrix} -\widehat{\mathcal{Q}}(\lambda)^* & \widehat{\mathcal{P}}(\lambda)^* \end{pmatrix},$$

is an $\mathfrak{L}$-resolvent matrix of A. $W_\Pi(\lambda)$ is called the $\Pi\mathfrak{L}$-resolvent matrix of *A corresponding to the boundary triplet* Π.

Theorem 2.5 (see [17, 9, 7]). *Suppose that $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ is a boundary triplet of the operator A such that $\ker \Gamma_0 = A \oplus \widehat{\mathfrak{N}}_\infty$. Let $M(\lambda)$ be the corresponding Weyl function and let $W_\Pi(\lambda) = (w_{ij}(\lambda))_1^2$ be the corresponding $\Pi\mathfrak{L}$-resolvent matrix. Then the formula*

$$P_{\mathfrak{L}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{L}} = (w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda))\,(w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda))^{-1}$$

for $\lambda \in \rho(A; \mathfrak{L})$ establishes a one-to-one correspondence between the set of all $\mathfrak{L}$-minimal self-adjoint extensions $\widetilde{A}$ of the operator A and the set of all functions $\tau \in \widetilde{\mathcal{N}}_{\mathfrak{L}}$. Moreover, the following conditions hold:

(i) $\operatorname{mul} \widetilde{A} = 0$ *if and only if*

$$\lim_{y\to\infty} \frac{\tau(iy)}{y} = 0.$$

(ii) $\ker \widetilde{A} = 0$ *if and only if*

$$\lim_{y\to 0} y(M(iy) + \tau(iy))^{-1} = 0.$$

3. An operator model

Proposition 3.1. *If the moment problem* (1) *is solvable, then the conditions*

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i \geq 0 \tag{8}$$

and

$$\sum_{i,j=-m}^{m-1} \xi_j^* S_{i+j} \xi_i = 0 \quad \textit{if and only if} \quad \sum_{i,j=-m}^{m-1} \xi_j^* S_{i+j+2} \xi_i = 0 \tag{9}$$

are valid for any sequence $\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N$.

Proof. Suppose $d\Sigma$ is a solution of the moment problem (1). Then

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i = \int_{-\infty}^{+\infty} \left(\sum_{j=-m}^{m} \xi_j t^j \right)^* d\Sigma(t) \left(\sum_{i=-m}^{m} \xi_i t^i \right) \geq 0$$

for any $\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N$. Thus (8) is valid.

Since

$$\int_{-\infty}^{+\infty} t^{-k}\, d\Sigma(t) = S_{-k} < \infty \qquad (k = 1, 2, \ldots, 2m)$$

the point $t = 0$ does not belong to the discrete spectrum of $d\Sigma$. Therefore the condition

$$\int_{-\infty}^{+\infty} \left(\sum_{j=-m}^{m-1} \xi_j t^j \right)^* d\Sigma(t) \left(\sum_{i=-m}^{m-1} \xi_i t^i \right) = 0$$

holds if and only if

$$\int_{-\infty}^{+\infty} \left(\sum_{j=-m}^{m-1} \xi_j t^{j+1} \right)^* d\Sigma(t) \left(\sum_{i=-m}^{m-1} \xi_i t^{i+1} \right) = 0$$

holds, which proves (9). $\square$

In the remainder of this section we assume that the conditions (8) *and* (9) *hold.*

Consider the linear space of N-vector Laurent polynomials of formal degree m

$$\mathfrak{H}_1 = \operatorname{span} \left\{ \phi z^k : \ \phi \in \mathbb{C}^N, \ k = -m, -m+1, \ldots, m \right\}.$$

In this space, we introduce the inner product defined by

$$(\phi z^i, \ \psi z^j) = \psi^* S_{i+j} \phi \qquad (\phi, \psi \in \mathbb{C}^N, \ i, j = -m, -m+1, \ldots, m). \tag{10}$$

Put $\mathfrak{H}_0 = \{ f \in \mathfrak{H}_1 : \ (f, f) = 0 \}$.

It follows from (8) that the inner product (10) is non-negative. Therefore the factor space $\mathfrak{H} = \mathfrak{H}_1 / \mathfrak{H}_0$ is a Hilbert space. We denote by $\hat{\phi} z^k$ the equivalence class $(\phi z^k + \mathfrak{H}_0) \in \mathfrak{H}$. It follows from (9) that

$$z(\hat{\phi} z^k) = \hat{\phi} z^{k+1}, \quad z^{-1}(\hat{\phi} z^{-k}) = \hat{\phi} z^{-k-1} \qquad (k = -m, -m+1, \ldots, m-1).$$

Therefore the multiplication operator

$$A(\hat{\phi} z^k) = \hat{\phi} z^{k+1},$$

$$\operatorname{dom} A = \operatorname{span} \left\{ \hat{\phi} z^k : \ \phi \in \mathbb{C}^N, \ k = -m, -m+1, \ldots, m-1 \right\}$$

is well defined and $\ker A = 0$.

A is a Hermitian operator and its domain, in general, is not dense in $\mathfrak{H}$. Therefore A^* is a linear relation. Note that

$$\operatorname{mul} A^* = \mathfrak{H} \ominus \operatorname{dom} A, \quad \ker A^* = \mathfrak{H} \ominus \operatorname{ran} A.$$

Put

$$\mathfrak{L} = \left\{ \hat{\phi} : \ \phi \in \mathbb{C}^N \right\}.$$

Proposition 3.2. *Let a linear relation $\widetilde{A}$ be an $\mathfrak{L}$-minimal self-adjoint extension of the operator A in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and let $E_t = E_t(\widetilde{A})$ be the spectral measure of $\widetilde{A}$. Then there exist some self-adjoint matrices X and Y obeying*

$$0 \le X \le S_{2m}, \quad 0 \le Y \le S_{-2m}$$

such that the equalities

$$\int_{-\infty}^{+\infty} t^k \, d(E_t\hat{\phi}, \hat{\psi}) = \psi^* S_k \phi \qquad (k = 0, \pm 1, \ldots, \pm(2m-1)), \tag{11}$$

$$\int_{-\infty}^{+\infty} t^{2m} \, d(E_t\hat{\phi}, \hat{\psi}) = \psi^* (S_{2m} - X)\phi, \tag{12}$$

$$\int_{-\infty}^{+\infty} t^{-2m} \, d(E_t\hat{\phi}, \hat{\psi}) = \psi^* (S_{-2m} - Y)\phi \tag{13}$$

hold for any $\phi, \psi \in \mathbb{C}^N$. Moreover,

$$X = 0 \quad \textit{if and only if} \quad \operatorname{mul} \widetilde{A} = 0,$$

$$Y = 0 \quad \textit{if and only if} \quad \ker \widetilde{A} = 0.$$

Proof. First, let us prove equality (11) for $k = 0, 1, \ldots, 2m-1$ and equality (12). Note that

$$\operatorname{mul} \widetilde{A} \subset \operatorname{mul} A^* \oplus (\widetilde{\mathfrak{H}} \ominus \mathfrak{H}) = (\mathfrak{H} \ominus \operatorname{dom} A) \oplus (\widetilde{\mathfrak{H}} \ominus \mathfrak{H}).$$

Since the extension $\widetilde{A}$ is $\mathfrak{L}$-minimal, $\operatorname{mul} \widetilde{A} = 0$ if and only if $\operatorname{mul} \widetilde{A} \perp (\mathfrak{H} \ominus \operatorname{dom} A)$.

The relation $\widetilde{A}$ can be uniquely represented in the form

$$\widetilde{A} = A' \oplus \widehat{\operatorname{mul}} \, \widetilde{A},$$

where A' is the operator part of $\widetilde{A}$. In particular, $A'f \perp \operatorname{mul} \widetilde{A}$ for any $f \in \operatorname{dom} A'$.

Put $P_{\mathcal{M}} = P_{\operatorname{mul} \widetilde{A}}$. Let us show that

$$(A')^k \hat{\phi} = A^k \hat{\phi} = z^k \hat{\phi} \qquad (\phi \in \mathbb{C}^N, \ k = 0, 1, \ldots, m-1). \tag{14}$$

Indeed, assume that this assertion is proven for $k \le n < m-1$. Then

$$(A')^{n+1}\hat{\phi} = A' A^n \hat{\phi} = A^{n+1}\hat{\phi} - P_{\mathcal{M}} A^{n+1}\hat{\phi} = A^{n+1}\hat{\phi}$$

since $A^{n+1}\hat{\phi} = z^{n+1}\hat{\phi} \in \operatorname{dom} A \perp \operatorname{mul} \widetilde{A}$ for $0 < n+1 < m$. Finally, the vector $(A')^m\hat{\phi}$ has the form

$$(A')^m\hat{\phi} = A'A^{m-1}\hat{\phi} = A^m\hat{\phi} - P_{\mathcal{M}}A^m\hat{\phi}.$$

Now it is clear that conditions (11) hold for $k = 0, 1, \dots, 2m-1$. Let us show that (12) is valid. Indeed,

$$\int_{-\infty}^{+\infty} t^{2m}\, d(E_t\hat{\phi}, \hat{\psi}) = ((A')^m\hat{\phi}, (A')^m\hat{\psi})$$

$$= (A^m\hat{\phi} - P_{\mathcal{M}}A^m\hat{\phi}, A^m\hat{\psi} - P_{\mathcal{M}}A^m\hat{\psi}) = (A^m\hat{\phi}, A^m\hat{\psi}) - (P_{\mathcal{M}}A^m\hat{\phi}, P_{\mathcal{M}}A^m\hat{\psi})$$
$$= \psi^* S_{2m}\phi - \psi^* X\phi,$$

where X is a self-adjoint matrix defined by

$$\psi^* X\phi = (P_{\mathcal{M}}A^m\hat{\phi}, P_{\mathcal{M}}A^m\hat{\psi}) \qquad (\phi, \psi \in \mathbb{C}^N).$$

It is clear that $0 \le X \le S_{2m}$ and $X = 0$ if and only if

$$A^m\mathfrak{L} = \left\{\hat{\phi}z^m : \ \phi \in \mathbb{C}^N\right\} \perp \operatorname{mul} \widetilde{A}.$$

Since $\mathfrak{H} = \operatorname{dom} A + A^m\mathfrak{L}$ and $\operatorname{dom} A \perp \operatorname{mul} \widetilde{A}$, this implies that $X = 0$ if and only if $\operatorname{mul} \widetilde{A} = 0$.

Now we will prove equality (11) for $k = -1, -2, \dots, -2m+1$ and equality (13). Since the extension $\widetilde{A}$ is $\mathfrak{L}$-minimal and

$$\ker \widetilde{A} \subset \ker A^* \oplus (\widetilde{\mathfrak{H}} \ominus \mathfrak{H}) = (\mathfrak{H} \ominus \operatorname{ran} A) \oplus (\widetilde{\mathfrak{H}} \ominus \mathfrak{H}),$$

$\ker \widetilde{A} = 0$ if and only if $\ker \widetilde{A} \perp (\mathfrak{H} \ominus \operatorname{ran} A)$.

The relation $\widetilde{A}$ can be uniquely represented in the form

$$\widetilde{A} = A'' \oplus \widehat{\ker}\, \widetilde{A},$$

where

$$\ker A'' = 0, \quad \widehat{\ker}\, \widetilde{A} = \left\{\{f, 0\} \in \mathfrak{H} \oplus \mathfrak{H} : \ f \in \ker \widetilde{A}\right\}.$$

In particular, $\operatorname{dom} A'' \perp \ker \widetilde{A}$.

Put $P_{\mathcal{K}} = P_{\ker \widetilde{A}}$. Now let us show that

$$(A'')^{-k}\hat{\phi} = A^{-k}\hat{\phi} = z^{-k}\hat{\phi} \qquad (\phi \in \mathbb{C}^N, \ k = 0, 1, \dots, m-1). \tag{15}$$

Indeed, assume that this assertion is proven for $k \le n < m-1$. Then

$$(A'')^{-n-1}\hat{\phi} = (A'')^{-1}A^{-n}\hat{\phi} = A^{-n-1}\hat{\phi} - P_{\mathcal{K}}A^{-n-1}\hat{\phi} = A^{-n-1}\hat{\phi}$$

since $A^{-n-1}\hat{\phi} = z^{-n-1}\hat{\phi} \in \operatorname{ran} A \perp \ker \widetilde{A}$ for $0 < n+1 < m$. Finally, the vector $(A'')^{-m}\hat{\phi}$ has the form

$$(A'')^{-m}\hat{\phi} = (A'')^{-1}A^{-m+1}\hat{\phi} = A^{-m}\hat{\phi} - P_{\mathcal{M}}A^{-m}\hat{\phi}.$$

It is clear that conditions (11) hold for $k = -1, -2, \dots, -2m+1$. Let us show that (13) is valid. Indeed,

$$\begin{aligned}\int_{-\infty}^{+\infty} t^{-2m}\, d(E_t\hat{\phi}, \hat{\psi}) &= ((A'')^{-m}\hat{\phi}, (A'')^{-m}\hat{\psi}) \\ &= (A^{-m}\hat{\phi} - P_{\mathcal{K}}A^{-m}\hat{\phi}, A^{-m}\hat{\psi} - P_{\mathcal{K}}A^{-m}\hat{\psi}) \\ = (A^{-m}\hat{\phi}, A^{-m}\hat{\psi}) - (P_{\mathcal{K}}A^{-m}\hat{\phi}, &P_{\mathcal{K}}A^{-m}\hat{\psi}) = \psi^* S_{-2m}\phi - \psi^* Y\phi,\end{aligned}$$

where Y is a self-adjoint matrix defined by

$$\psi^* Y \phi = (P_{\mathcal{K}}A^{-m}\hat{\phi}, P_{\mathcal{K}}A^{-m}\hat{\psi}) \qquad (\phi, \psi \in \mathbb{C}^N).$$

It is clear that $0 \le Y \le S_{-2m}$ and $Y = 0$ if and only if $A^{-m}\mathfrak{L} \perp \ker \widetilde{A}$. This implies that $Y = 0$ if and only if $\ker \widetilde{A} = 0$. □

Now combining Proposition 3.1 and Proposition 3.2, we obtain a solvability criterion for the moment problem (1).

Theorem 3.3. *The moment problem* (1) *is solvable if and only if the conditions* (8) *and* (9) *hold.*

Elaborating Proposition 3.2, we can describe all the solutions of (1).

Theorem 3.4. *There exists a one-to-one correspondence between the set of all solutions $d\Sigma$ of the moment problem* (1) *and the set of all $\mathfrak{L}$-minimal self-adjoint extensions $\widetilde{A}$ of the operator A obeying the conditions*

$$\operatorname{mul} \widetilde{A} = 0, \quad \ker \widetilde{A} = 0. \tag{16}$$

This correspondence is given by

$$\psi^*\Sigma(t)\phi = (E_t(\widetilde{A})\hat{\phi}, \hat{\psi}) \qquad (\phi, \psi \in \mathbb{C}^N), \tag{17}$$

where $E_t(\widetilde{A})$ is the spectral measure of $\widetilde{A}$.

Proof. It follows from Proposition 3.2 that (17) is a solution of the moment problem (1) if $\widetilde{A}$ is an $\mathfrak{L}$-minimal self-adjoint extension satisfying (16) and E_t is the spectral measure of $\widetilde{A}$. Thus we only need to prove the converse assertion of the theorem.

Suppose that $d\Sigma(t)$ is a solution of the moment problem (1). Let us define a linear bounded self-adjoint operator $e(t)$ in $\mathfrak{L}$ by

$$(e(t)\hat{\phi}, \hat{\psi}) = \psi^*\Sigma(t)\phi.$$

Then $e(t)$ obeys the conditions

$$e(-\infty) = 0_{\mathfrak{L}}, \quad e(+\infty) = I_{\mathfrak{L}}, \quad e(t-0) = e(t) \qquad (t \in \mathbb{R}).$$

By the Naimark dilation theorem (see [2, 4]), there exists a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{L}$ and a resolution of identity $E_t : \widetilde{\mathfrak{H}} \to \widetilde{\mathfrak{H}}$ such that

$$e(t) = P_{\mathfrak{L}} E_t|_{\mathfrak{L}}, \quad \overline{\operatorname{span}} \left\{ E_t \hat{\phi} : \hat{\phi} \in \mathfrak{L} \right\} = \widetilde{\mathfrak{H}}.$$

The resolution of identity E_t defines the self-adjoint operator

$$\widetilde{A} = \int_{-\infty}^{+\infty} t \, dE_t$$

in the space $\widetilde{\mathfrak{H}}$. By construction, $\widetilde{A}$ is $\mathfrak{L}$-mininal. Let us show that there exists an isometric embedding $V : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ such that $VAV^{-1} \subset \widetilde{A}$. Indeed,

$$\int_{-\infty}^{+\infty} t^{\pm 2k} \, d(E_t \hat{\phi}, \hat{\phi}) = \int_{-\infty}^{+\infty} t^{\pm 2k} \, d(e(t) \hat{\phi}, \hat{\phi}) = \phi^* S_{\pm 2k} \phi < \infty,$$

and therefore $\mathfrak{L} \subset \operatorname{dom} \widetilde{A}^k$ for each $k = 0, \pm 1, \ldots, \pm m$. Put

$$V(z^k \hat{\phi}) = V(A^k \hat{\phi}) = \widetilde{A}^k \hat{\phi} \qquad (\hat{\phi} \in \mathfrak{L}, \ k = 0, \pm 1, \ldots, \pm m).$$

Note that V maps the space $\mathfrak{L}$ onto itself. The mapping V is isometric since

$$(V(z^i \hat{\phi}), V(z^j \hat{\psi}))_{\widetilde{\mathfrak{H}}} = (\widetilde{A}^i \hat{\phi}, \widetilde{A}^j \hat{\psi})_{\widetilde{\mathfrak{H}}} = \int_{-\infty}^{+\infty} t^{i+j} \, d(E_t \hat{\phi}, \hat{\psi})$$
$$= \int_{-\infty}^{+\infty} t^{i+j} \, \psi^* d\Sigma(t) \phi = \psi^* S_{i+j} \phi = (z^i \hat{\phi}, z^j \hat{\psi})$$

for $\phi, \psi \in \mathbb{C}^N$, $i, j = 0, \pm 1, \ldots, \pm 2m$, and the inclusion $VAV^{-1} \subset \widetilde{A}$ holds by construction.

Obviously, $\operatorname{mul} \widetilde{A} = 0$ since $\widetilde{A}$ is an operator. It follows from Proposition 3.2 that $\ker \widetilde{A} = 0$. □

Corollary 3.4.1. *The moment problem* (1) *has a unique solution if and only if the operator A is self-adjoint.*

4. Orthogonal Laurent polynomials

Definition 4.1. A sequence of self-adjoint $N \times N$-matrices $\{S_k\}_{-2m}^{2m}$ is called *strictly positive* if the quadratic form

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i \qquad (\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N)$$

is strictly positive definite.

A strictly positive sequence $\{S_k\}_{-2m}^{2m}$ is called *normalized* if $S_0 = I$.

Any strictly positive sequence $\{\widetilde{S}_k\}_{-2m}^{2m}$ can be normalized by the rule

$$S_k = \widetilde{S}_0^{-\frac{1}{2}} \widetilde{S}_k \widetilde{S}_0^{-\frac{1}{2}} \qquad (k = 0, \pm 1, \ldots, \pm 2m).$$

Definition 4.2. The moment problem (1) is called *nondegenerate* if the given sequence of moments $\{S_k\}_{-2m}^{2m}$ is strictly positive.

The following assertion is well known.

Proposition 4.1. *If a sequence $\{S_k\}_{-2m}^{2m}$ is strictly positive, then there exist self-adjoint matrices*

$$S_{-2m-2},\ S_{-2m-1},\ S_{2m+1},\ S_{2m+2}$$

such that the sequence $\{S_k\}_{-2m-2}^{2m+2}$ is also strictly positive.

In the rest of the paper, we assume that the given sequence $\{S_k\}_{-2m}^{2m}$ is strictly positive and normalized. Proposition 4.1 allows us to regard $\{S_k\}_{-2m}^{2m}$ as a part of some infinite positive bisequence $\{S_k\}_{-\infty}^{+\infty}$. Further we will use the extended coefficients $S_{\pm(2m+1)},\ S_{\pm(2m+2)},\ \ldots$ in our calculations. While the bisequence $\{S_k\}_{-\infty}^{+\infty}$ is not uniquely determined by the original matrices $\{S_k\}_{-2m}^{2m}$, its variation does not significantly change the final result.

Consider the Hilbert space of N-vector Laurent polynomials

$$\widehat{\mathfrak{H}} = \overline{\operatorname{span}} \left\{ \phi z^k :\ \phi \in \mathbb{C}^N,\ k = 0, \pm 1, \pm 2, \ldots \right\}$$

with the inner product

$$(\phi z^i, \psi z^j) = \psi^* S_{i+j} \phi \qquad (\phi, \psi \in \mathbb{C}^N,\ i, j = 0, \pm 1, \pm 2, \ldots).$$

The finite-dimentional Hilbert spaces $\mathfrak{H}$ and $\mathfrak{L}$ introduced in Section 3 are subspaces of $\widehat{\mathfrak{H}}$. Since the bisequence $\{S_k\}_{-\infty}^{+\infty}$ is normalized, the subspace $\mathfrak{L}$ is naturally isomorphic to the space $\mathbb{C}^N$. Further we will use $\mathfrak{L}$ and $\mathbb{C}^N$ interchangeably.

Definition 4.3 (see [24])**.** A sequence of $N\times N$-matrix Laurent polynomials $\{P_k(z)\}_0^\infty$ of the form

$$P_{2k}(z) = \sum_{j=-k}^{k} P_{2k}^{(j)} z^j, \quad P_{2k+1}(z) = \sum_{j=-k-1}^{k} P_{2k+1}^{(j)} z^j \qquad (P_k^{(j)} \in \mathbb{C}^{N\times N})$$

is called *the sequence of orthogonal Laurent polynomials of the first kind* if the following conditions hold:

(A) The coefficients $P_{2k}^{(k)}$ and $P_{2k+1}^{(-k-1)}$ are strictly positive matrices.

(B) The Laurent polynomials $\{P_k(z)\}_0^\infty$ are orthonormal, i. e.,

$$(P_i(z)\xi, P_j(z)\eta) = 0, \quad (P_k(z)\xi, P_k(z)\eta) = \eta^*\xi \qquad (\xi, \eta \in \mathbb{C}^N,\ i, j, k = 0, 1, \ldots,\ i \neq j).$$

Conditions (A) and (B) uniquely determine the sequence $\{P_k(z)\}_0^\infty$.

Definition 4.4 (see [24])**.** The sequence of $N \times N$-matrix Laurent polynomials $\{Q_k(z)\}_0^\infty$ defined by

$$\eta^* Q_k(z)\xi = (R_k(\cdot, z)\xi, \eta) \qquad (\xi, \eta \in \mathbb{C}^N,\ k = 0, 1, 2, \ldots),$$

where

$$R_k(\zeta, z) = \frac{P_k(\zeta) - P_k(z)}{\zeta - z} \qquad (k = 0, 1, 2, \ldots),$$

is called *the sequence of Laurent polynomials of the second kind.*

Extending Definitions 4.3 and 4.4, put

$$P_{-2}(z) = 0, \quad P_{-1}(z) = 0, \quad Q_{-2}(z) = -I, \quad Q_{-1}(z) = 0.$$

If we denote by $\{\epsilon_j\}_1^N$ the standard basis in $\mathbb{C}^N$, then the sequence

$$\{P_i(z)\epsilon_j\}_{i=0}^\infty = \{P_0(z)\epsilon_1, \ldots, P_0(z)\epsilon_N, P_1(z)\epsilon_1, \ldots, P_1(z)\epsilon_N,\ \ldots\}$$

forms an orthonormal basis in the space $\widehat{\mathfrak{H}}$. Therefore, any element $f \in \widehat{\mathfrak{H}}$ can be uniquely represented as a Fourier series

$$f(z) = \sum_{k=0}^\infty P_k(z)\phi_k, \tag{18}$$

where the Fourier coefficients $\phi_k \in \mathbb{C}^N$ are determined by the equalities

$$\epsilon_j^*\phi_k = (f(z), P_k(z)\epsilon_j) \qquad (j = 1, \ldots, N).$$

The coefficients $\{\phi_k\}_0^\infty$ obey the condition

$$\|f\|^2 = \sum_{k=0}^\infty \|\phi_k\|^2_{\mathbb{C}^N} < \infty. \tag{19}$$

Conversely, any vector f of the form (18) satistying (19) belongs to the space $\widehat{\mathfrak{H}}$.

Theorem 4.2 (see [24])**.** *The Laurent polynomials $\{P_k(z)\}_0^\infty$ and $\{Q_k(z)\}_0^\infty$ obey the following recurrence relations:*

$$
\begin{aligned}
zP_{2k}(z) &= P_{2k-2}(z)C^*_{2k-2} + P_{2k-1}(z)B^*_{2k-1} \\
&\quad + P_{2k}(z)A_{2k} + P_{2k+1}(z)B_{2k} + P_{2k+2}(z)C_{2k}, \\
zQ_{2k}(z) &= Q_{2k-2}(z)C^*_{2k-2} + Q_{2k-1}(z)B^*_{2k-1} \\
&\quad + Q_{2k}(z)A_{2k} + Q_{2k+1}(z)B_{2k} + Q_{2k+2}(z)C_{2k}, \\
zP_{2k+1}(z) &= P_{2k}(z)B^*_{2k} + P_{2k+1}(z)A_{2k+1} + P_{2k+2}(z)B_{2k+1}, \\
zQ_{2k+1}(z) &= Q_{2k}(z)B^*_{2k} + Q_{2k+1}(z)A_{2k+1} + Q_{2k+2}(z)B_{2k+1} \\
&\qquad (k = 0, 1, 2, \ldots)
\end{aligned}
\tag{20}
$$

with the initial conditions

$$
P_{-2}(z) = 0, \quad P_0(z) = I, \quad Q_{-2}(z) = -I, \quad Q_0(z) = 0, \tag{21}
$$

where the coefficients $\{A_k\}_0^\infty$, $\{B_k\}_{-1}^\infty$, $\{C_k\}_{-2}^\infty$ *are some* $N \times N$*-matrices.*

Proposition 4.3 (see [24]). *The coefficients* $\{A_k\}_0^\infty$, $\{B_k\}_{-1}^\infty$, $\{C_k\}_{-2}^\infty$ *of the recurrence relations* (20) *obey the following conditions.*

(i) $C_{-2} = I,\ B_{-1} = 0,\ C_{2k-1} = 0 \qquad (k = 0, 1, 2, \ldots)$;

(ii) *The following matrices are well defined:*

$$
C_{2k}^{-1}, \quad \widetilde{B}_0 = (B_0^* - A_0 C_0^{-1} B_1)^{-1},
$$

$$
\widetilde{C}_{2k+1} = -\left[\begin{pmatrix} B_{2k} & B^*_{2k+1} \end{pmatrix} \begin{pmatrix} C_{2k} & A_{2k+2} \\ 0 & C_{2k+2} \end{pmatrix}^{-1} \begin{pmatrix} B^*_{2k+2} \\ B_{2k+3} \end{pmatrix} \right]^{-1} \quad (k = 0, 1, 2 \ldots);
$$

(iii) *The following inequalities hold:*

$$
C_{2k}C_{2k-2}\cdots C_0 > 0, \quad \widetilde{C}_{2k+1}\widetilde{C}_{2k-1}\cdots \widetilde{C}_1 \widetilde{B}_0 > 0 \qquad (k = 0, 1, 2, \ldots);
$$

(iv) *The matrices* A_k *are self-adjoint and obey the identities*

$$
A_{2k+1} = B_{2k} C_{2k}^{-1} B_{2k+1} \qquad (k = 0, 1, 2, \ldots).
$$

Theorem 4.4 (see [24]). *Let* $\{A_k\}_0^\infty$, $\{B_k\}_{-1}^\infty$, $\{C_k\}_{-2}^\infty$ *be arbitrary matrices satisfying conditions* (i)–(iv). *Then there exists a unique positive and normalized bisequence of moments* $\{S_k\}_{-\infty}^{+\infty}$ *such that the corresponding Laurent polynomials* $\{P_k(z)\}_0^\infty$ *and* $\{Q_k(z)\}_0^\infty$ *obey* (20) *with the given coefficients.*

The sequence

$$
\{P_i(z)\epsilon_j\}_{i=0}^{2m} = \{P_0(z)\epsilon_1, \ldots, P_0(z)\epsilon_N,\ \ldots,\ P_{2m}(z)\epsilon_1, \ldots, P_{2m}(z)\epsilon_N\}
$$

forms an orthonormal basis in the space $\mathfrak{H}$. Any element $f \in \mathfrak{H}$ can be uniquely represented as a finite sum

$$
f(z) = \sum_{k=0}^{2m} P_k(z)\phi_k. \tag{22}
$$

Recall that the linear operator A was given by

$$\operatorname{dom} A = \operatorname{span}\left\{\phi z^k :\ \phi \in \mathbb{C}^N,\ k = -m, -m+1, \ldots, m-1\right\},$$
$$Af(z) = zf(z) \qquad (f \in \operatorname{dom} A).$$

In the basis $\{P_i(z)\epsilon_j\}_{i=0}^{2m}$, the operator A has the following block-matrix form

$$\begin{pmatrix} A_0 & B_0^* & C_0^* & & & & * \\ B_0 & A_1 & B_1^* & & & & * \\ C_0 & B_1 & A_2 & & & & * \\ & & & \ddots & & & \vdots \\ & & & & A_{2m-2} & B_{2m-2}^* & * \\ & & & & B_{2m-2} & A_{2m-1} & * \\ & & & & C_{2m-2} & B_{2m-1} & * \end{pmatrix}$$

By the symbol $*$ we denote undefined values.

Denote by $\widetilde{A}_1$ the self-adjoint extension of A in $\mathfrak{H}$ given by

$$\widetilde{A}_1 P_{2m}(z)\xi = P_{2m-2}(z)C_{2m-2}^*\xi + P_{2m-1}(z)B_{2m-1}^*\xi + P_{2m}(z)A_{2m}\xi. \tag{23}$$

The operator $\widetilde{A}_1$ has the following block-matrix form

$$\begin{pmatrix} A_0 & B_0^* & C_0^* & & & & \\ B_0 & A_1 & B_1^* & & & & \\ C_0 & B_1 & A_2 & & & & \\ & & & \ddots & & & \\ & & & & A_{2m-2} & B_{2m-2}^* & C_{2m-2}^* \\ & & & & B_{2m-2} & A_{2m-1} & B_{2m-1}^* \\ & & & & C_{2m-2} & B_{2m-1} & A_{2m} \end{pmatrix}$$

5. Solutions of the moment problem

In this section we continue to study the Hermitian operator A in the Hilbert space $\mathfrak{H}$, which was defined in Section 3, assuming that the bisequence $\{S_k\}_{-\infty}^{+\infty}$ is strictly positive and normalized. Since $\mathfrak{H}$ is finite dimensional, the deficiency indices of A are equal to $\dim(\mathfrak{H} \ominus \operatorname{dom} A) = N$.

Theorem 5.1. *The adjoint relation A^* has the form*

$$A^* = \left\{\left\{f, \widetilde{A}_1 f + P_{2m}\delta\right\} : f \in \mathfrak{H},\ \delta \in \mathfrak{L}\right\}, \tag{24}$$

where $\widetilde{A}_1$ is the self-adjoint extension of the operator A defined by (23). *The deficiency subspaces of A has the form*

$$\begin{aligned} \mathfrak{N}_\lambda &= \left\{ f_{\lambda,\phi}(z) = \sum_{k=0}^{2m} P_k(z) P_k(\overline{\lambda})^* \phi : \ \phi \in \mathfrak{L} \right\} \qquad (\lambda \in \mathbb{C} \setminus \{0\}), \\ \mathfrak{N}_\infty &= \{ P_{2m}(z)\phi : \ \phi \in \mathfrak{L} \}. \end{aligned} \tag{25}$$

Proof. A vector $\hat{f} = \{f, f'\}$ belongs to A^* if and only if it obeys the equalities

$$(f(z), AP_k(z)\epsilon_j) = (f'(z), P_k(z)\epsilon_j) \qquad (k = 0, 1, \dots, 2m-1, \ j = 1, \dots, N). \tag{26}$$

Suppose the Laurent polynomials $f(z)$ and $f'(z)$ have the form

$$f(z) = \sum_{k=0}^{2m} P_k(z) f_k, \quad f'(z) = \sum_{k=0}^{2m} P_k(z) f'_k.$$

Then the equalities (26) can be expressed as

$$\begin{aligned} f'_{2k} &= C_{2k-2} f_{2k-2} + B_{2k-1} f_{2k-1} \\ &\quad + A_{2k} f_{2k} + B^*_{2k} f_{2k+1} + C^*_{2k} f_{2k+2}, \qquad (k = 0, 1, \dots, m-1), \\ f'_{2k+1} &= B_{2k} f_{2k} + A_{2k+1} f_{2k+1} + B^*_{2k+1} f_{2k+2} \end{aligned} \tag{27}$$

assuming that $f_{-2} = f_{-1} = 0$. Note that the leading coefficient f'_{2m} is not constrained by (27). Put

$$\delta = f'_{2m} - C_{2m-2} f_{2m-2} - B_{2m-1} f_{2m-1} - A_{2m} f_{2m}.$$

Then (27) can be expressed in the form

$$f'(z) = \widetilde{A}_1 f(z) + P_{2m}(z)\delta. \tag{28}$$

Conversely, any vector $\hat{f} = \{f, f'\}$ satisfying (28) for some $\delta \in \mathbb{C}^N$ obeys (26), and hence it belongs to A^*.

Now let us prove (25). The form of $\mathfrak{N}_\infty$ is obvious, so we only need to find $\mathfrak{N}_\lambda$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

Let us show that any Laurent polynomial $f_{\lambda,\phi}$ belongs to $\mathfrak{N}_\lambda = \ker(A^* - \lambda)$. In other words, it means that any vector $\hat{f}_{\lambda,\phi} = \{f_{\lambda,\phi}, \lambda f_{\lambda,\phi}\}$ belongs to A^*. Using (26), this condition can be expressed in the form of recurrence relations

$$\begin{aligned} \lambda P_{2k}(\overline{\lambda})^* &= C_{2k-2} P_{2k-2}(\overline{\lambda})^* + B_{2k-1} P_{2k-1}(\overline{\lambda})^* \\ &\qquad + A_{2k} P_{2k}(\overline{\lambda})^* + B^*_{2k} P_{2k+1}(\overline{\lambda})^* + C^*_{2k} P_{2k+2}(\overline{\lambda})^*, \\ \lambda P_{2k+1}(\overline{\lambda})^* &= B_{2k} P_{2k}(\overline{\lambda})^* + A_{2k+1} P_{2k+1}(\overline{\lambda})^* + B^*_{2k+1} P_{2k+2}(\overline{\lambda})^* \\ &\qquad\qquad (k = 0, 1, \dots, m-1), \end{aligned}$$

which follow from Theorem 4.2. Therefore $f_{\lambda,\phi} \in \mathfrak{N}_\lambda$.

Since the set $\{f_{\lambda,\phi}(z) :\ \phi \in \mathfrak{L}\}$ is an N-dimensional linear subspace in $\mathfrak{H}$, it coincides with $\mathfrak{N}_\lambda$. ☐

Corollary 5.1.1. *The operator A is simple, i. e.,*

$$\bigcap_{\lambda\in\mathbb{C}\setminus\mathbb{R}} \mathfrak{M}_\lambda = \{0\}\,.$$

Proof. Let $f \in \mathfrak{H}$ be a Laurent polynomial satisfying

$$(f, f_{\lambda,\phi}) = 0 \qquad (\lambda \in \mathbb{C}\setminus\mathbb{R},\ \phi \in \mathfrak{L}). \tag{29}$$

Suppose that f has the form

$$f(z) = \sum_{k=0}^{2m} P_k(z) f_k \qquad (\{f_k\}_0^{2m} \subset \mathbb{C}^N).$$

Then (29) is transformed to

$$P_k(\lambda) f_k = 0 \qquad (\lambda \in \mathbb{C}\setminus\mathbb{R},\ k = 0, 1, \ldots, 2m).$$

Hence $f_k = 0$ for $k = 0, 1, \ldots, 2m$, and $f = 0$. ☐

Proposition 5.2. *The following condition holds:*

$$\mathfrak{L} \cap \operatorname{ran}(A-\lambda) = \{0\} \qquad (\lambda \in \mathbb{C}\setminus\{0\}).$$

Proof. Suppose that α is the angle between the subspaces $\mathfrak{L}$ and $\operatorname{ran}(A-\lambda)$. We claim that $\alpha > 0$. Indeed,

$$\sin\alpha = \inf_{\substack{\|\phi\|_{\mathfrak{L}}=1\\ f\in\operatorname{dom}A}} \{\|\phi - (A-\lambda)f\|\} = \inf_{\substack{\|\phi\|_{\mathfrak{L}}=1\\ g(\lambda)=0}} \{\|\phi - g\|\} = \inf_{\substack{\|\phi\|_{\mathfrak{L}}=1\\ h(\lambda)=\phi}} \{\|h\|\}$$

$$= \inf_{h\in\mathfrak{H}} \left\{ \frac{\|h\|}{\|h(\lambda)\|_{\mathfrak{L}}} \right\} = \inf_{\{h_k\}_0^{2m}\subset\mathbb{C}^N} \left\{ \frac{\left(\sum_{k=0}^{2m} \|h_k\|_{\mathfrak{L}}^2\right)^{\frac12}}{\left\|\sum_{k=0}^{2m} P_k(\lambda) h_k\right\|_{\mathfrak{L}}} \right\}.$$

Using Cauchy's inequality, we obtain

$$\sin\alpha \geq \frac{1}{\left(\sum_{k=0}^{2m} \|P_k(\lambda)\|_{\mathfrak{L}}^2\right)^{\frac12}} > 0.$$ ☐

The operator A is a simple Hermitian operator with deficiency indices (N, N), the decomposition

$$\mathfrak{H} = \mathfrak{M}_\lambda \dotplus \mathfrak{L} \qquad (\lambda \in \mathbb{C}\setminus\{0\})$$

holds, and the set of $\mathfrak{L}$-regular points of A coincides with the domain $\mathbb{C}\setminus\{0\}$. Let us construct the representation of A with the module $\mathfrak{L}$. Denote by $\mathcal{P}(\lambda)$ the skew projection onto $\mathfrak{L}$ parallel $\mathfrak{M}_\lambda$ in the space $\mathfrak{H}$. Put

$$\mathcal{Q}(\lambda) = P_{\mathfrak{L}}(A-\lambda)^{-1}(I - \mathcal{P}(\lambda)).$$

Then any vector

$$f(z) = \sum_{k=0}^{2m} P_k(z) f_k \in \mathfrak{H}$$

obeys

$$\mathcal{P}(\lambda) f = f(\lambda) = \sum_{k=0}^{2m} P_k(\lambda) f_k, \quad \mathcal{Q}(\lambda) f = \sum_{k=0}^{2m} Q_k(\lambda) f_k.$$

Proposition 5.3. *The following equalities hold:*

$$\left.\begin{aligned} \left(\mathcal{P}(\lambda)^* \phi\right)(z) &= \sum_{k=0}^{2m} P_k(z) P_k(\lambda)^* \phi, \\ \left(\mathcal{Q}(\lambda)^* \phi\right)(z) &= \sum_{k=0}^{2m} P_k(z) Q_k(\lambda)^* \phi \end{aligned}\right. \qquad (\phi \in \mathfrak{L}).$$

Proof. Let us prove the first equality, the second equality can be proved similarly. Expand the vector $\mathcal{P}(\lambda)^* \phi$ as a Fourier series

$$\mathcal{P}(\lambda)^* \phi = \sum_{k=0}^{2m} P_k(z) f_k.$$

Then the coefficients f_k are determined from the equalities

$$\begin{aligned} \epsilon_j^* f_k &= (\mathcal{P}(\lambda)^* \phi, P_k(z) \epsilon_j) = (\phi, \mathcal{P}(\lambda) P_k(z) \epsilon_j) \\ &= (\phi, P_k(\lambda) \epsilon_j)_{\mathbb{C}_N} = \epsilon_j^* P_k(\lambda)^* \phi \qquad (k = 0, 1, \dots, 2m,\ j = 1, \dots, N). \end{aligned}$$

□

Now let us introduce a boundary triplet of A^*.

Theorem 5.4 (cf. [7, Proposition 10.1])**.** *Let*

$$\hat{f} = \left\{ f, \widetilde{A}_1 f + P_{2m} \delta \right\} \in A^*, \quad f = \sum_{k=0}^{2m} P_k f_k \in \mathfrak{H}.$$

The triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ *given by*

$$\Gamma_0 \hat{f} = f_{2m}, \quad \Gamma_1 \hat{f} = \delta$$

is a boundary triplet of the operator A^*.

Proof. The proof is a straightforward check of the Green formula (4). □

The boundary triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ defines two self-adjoint extensions $\widetilde{A}_0 = \ker \Gamma_0$ and $\widetilde{A}_1 = \ker \Gamma_1$. Note that the extension $\widetilde{A}_0 = \ker \Gamma_0$ obeys the condition $\widetilde{A}_0 = A \oplus \widehat{\mathfrak{N}}_\infty$ and the extension $\widetilde{A}_1 = \ker \Gamma_1$ coincides with the extension $\widetilde{A}_1$ defined by (23).

Theorem 5.5 (cf. [7, Proposition 10.1])**.** *Let* $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ *be the boundary triplet of* A *defined in Theorem* 5.4*. Then the matrix function*

$$W(\lambda) = \begin{pmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} -Q_{2m}(\lambda) & -Q_{2m+1}(\lambda)B_{2m} - Q_{2m+2}(\lambda)C_{2m} \\ P_{2m}(\lambda) & P_{2m+1}(\lambda)B_{2m} + P_{2m+2}(\lambda)C_{2m} \end{pmatrix} \quad (30)$$

is the corresponding $\Pi\mathfrak{L}$*-resolvent matrix.*

Proof. It is easy to check that

$$\widetilde{A}_1 \sum_{k=0}^{2m} P_k(z)P_k(\lambda)^*\phi$$
$$= \overline{\lambda} \sum_{k=0}^{2m} P_k(z)P_k(\lambda)^*\phi - P_{2m}(z)(P_{2m+1}(\lambda)B_{2m} + P_{2m+2}(\lambda)C_{2m})^*\phi,$$

$$\widetilde{A}_1 \sum_{k=0}^{2m} P_k(z)Q_k(\lambda)^*\phi$$
$$= \overline{\lambda} \sum_{k=0}^{2m} P_k(z)Q_k(\lambda)^*\phi + \phi - P_{2m}(z)(Q_{2m+1}(\lambda)B_{2m} + Q_{2m+2}(\lambda)C_{2m})^*\phi$$

for any $\phi \in \mathfrak{L}$. Therefore

$$\widehat{\mathcal{P}}(\lambda)^*\phi = \left\{ \sum_{k=0}^{2m} P_k(z)P_k(\lambda)^*\phi,\ \overline{\lambda} \sum_{k=0}^{2m} P_k(z)P_k(\lambda)^*\phi \right\} =$$
$$\left\{ \sum_{k=0}^{2m} P_k(z)P_k(\lambda)^*\phi,\ \widetilde{A}_1 \sum_{k=0}^{2m} P_k(z)P_k(\lambda)^*\phi \right.$$
$$\left. + P_{2m}(z)(P_{2m+1}(\lambda)B_{2m} + P_{2m+2}(\lambda)C_{2m})^*\phi \right\},$$

$$\widehat{\mathcal{Q}}(\lambda)^*\phi = \left\{ \sum_{k=0}^{2m} P_k(z)Q_k(\lambda)^*\phi,\ \overline{\lambda} \sum_{k=0}^{2m} P_k(z)Q_k(\lambda)^*\phi + \phi \right\}$$
$$= \left\{ \sum_{k=0}^{2m} P_k(z)Q_k(\lambda)^*\phi,\ \widetilde{A}_1 \sum_{k=0}^{2m} P_k(z)Q_k(\lambda)^*\phi \right.$$
$$\left. + P_{2m}(z)(Q_{2m+1}(\lambda)B_{2m} + Q_{2m+2}(\lambda)C_{2m})^*\phi \right\}.$$

Thus

$$
\begin{aligned}
w_{11}(\lambda)^* &= -\Gamma_0\widehat{\mathcal{Q}}(\lambda)^* = -Q_{2m}(\lambda)^*,\\
w_{12}(\lambda)^* &= -\Gamma_1\widehat{\mathcal{Q}}(\lambda)^* = -(Q_{2m+1}(\lambda)B_{2m} + Q_{2m+2}(\lambda)C_{2m})^*,\\
w_{21}(\lambda)^* &= \Gamma_0\widehat{\mathcal{P}}(\lambda)^* = P_{2m}(\lambda)^*,\\
w_{22}(\lambda)^* &= \Gamma_1\widehat{\mathcal{P}}(\lambda)^* = (P_{2m+1}(\lambda)B_{2m} + P_{2m+2}(\lambda)C_{2m})^*. \qquad \square
\end{aligned}
$$

Corollary 5.5.1. *Let $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ be the boundary triplet of A defined in Theorem* 5.4*. The corresponding Weyl function $M(\lambda)$ has the form*

$$M(\lambda) = (P_{2m+1}(\lambda)B_{2m} + P_{2m+2}(\lambda)C_{2m})\, P_{2m}(\lambda)^{-1}. \tag{31}$$

Using Theorems 2.5, 3.4, and 5.5, we obtain our main result.

Theorem 5.6. *There exists a one-to-one correspondence between the set of all the solutions $d\Sigma$ of the moment problem* (1) *and the set of all the functions $\tau \in \widetilde{\mathcal{N}}_{\mathbb{C}^N}$ obeying*

$$\lim_{y\to\infty} \frac{\tau(iy)}{y} = 0, \quad \lim_{y\to 0} y(M(iy) + \tau(iy))^{-1} = 0,$$

where the function $M(\lambda)$ is defined by (31)*. The correspondence is given by the following Nevanlinna type formula*

$$\int_{-\infty}^{+\infty} \frac{d\Sigma(t)}{t-\lambda} = (w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda))\,(w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda))^{-1}, \tag{32}$$

where the functions $(w_{ij}(\lambda))_1^2$ are defined by (30)*.*

6. An example

In this section we consider a simple example, which illustrates the approach developed in the previous sections.

Denote by $\{U_k(z)\}_0^\infty$ the sequence of Chebyshev polynomials of the second kind on $[-1, 1]$. The polynomials $\{U_k(z)\}_0^\infty$ obey the recurrence relations

$$U_0(z) = 1, \quad 2zU_k(z) = U_{k-1}(z) + U_{k+1}(z) \qquad (k = 1, 2, \ldots)$$

and can be expressed explicitly in the form

$$U_k(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{when} \quad x = \cos\theta.$$

The polynomials $\{U_k(z)\}_0^\infty$ are orthogonal polynomials of the first kind corresponding to the Jacobi matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Now consider a generalized Jacobi matrix for the strong moment problem

$$\begin{pmatrix} 0 & I & I & & & \\ I & 0 & 0 & & & \\ I & 0 & 0 & I & I & \\ & & I & 0 & 0 & \ddots \\ & & I & 0 & 0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{33}$$

with the coefficients

$$\begin{matrix} A_k = 0, \quad B_{2k+1} = 0, \quad C_{2k+1} = 0, \\ B_{2k} = I, \quad C_{2k} = I \end{matrix} \qquad (k = 0, 1, 2, \ldots).$$

In view of Theorem 4.4, the matrix (33) uniquely determines a bisequence of moments $\{S_k\}_{-\infty}^{+\infty}$.

The Laurent polynomials of the first kind $\{P_k(z)\}_0^\infty$ corresponding to (33) satisfy the conditions

$$\begin{aligned} &P_{-2}(z) = P_{-1}(z) = 0, \quad P_0(z) = I, \\ &\left(z - \frac{1}{z}\right) P_{2k}(z) = P_{2k-2}(z) + P_{2k+2}(z), \\ &\qquad\quad P_{2k+1} = \frac{1}{z} P_{2k}(z) \end{aligned} \qquad (k = 0, 1, 2, \ldots).$$

Therefore,

$$\begin{aligned} P_{2k}(z) &= U_k\left(\frac{1}{2}\left(z - \frac{1}{z}\right)\right), \\ P_{2k+1}(z) &= \frac{1}{z} U_k\left(\frac{1}{2}\left(z - \frac{1}{z}\right)\right) \end{aligned} \qquad (k = 0, 1, 2, \ldots).$$

The sequence of Laurent polynomials of the second kind $\{Q_k(z)\}_0^\infty$ coincides with the sequence $\{P_k(z)\}_0^\infty$ shifted left by 2 positions, that is,

$$Q_k(z) = P_{k-2}(z) \qquad (k = 0, 1, 2, \ldots).$$

Now using Theorem 5.5 we can obtain the resolvent matrix $W(\lambda)$ and the Weyl function $M(\lambda)$ corresponding to the strong truncated moment problem (1) with the moments $\{S_k\}_{-2m}^{2m}$ determined by the generalized Jacobi matrix (33). They have the form

$$W(\lambda) = \begin{pmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} -U_{m-1}(\omega) & -\frac{1}{\lambda}U_{m-1}(\omega) - U_m(\omega) \\ U_m(\omega) & \frac{1}{\lambda}U_m(\omega) + U_{m+1}(\omega) \end{pmatrix}, \tag{34}$$

$$M(\lambda) = \frac{1}{\lambda}I + U_{m+1}(\omega)U_m(\omega)^{-1},$$

where $\omega = \dfrac{1}{2}\left(\lambda - \dfrac{1}{\lambda}\right)$. It is easy to check that the point $\lambda = 0$ is a regular point of $M(\lambda)$ and $M(0) = 0$.

Finally we can describe the set of the solutions $d\Sigma$ of the moment problem using Theorem 5.6. The formula (32), where $w_{ij}(\lambda)$ are elements of the resolvent matrix (34), gives a one-to-one correspondence between the set of the solutions $d\Sigma$ of the strong truncated moment problem given by the Jacobi matrix (33) and the set of matrix functions $\tau \in \mathcal{N}_{\mathbb{C}^N}$ obeying the conditions

$$\lim_{y\to\infty} y^{-1}\tau(iy) = 0, \quad \lim_{y\to 0} y\tau(iy)^{-1} = 0.$$

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