ON THE NON-COMMUTATIVE NEUTRIX PRODUCT OF THE DISTRIBUTIONS $\delta^{(r)}(x)$ AND $x^{-s}\ln^m|x|$

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Abstract. It is proved that the non-commutative neutrix product of the distributions $\delta^{(r)}(x)$ and $x^{-s}\ln^m|x|$ exists and

$$\delta^{(r)}(x) \circ x^{-s}\ln^m|x| = 0$$

for $r, m = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$.

In the following, we let $D$ be the space of infinitely differentiable functions with compact support and let $D'$ be the space of distributions defined on $D$.

We now let $\rho$ be a function in $D$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now $f$ is an arbitrary distribution in $D'$, we define

$$f_n(x) = (f \ast \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. Let $f$ and $g$ be distributions in $D'$ for which on the interval $(a, b)$, $f$ is the $k$-th derivative of a locally summable function $F$ in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of $f$ and $g$ is defined on the interval $(a, b)$ by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$
The distribution \( x^{-1} \ln^m |x| \) is defined by
\[
x^{-1} \ln^m |x| = \frac{(\ln^{m+1} |x|)'}{m + 1}
\]
for \( m = 0, 1, 2, \ldots \). The distribution \( x^{-s} \) is then defined by
\[
x^{-s} = \frac{(-1)^s (\ln |x|)^{(s)}}{(s - 1)!}
\]
for \( s = 1, 2, \ldots \) and the distribution \( x^{-s} \ln^m |x| \) is then defined inductively by the equation
\[
(x^{-s+1} \ln^m |x|)' = -(s - 1)x^{-s} \ln^m |x| + mx^{-s} \ln^{m-1} |x|
\]
for \( s, m = 0, 1, 2, \ldots \).

It follows that
\[
\langle x^{-2s+1} \ln^m |x|, \varphi(x) \rangle = \int_0^\infty x^{-2s+1} \ln^m x \left[ \varphi(x) - \varphi(-x) \right]
\]
It is obvious that if the product \( fg \) exists, then the neutrix product \( f \circ g \) exists and \( fg = f \circ g \).

The following theorem is easily proved.

**Theorem 1.** Let \( f \) and \( g \) be distributions in \( D' \) and suppose that the neutrix product \( f \circ g' \) (or \( f' \circ g \)) exists. Then the neutrix product \( f' \circ g \) (or \( f \circ g' \)) exists and

\[
(f \circ g)' = f \circ g' + f' \circ g. \tag{1}
\]

Using the neutrix product, the next theorem was proved in [3].

**Theorem 2.** The neutrix products \( \delta^{(r)}(x) \circ x^{-s} \) and \( x^{-s} \circ \delta^{(r)}(x) \) exist and

\[
x^{-s} \circ \delta^{(r)}(x) = \frac{(-1)^r r!}{(r+s)!} \delta^{(r+s)}(x) \tag{2}
\]

for \( r = 0, 1, 2, \ldots \) and \( s = 1, 2, \ldots \).

We first of all prove the following theorem.

**Theorem 3.** The neutrix product \( \delta^{(r)}(x) \circ \ln^m |x| \) exists and

\[
\delta^{(r)}(x) \circ \ln^m |x| = 2c_m \delta^{(r)}(x) \tag{4}
\]

for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \), where

\[
c_m = \int_0^1 \ln^m u \delta(u) \, du
\]

for \( m = 1, 2, \ldots \).

**Proof.** Putting

\[
(ln^m |x|)_n = \ln^m |x| * \delta_n(x) = \int_{-1/n}^{1/n} \ln^m |x-t| \delta_n(t) \, dt,
\]

we have

\[
\langle \delta^{(r)}(x), x^k (\ln^m |x|)_n \rangle = (-1)^r \int \langle \delta(x), \delta^{(r)}(x) \rangle \delta(x), x^k (\ln^m |x|)_n \rangle \, dx
\]

\[
= (-1)^r \sum_{i=0}^{k} \binom{r}{i} \frac{k!}{(k-i)!} \langle \delta(x), x^{k-i} (\ln^m |x|)_n \rangle \delta^{(r-k)}(t) \, dt
\]

\[
= (-1)^r k! \binom{r}{k} \int_{-1/n}^{1/n} \ln^m |t| \delta^{(r-k)}(t) \, dt
\]

\[
= (-1)^r k! n^{r-k} \binom{r}{k} \int_{-1}^{1} \ln^m |u/n| \rho^{(r-k)}(u) \, du,
\]
for \( k = 0, 1, 2, \ldots r - 1 \). It follows that
\[
N \lim_{n \to \infty} \langle \delta^{(r)}(x), x^k (\ln^m |x|)_n \rangle = 0,
\] (5)
for \( k = 0, 1, 2, \ldots, r - 1 \).

When \( k = r \), we have
\[
\langle \delta^{(r)}(x), x^r (\ln^m |x|)_n \rangle = (-1)^r r! \int_{-1}^{1} \ln^m |u/n| \rho(u) \, du
\]
and it follows that
\[
N \lim_{n \to \infty} \langle \delta^{(r)}(x), x^r (\ln^m |x|)_n \rangle = (-1)^r r! \int_{-1}^{1} \ln^m |u| \rho(u) \, du = 2(-1)^r r! c_m.
\] (6)

When \( k = r + 1 \), we have for an arbitrary infinitely differentiable function \( \psi \),
\[
\langle \delta^{(r)}(x), x^{r+1} (\ln^m |x|)_n \psi(x) \rangle = (-1)^r \langle \delta(x), [x^{r+1} (\ln^m |x|)_n \psi(x)]^{(r)} \rangle = 0.
\] (7)

If now \( \varphi \) is an arbitrary function in \( D \), we have
\[
\varphi(x) = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} x^{r+1},
\]
where \( 0 < \xi < 1 \). It follows that
\[
\langle \delta^{(r)}(x) (\ln^m |x|)_n, \varphi(x) \rangle = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} \langle \delta^{(r)}(x), x^k (\ln^m |x|)_n \rangle + \frac{1}{(r+1)!} \langle \delta^{(r)}(x), x^{r+1} (\ln^m |x|)_n \varphi^{(r+1)}(\xi x) \rangle
\]
and it now follows from equations (5) to (7) that
\[
N \lim_{n \to \infty} \langle \delta^{(r)}(x) (\ln^m |x|)_n, \varphi(x) \rangle = 2(-1)^r c_m \varphi^{(r)}(0)
= 2c_m \langle \delta^{(r)}(x), \varphi(x) \rangle,
\]
proving equation (4) for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \). This completes the proof of the theorem. \( \square \)

We now prove...

**Theorem 4.** The neutrix product \( \delta^{(r)}(x) \circ (x^{-s} \ln^m |x|) \) exists and
\[
\delta^{(r)}(x) \circ (x^{-s} \ln^m |x|) = 0,
\] (8)
for \( r = 0, 1, 2, \ldots \) and \( s, m = 1, 2, \ldots \).
Proof. We first of all prove that

$$\delta^{(r)}(x) \circ (x^{-1} \ln^m |x|) = 0$$  \hspace{1cm} (9)

for \( r, m = 0, 1, 2, \ldots \). Differentiating the equation

$$\delta^{(r)}(x) \circ \ln^{m+1} |x| = 2c_{m+1}\delta^{(r)}(x)$$

and using Theorem 3, we have

$$\delta^{(r+1)}(x) \circ \ln^{m+1} |x| + (m + 1)\delta^{(r)}(x) \circ (x^{-1} \ln^m |x|) = 2c_{m+1}\delta^{(r+1)}(x)$$

and equation (9) follows.

Next, suppose that \( \delta^{(r)}(x) \circ (x^{-2} \ln^m |x|) \) exists and

$$\delta^{(r)}(x) \circ (x^{-2} \ln^m |x|) = 0$$  \hspace{1cm} (10)

for \( r = 0, 1, 2, \ldots \) and some \( m \). This is true when \( m = 0 \). Differentiating the equation

$$\delta^{(r)}(x) \circ (x^{-2} \ln^m |x|) = 0$$

we get

$$\delta^{(r+1)}(x) \circ (x^{-1} \ln^{m+1} |x|) - \delta^{(r)}(x) \circ (x^{-2} \ln^{m+1} |x|) + (m + 1)\delta^{(r)}(x)$$

$$\circ (x^{-2} \ln^m |x|) = -\delta^{(r)}(x) \circ (x^{-2} \ln^{m+1} |x|) = 0$$

on using equation (9) and our assumption. Equation (10) follows by induction for \( r, m = 0, 1, 2, \ldots \).

We have therefore proved that

$$\delta^{(r)}(x) \circ (x^{-i} \ln^m |x|) = 0$$  \hspace{1cm} (11)

for \( i = 1, 2 \) and \( r, m = 0, 1, 2, \ldots \).

We can now prove similarly that

$$\delta^{(r)}(x) \circ (x^{-3} \ln^m |x|) = 0$$

for \( r, m = 0, 1, 2, \ldots \) and so on.

In general, suppose that equation (11) holds for \( i = 1, 2, \ldots, s \) and some \( s \) and \( r, m = 0, 1, 2, \ldots \). This is true when \( s = 1 \) or \( 2 \). Then suppose that

$$\delta^{(r)}(x) \circ (x^{-s-1} \ln^m |x|) = 0,$$  \hspace{1cm} (12)

for \( r = 0, 1, 2, \ldots \) and some \( m \). This is true when \( m = 0 \). From our assumption on equation (11) with \( i = s \) and \( m + 1 \) for \( m \), we have

$$\delta^{(r)}(x) \circ (x^{-s} \ln^{m+1} |x|) = 0.$$  \hspace{1cm} (13)
Differentiating equation (13), we have
\[
\delta^{(r+1)}(x) \circ (x^{-s} \ln^{m+1} |x|) - s\delta^{(r)}(x) \circ (x^{-s-1} \ln^{m+1} |x|) + (m+1)\delta^{(r)}(x) \circ (x^{-s-1} \ln^m |x|) = 0
\]
and it follows from our assumptions that
\[
\delta^{(r)}(x) \circ (x^{-s-1} \ln^{m+1} |x|) = 0.
\]
Equation (12) follows by induction for \( r, m = 0, 1, 2, \ldots \).

Equation (11) therefore holds \( i = 1, 2, \ldots, s + 1 \) and \( r, m = 0, 1, 2, \ldots \) and so follows by induction for \( r, m = 0, 1, 2, \ldots \) and \( s = 1, 2, \ldots \). This completes the proof of the theorem. \( \square \)

In the next theorem we put
\[
c_{r,m} = \int_0^1 u^r \ln^m u \rho^{(r)}(u) \, du
\]
for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \).

Integrating by parts, we have
\[
c_{r,m} = -\int_0^1 [mu^{r-1} \ln^{m-1} u + ru^{r-1} \ln^m u] \rho^{(r-1)}(u) \, du
\]
\[
= -mc_{r-1,m-1} - r c_{r-1,m},
\]
\[
c_{1,m} = -mc_{0,m-1} - c_{0,m},
\]
\[
= -mc_{m-1} - c_m.
\]

It follows by induction that
\[
c_{r,m} = (-1)^r r! c_m - mr! \sum_{i=1}^{r} \frac{(-1)^i c_{r-i,m-1}}{(r-i+1)!}
\]
\[
= (-1)^r r! c_m + (-1)^r mr! c_{m-1} + mr! \sum_{i=1}^{r-1} \frac{(-1)^i c_{r-i,m-1}}{(r-i+1)!}
\]
and so each \( c_{r,m} \) can be expressed as a linear sum of \( c_1, c_2, \ldots, c_m \) for \( r, m = 1, 2, \ldots \).

**Theorem 5.** The neutrix product \( \ln^m |x| \circ \delta^{(r)}(x) \) exists and
\[
\ln^m |x| \circ \delta^{(r)}(x) = \frac{2(-1)^r c_{r,m} \delta^{(r)}(x)}{r!} \quad (14)
\]
for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \).
Proof. We have

$$\langle \ln^m |x|, x^k \delta_n^{(r)}(x) \rangle = \int_{-1/n}^{1/n} \ln^m |x| x^k \delta_n^{(r)}(x) \, dx$$

$$= n^{r-k} \int_{-1}^{1} \ln^m |u/n| u^k \rho^{(r)}(u) \, du$$

for $k = 0, 1, 2, \ldots$. It follows that

$$\text{N} - \lim_{n \to \infty} \langle \ln^m |x|, x^k \delta_n^{(r)}(x) \rangle = 0 \quad (15)$$

for $k = 0, 1, 2, \ldots, r - 1$.

When $k = r$, we have

$$\langle \ln^m |x|, x^r \delta_n^{(r)}(x) \rangle = \int_{-1/n}^{1/n} u^r \ln^m |u/n| \rho^{(r)}(u) \, du$$

and it follows that

$$\text{N} - \lim_{n \to \infty} \langle \ln^m |x|, x^r \delta_n^{(r)}(x) \rangle = \int_{-1}^{1} u^r \ln^m |u| \rho^{(r)}(u) \, du = 2c_{r,m} \quad (16)$$

When $k = r + 1$, we have for an arbitrary infinitely differentiable function $\psi$,

$$\langle \ln^m |x|, x^{r+1} \delta_n^{(r)}(x) \psi(x) \rangle = n^{-1} \int_{-1/n}^{1/n} u^{r+1} \ln^m |u/n| \rho^{(r)}(u) \psi(u/n) \, du$$

$$= O(n^{-1} \ln^m n). \quad (17)$$

If now $\varphi$ is an arbitrary function in $\mathcal{D}$, we have

$$\varphi(x) = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(r+1)}(\xi x)}{(r + 1)!} x^{r+1},$$

where $0 < \xi < 1$. It follows that

$$\langle \ln^m |x|, \delta_n^{(r)}(x) \varphi(x) \rangle = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} \langle \ln^m |x|, x^k \delta_n^{(r)}(x) \rangle$$

$$+ \frac{1}{(r + 1)!} \langle \ln^m |x|, x^{r+1} \delta_n^{(r)}(x), \varphi^{(r+1)}(\xi x) \rangle$$

and it now follows from equations (15) to (17) that

$$\text{N} - \lim_{n \to \infty} \langle \ln^m |x|, \delta_n^{(r)}(x) \varphi(x) \rangle = \frac{2c_{r,m}}{r!} \varphi^{(r)}(0)$$

$$= \frac{2(-1)^r c_{r,m}}{r!} \langle \delta^{(r)}(x), \varphi(x) \rangle,$$
proving equation (14) for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \). This completes the proof of the theorem. \( \square \)

We finally prove the following generalization of equation (3).

**Theorem 6.** The neutrix product \((x^{-s} \ln^m |x|) \circ \delta^{(r)}(x)\) exists for \( r = 0, 1, \ldots \) and \( s, m = 1, 2, \ldots \).

In particular,

\[
(x^{-1} \ln^m |x|) \circ \delta^{(r)}(x) = \frac{2(-1)^{r+1} c_{r,m}}{(r+1)!} \delta^{(r+1)}(x)
\]

for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \).

**Proof.** We first of all prove equation (18). We have

\[
(m + 1)\langle x^{-1} \ln^m |x|, x^k \delta_n^{(r)}(x) \rangle = - \int_{-1/n}^{1/n} \ln^{m+1} |x|[x^k \delta_n^{(r)}(x)]' \, dx
\]

\[= -n^{-k+1} \int_{-1}^{1} \left[ \ln |u| - \ln n \right]^{m+1} [u^k \rho^{(r)}(u)]' \, du
\]

on making the substitution \( nx = u \). It follows that

\[
N\lim_{n \to \infty} \langle x^{-1} \ln^m |x|, x^k \delta_n^{(r)}(x) \rangle = 0
\]

for \( k = 0, 1, 2, \ldots, r, \)

\[
N\lim_{n \to \infty} \langle x^{-1} \ln^m |x|, x^{r+1} \delta_n^{(r+1)}(x) \rangle = -(m + 1)^{-1} \int_{-1}^{1} \ln^{m+1} |u|[u^{r+1} \rho^{(r)}(u)]' \, du
\]

\[= \int_{-1}^{1} u^r \ln^m |u| \rho^{(r)}(u) \, du
\]

\[= 2c_{r,m}
\]

when \( k = r + 1 \) and

\[
N\lim_{n \to \infty} \langle x^{-1} \ln^m |x|, x^{r+1} \delta_n^{(r)}(x) \psi(x) \rangle = 0
\]

for any infinitely differentiable function \( \psi(x) \), when \( k = r + 2 \).

If now \( \varphi \) is an arbitrary function in \( D \), we have

\[
\varphi(x) = \sum_{k=0}^{r+1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(r+1)}(\xi x)}{(r+2)!} x^{r+2},
\]

where \( \xi \) is some constant in \( (0, 1) \).
where \( 0 < \xi < 1 \). It follows that

\[
\langle x^{-1} \ln^m |x|, \delta_n^{(r)}(x) \phi(x) \rangle = \sum_{k=0}^{r+1} \frac{\varphi^{(k)}(0)}{k!} \langle x^{-1} \ln^m |x|, x^k \delta_n^{(r)}(x) \rangle + \frac{1}{(r+2)!} \langle \ln^m |x|, x^{r+2} \delta_n^{(r+1)}(x), \varphi^{(r+2)}(\xi x) \rangle
\]

and it now follows from equations (20) to (22) that

\[
\text{N-lim}_{n \to \infty} \langle x^{-1} \ln^m |x|, \delta_n^{(r)}(x) \phi(x) \rangle = \frac{2c_{r,m}}{(r+1)!} \varphi^{(r+1)}(0) = \frac{2(-1)^{r+1} c_{r,m}}{(r+1)!} \delta^{(r+1)}(x), \phi(x),
\]

proving equation (18) for \( r = 0, 1, 2, \ldots \) and \( m = 1, 2, \ldots \).

Next, suppose that \( (x^{-2} \ln^m |x|) \circ \delta^{(r)}(x) \) exists and

\[
(x^{-2} \ln^m |x|) \circ \delta^{(r)}(x) = c_{r,2m} \delta^{(r+2)}(x)
\]

for \( r = 0, 1, 2, \ldots \) and some \( m \). This is true when \( m = 0 \) with \( c_{r,2,0} = r!/(r+2)! \). Differentiating the equation

\[
(x^{-1} \ln^{m+1} |x|) \circ \delta^{(r)}(x) = \frac{2(-1)^{r+1} c_{r,m}}{(r+1)!} \delta^{(r+1)}(x)
\]

we get

\[
(x^{-1} \ln^{m+1} |x|) \circ \delta^{(r+1)}(x) - (x^{-2} \ln^{m+1} |x|) \circ \delta^{(r)}(x) + (m+1)(x^{-2} \ln^m |x|) \circ \delta^{(r)}(x) = \frac{2(-1)^{r+1} c_{r,m}}{(r+1)!} \delta^{(r+2)}(x)
\]

\[
= \frac{2(-1)^{r+2} c_{r+1,2m}}{(r+2)!} \delta^{(r+2)}(x) - (x^{-2} \ln^{m+1} |x|) \circ \delta^{(r)}(x)
\]

\[
+ (m+1) c_{r,2m} \delta^{(r+2)}(x)
\]

on using equation (18) and our assumption. This proves the existence of \( (x^{-2} \ln^{m+1} |x|) \circ \delta^{(r)}(x) \) and

\[
(x^{-2} \ln^{m+1} |x|) \circ \delta^{(r)}(x) = c_{r,2m+1} \delta^{(r+2)}(x)
\]

with

\[
c_{r,2m+1} = \frac{2(-1)^r c_{r,m}}{(r+1)!} + \frac{2(-1)^r c_{r+1,m}}{(r+2)!} + (m+1) c_{r,2m}.
\]

Therefore \( (x^{-2} \ln^{m+1} |x|) \circ \delta^{(r)}(x) \) exists and equation (23) follows by induction for \( r, m = 0, 1, 2, \ldots \).
We have therefore proved that
\[(x^{-i} \ln^m |x|) \circ \delta^{(r)}(x) = c_{r,i,m} \delta^{(r+i)}(x)\] (24)
for \(i = 1, 2\) and \(r, m = 0, 1, 2, \ldots\)

We can now prove similarly that
\[(x^{-3} \ln^m |x|) \circ \delta^{(r)}(x) = c_{r,3,m} \delta^{(r+3)}(x)\]
for \(r, m = 0, 1, 2, \ldots\) and so on.

In general, suppose that equation (24) holds for \(i = 1, 2, \ldots, s\) and some \(s\) and \(r, m = 0, 1, 2, \ldots\). This is true when \(s = 1\) or \(2\). Then suppose that \((x^{-s-1} \ln^m |x|) \circ \delta^{(r)}(x)\) exists and
\[(x^{-s-1} \ln^m |x|) \circ \delta^{(r)}(x) = c_{r,s+1,m} \delta^{(r+s+1)}(x),\] (25)
for \(r = 0, 1, 2, \ldots\) and some \(m\). This is true with
\[c_{r,s+1,0} = \frac{(-1)^{s+1} r!}{(r + s + 1)!},\]
when \(m = 0\). From our assumption on equation (24) with \(i = s\) and \(m + 1\) for \(m\), we have
\[(x^{-s} \ln^{m+1} |x|) \circ \delta^{(r)}(x) = c_{r,s,m+1} \delta^{(r+s)}(x).\] (26)

Differentiating equation (26), we have
\[(x^{-s} \ln^{m+1} |x|) \circ \delta^{(r+1)}(x) = x^{-s-1} \ln^{m+1} |x| \circ \delta^{(r)}(x) + (m + 1)(x^{-s-1} \ln^m |x|) \circ \delta^{(r)}(x)\]
\[= [c_{r+1,s,m+1} + (m + 1)c_{r,s+1,m}] \delta^{(r+s+1)}(x) - s(x^{-s-1} \ln^{m+1} |x|) \circ \delta^{(r)}(x)\]
\[= c_{r,s+1,m+1} \delta^{(r+s+1)}(x),\]
from our assumptions and it follows that
\[(x^{-s-1} \ln^{m+1} |x|) \circ \delta^{(r)}(x) = s^{-1}[c_{r+1,s,m+1} + (m + 1)c_{r,s+1,m} - c_{r,s,m+1}] \delta^{(r+s+1)}(x)\]
\[= c_{r,s+1,m+1} \delta^{(r+s+1)}(x).\]
Equation (25) therefore holds for \(m + 1\), with
\[c_{r,s+1,m+1} = s^{-1}[c_{r+1,s,m+1} + (m + 1)c_{r,s+1,m} - c_{r,s,m+1}],\]
and so follows by induction for \(r, m = 0, 1, 2, \ldots\). Then equation (24) holds for \(i = 1, 2, \ldots, s+1\). Equation (24) follows by induction for \(r, m = 0, 1, 2, \ldots\) and \(s = 1, 2, \ldots\). This completes the proof of the theorem. \(\square\)
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REFERENCES