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BANACH-MAZUR DISTANCE BETWEEN TWO DIMENSIONAL BANACH SPACES

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Abstract. The purpose of the present paper is to investigate geometric properties of two-dimensional Banach spaces. We are also concerned with the Banach-Mazur distance between Banach spaces. For real or complex spaces $d(l_1^2, l_p^2) = 2^{1-\frac{1}{p}}$, if $1 \le p \le 2$ and if $1 \le p \le \infty$ and l_p^2 is two-dimensional real space, then $d(l_1^2, l_p^2) = 2^{\frac{1}{p}}$.

1. Introduction

For isomorphic Banach spaces E and F, $d(E, F)$ measures how far the unit ball of E is from an image of the unit ball of F.

In general, it can be rather difficult to compute the Banach-Mazur distance between two given spaces. The main problem lies in finding isomorphisms with small norms. Let us recall the definition of the Banach-Mazur distance.

Definition 1. *Let E and F be Banach spaces . We define the Banach-Mazur distance between E and F by*

$$d(E, F) = \inf \Big\{ \|T\| \; \|T^{-1}\| \; : \; T : E \to F \text{ isomorphism} \Big\}. \qquad (1)$$

Geometrically, let B_E and B_F denote the unit balls of the spaces E and F respectively. Then $d(E, F) < d$, if there exists an isomorphism $T : E \to F$ such that

$$B_F \subset T(B_E) \subset d\,B_F.$$

Obviously, for Banach spaces E, F and G one has

$$d(E, F) \le d(E, G)\, d(G, F). \qquad (2)$$

If E and F are finite-dimensional and dim E = dim F, then there exists an isomorphism $T : E \to F$ such that $\|T\|\,\|T^{-1}\| = d(E, F)$. For infinite dimensional spaces this may not be true.

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If E and F are not isomorphic, then one defines their Banach-Mazur distance to be infinity.

Example 1. If $E = (V, \| \, . \, \|)$ is an n-dimensional normed space, then

$$d(E, l_1^n) \leq n.$$

In 1948 for finite dimensional spaces Fritz John [2] proved an essentially best-possible upper bound for $d(E, F)$ by first bounding the distance of an n-dimensional space from l_2^n. The following theorem of John shows that one can obtain a good upper bound for $d(E, l_2^n)$ be taking the ellipsoid D of minimal volume containing B_E. An ellipsoid is the set of the form $\{x : \langle x, x \rangle \leq 1\}$ for some inner product $\langle ., . \rangle$ on $\mathbb{R}^n$.

Theorem 1 (John's theorem). *Let E be a normed space with unit ball B_E. Then there is a unique ellipsoid D of minimal (Euclidean) volume containing B_E. Furthermore,*

$$n^{-\frac{1}{2}} D \subset B_E \subset D.$$

In particular, $d(E, l_2^n) \leq n^{\frac{1}{2}}$.

Let E and F be n-dimensional normed spaces by Theorem 1. Then

$$d(E, F) \leq d(E, l_2^n)\, d(l_2^n, F) \leq n^{\frac{1}{2}} n^{\frac{1}{2}} = n.$$

The following lemma was proved by J. Lamperti [3].

Lemma 1. *If x and y are complex numbers, then if $2 < p \leq \infty$, then*

$$|x + y|^p + |x - y|^p \geq 2|x|^p + 2|y|^p. \tag{3}$$

If $1 \leq p < 2$, then

$$|x + y|^p + |x - y|^p \leq 2|x|^p + 2|y|^p. \tag{4}$$

For $p = 2$, of course, we get equality for any x and y.

The two-dimensional real spaces l_∞^2 and l_1^2 are isometric because the unit balls in both l_∞^2 and l_1^2 are square. We can see that by rotating l_1^2 into l_∞^2. Define a map $T : l_1^2 \to l_\infty^2$ by

$$T(x, y) = \begin{pmatrix} (x + y) \\ (x - y) \end{pmatrix}.$$

Then T is an isometry, since $\max(|x+y|, |x-y|) = |x| + |y|$, for all $x, y \in \mathbb{R}$. But in three-dimensional spaces l_∞^3 and l_1^3 are not isometric because the unit balls in both l_∞^3 and l_1^3 are different, the unit ball in l_1^3 is octahedron while the unit ball in l_∞^3 is a cube. From these results we calculate the Banach-Mazur distance between l_1^2 and l_p^2 for the real (complex) two-dimensional spaces.

Theorem 2. *Let l_p^2 be the real or complex two-dimensional space. Then*

$$d(l_1^2, l_p^2) = 2^{1-\frac{1}{p}}, \;\; \textit{for } 1 \le p \le 2. \tag{5}$$

Proof. Let $T\colon l_p^2 \to l_1^2$ be the identity map

$$T \,:\, (x_1, x_2) \,\to (x_1, x_2).$$

By Holder's inequality:

$$\begin{aligned} |x_1| + |x_2| &\le (1^q + 1^q)^{\frac{1}{q}} \, (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= 2^{1-\frac{1}{p}} \|(x_1, x_2)\|_p. \end{aligned} \tag{6}$$

But if $x_1 = x_2 = 2^{-\frac{1}{p}}, \quad \|x\| = 1$. Then

$$\|T(x)\| = \|(x_1, x_2)\|_1 = |x_1| + |x_2| = 2^{1-\frac{1}{p}}. \tag{7}$$

Hence, $\|T\| = 2^{1-\frac{1}{p}}$. Next, we show that $\| \, T^{-1} \, \| \, = 1$.
Let $T^{-1}\colon l_1^2 \to l_p^2$ and let $y = (y_1, y_2) \in l_1^2$, then $\|y\|_1 = |y_1| + |y_2|$. Since $(|y_1|^p + |y_2|^p)^{\frac{1}{p}} \le |y_1| + |y_2|$, then $\|T^{-1}\| \le 1$. But if $y_1 = 1$ and $y_2 = 0$, then $\|T^{-1}y\|_1 = 1 = \|y\|_1$. Therefore $\|T^{-1}\| = 1$. Hence,

$$d(l_1^2, l_p^2) \le 2^{1-\frac{1}{p}}, \;\; \text{for all } p. \tag{8}$$

Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L(l_1^2, l_p^2)$ be an isomorphism with $\|T^{-1}\| = 1$ such that $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$ and (e_i) be the unit vector in l_1^2, for $i = 1, 2$.

We show that $\|T\| \ge 2^{1-\frac{1}{p}}$, when $1 \le p < 2$.

Then $T(x_1, x_2) = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$ and in particular

$$T(e_1) = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } T(e_2) = \begin{pmatrix} b \\ d \end{pmatrix} \tag{9}$$

$T(1,1) = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$ and $T(1,-1) = \begin{pmatrix} a-b \\ c-d \end{pmatrix}$.

Since $\|T^{-1}\| = 1, \;\; \|T(1,1)\| \ge 2$ and $\|T(1,-1)\| \ge 2$ i.e.,

$$|a+b|^p + |c+d|^p \ge 2^p, \tag{10}$$

$$|a-b|^p + |c-d|^p \ge 2^p. \tag{11}$$

Adding (10) and (11) we get that

$$|a+b|^p + |a-b|^p + |c+d|^p + |c-d|^p \ge 2^{p+1}.$$

Therefore,

$$2|a|^p + 2|b|^p + 2|c|^p + 2|d|^p \ge 2^{p+1}, \;\; (\text{ from } (4)\ )$$

$$|a|^p + |b|^p + |c|^p + |d|^p \geq 2^p.$$

Then either $|a|^p + |c|^p \geq 2^{p-1}$ or $|b|^p + |d|^p \geq 2^{p-1}$.
In the first case $\|T(e_1)\| = \big(|a|^p + |c|^p\big)^{\frac{1}{p}} \geq (2^{p-1})^{\frac{1}{p}} = 2^{1-\frac{1}{p}}$.
Similarly, the second case gives $\|T(e_2)\| \geq 2^{1-\frac{1}{p}}$. Hence

$$\|T\| \geq 2^{1-\frac{1}{p}}. \tag{12}$$

Therefore, $d(l_1^2, l_p^2) = 2^{1-\frac{1}{p}}$, when $1 \leq p \leq 2$. □

In the next theorem, we calculate the Banach-Mazur distance between the real spaces l_1^2 and l_p^2, when $p \geq 2$.

Theorem 3. *Let $l_p^2 (2 \leq p \leq \infty)$ be two-dimensional real space. Then*

$$d(l_1^2, l_p^2) = 2^{\frac{1}{p}}.$$

If $l_p^2 (p > 2)$ is the complex two-dimensional space. Then

$$d(l_1^2, l_p^2) = \sqrt{2}.$$

Proof. Since $d(E,F) = d(E^*, F^*)$ we have that

$$d(l_1^2, l_p^2) = d(l_\infty^2, l_q^2) \text{ where } \frac{1}{p} + \frac{1}{q} = 1$$

But l_∞^2 is isometric to l_1^2 and so $d(l_\infty^2, l_q^2) = d(l_1^2, l_q^2) = 2^{1-\frac{1}{q}}$ since $q < 2$. Therefore, $d(l_1^2, l_p^2) = 2^{1-\frac{1}{q}} = 2^{\frac{1}{p}}$, for $p \geq 2$.

In the complex case, if we define $T : l_1^2 \to l_p^2$ by

$$(\alpha, \beta) \to (\alpha + \beta, \alpha\beta).$$

Then

$$\begin{aligned}\|T(\alpha,\beta)\|_p &= \|(\alpha+\beta, \alpha-\beta)\|_p = (|\alpha+\beta|^p + |\alpha-\beta|^p)^{\frac{1}{p}} \\ &\leq (2(|\alpha| + |\beta|)^p)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}} \|(\alpha,\beta)\|_p.\end{aligned}$$

Hence, $\|T\| \leq 2^{\frac{1}{p}}$. But if $\alpha = 1$ and $\beta = 0$, then $\|T(\alpha,\beta)\|_p = 2^{\frac{1}{p}}$. Therefore,

$$\|T\| = 2^{\frac{1}{p}}. \tag{13}$$

Now, $T^{-1} : l_p^2 \to l_1^2$ and $(u,v) \to \big(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\big)$.

$$\|T^{-1}(u,v)\|_1 = \frac{1}{2}|u+v| + \frac{1}{2}|u-v| \leq (2)^{-\frac{1}{2}} \big(|u+v|^2 + |u-v|^2\big)^{\frac{1}{2}}$$

$$\leq (|u|^2 + |v|^2)^{\frac{1}{2}} \leq (2)^{\frac{1}{2}} \left(\frac{|u|^p + |v|^p}{2} \right)^{\frac{1}{p}}.$$

Hence, $\|T^{-1}\| \leq 2^{\frac{1}{2}-\frac{1}{p}}$. But if $(u,v) = (1,i)$, then

$$\| T^{-1}(1,i) \|_1 = \frac{1}{2}|1+i| + \frac{1}{2}|1-i| = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$$

and $\|(1,i)\|_p = 2^{\frac{1}{p}}$. Therefore,

$$\|T^{-1}\| = 2^{\frac{1}{2}-\frac{1}{p}}. \tag{14}$$

From (13) and (14) we deduce that $\|T\|\|T^{-1}\| = 2^{\frac{1}{2}}$. □

This suggests that $d(l_1^2, l_p^2) = \sqrt{2}$ for all $p > 2$. Estimates for $d(l_1^2, l_p^2)$ are given in [5], but there the precise values in the complex case are not given.

Now, we will estimate the Banach-Mazur distance between the spaces l_p^2 and l_q^2.

Lemma 2. *Let* $1 \leq p \leq q < 2$, *or* $2 \leq p \leq q \leq \infty$, *then* $d(l_p^2, l_q^2) = 2^{\frac{1}{p}-\frac{1}{q}}$.

Proof. Let $\mathrm{id} : l_p^2 \to l_q^2$ denotes the identity operator. Then

$$d(l_p^2, l_q^2) \leq \| \mathrm{id} \| \| \mathrm{id}^{-1} \| = 2^{\frac{1}{p}-\frac{1}{q}}. \tag{15}$$

Now we need to show that

$$d(l_p^2, l_q^2) \geq 2^{\frac{1}{p}-\frac{1}{q}}.$$

First, if $1 \leq p \leq q < 2$, from Theorem 2 we have that

$$d(l_1^2, l_p^2) = 2^{1-\frac{1}{p}} \quad \text{and} \quad d(l_1^2, l_q^2) = 2^{1-\frac{1}{q}}.$$

Next using the property (2) one gets

$$2^{1-\frac{1}{q}} = d(l_1^2, l_q^2) \leq d(l_1^2, l_p^2)\, d(l_p^2, l_q^2)$$
$$2^{1-\frac{1}{q}} \leq d(l_p^2, l_q^2)\, 2^{1-\frac{1}{p}}.$$

It follows that

$$2^{\frac{1}{p}-\frac{1}{q}} \leq d(l_p^2, l_q^2).$$

We have proved $d(l_p^2, l_q^2) = 2^{\frac{1}{p}-\frac{1}{q}}$.

Second, if $2 \leq p \leq q \leq \infty$, from [5] we have

$$d(l_2^2, l_p^2) = 2^{\frac{1}{2}-\frac{1}{p}} \text{ and } d(l_2^2, l_q^2) = 2^{\frac{1}{2}-\frac{1}{q}}.$$

Then

$$2^{\frac{1}{2}-\frac{1}{q}} = d(l_2^2, l_q^2) \leq d(l_2^2, l_p^2)\ d(l_p^2, l_q^2)$$

$$2^{\frac{1}{2}-\frac{1}{q}} \leq 2^{\frac{1}{2}-\frac{1}{p}} d(l_q^2, l_p^2).$$

Hence $d(l_p^2, l_q^2) \geq 2^{\frac{1}{p}-\frac{1}{q}}$, if $2 \leq p \leq q \leq \infty$. From (15) we deduce that

$$d(l_p^2, l_q^2) = 2^{\frac{1}{p}-\frac{1}{q}}.$$

□

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References

[1] H. Cohen, *A Bound-two isomorphism between $C(X)$ Banach spaces*, Proc. Amer. Math. Soc., 50 (1975), 215–217.

[2] F. John, *Extreme problems with inqualities as subsidiary conditions* , Courant Anniversary volume, (1948), 187–204, Interscience, New York.

[3] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math., 8 (1958), 459–466.

[4] S. Al-mezel, *Generalisation of the Banach-Stone theorem*, M.Phil Thesis, Mathematics Department, University of Wales Swansea, United Kingdom, 2000.

[5] V. E. Mazaev, V. E. Gurari and M. E. Kadec, *On the distance between isomorphic L_p spaces of finite dimension* (Russian), Matematiceskii Sbornik 70 (112) 4 (1966), 481–489.

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