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ON THE NON-COMMUTATIVE NEUTRIX PRODUCT OF THE DISTRIBUTIONS $\delta^{(r)}(x)$ AND $x^{-s}\ln^m|x|$

BRIAN FISHER, INCI EGE AND EMIN ÖZÇAḠ

Abstract. It is proved that the non-commutative neutrix product of the distributions $\delta^{(r)}(x)$ and $x^{-s}\ln^m|x|$ exists and

$$\delta^{(r)}(x) \circ x^{-s}\ln^m|x| = 0$$

for $r, m = 0, 1, 2, \dots$ and $s = 1, 2, \dots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

We now let ρ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

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The distribution $x^{-1}\ln^m|x|$ is defined by

$$x^{-1}\ln^m|x| = \frac{(\ln^{m+1}|x|)'}{m+1}$$

for $m = 0, 1, 2, \dots$. The distribution x^{-s} is then defined by

$$x^{-s} = \frac{(-1)^s(\ln|x|)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \dots$ and the distribution $x^{-s}\ln^m|x|$ is then defined inductively by the equation

$$(x^{-s+1}\ln^m|x|)' = -(s-1)x^{-s}\ln^m|x| + mx^{-s}\ln^{m-1}|x|$$

for $s, m = 0, 1, 2, \dots$.

It follows that

$$\langle x^{-2s+1}\ln^m|x|, \varphi(x)\rangle = \int_0^\infty x^{-2s+1}\ln^m x\Big[\varphi(x) - \varphi(-x) - 2\sum_{i=0}^{s-2}\frac{\varphi^{(2i+1)}(0)}{(2i+1)!}x^{2i+1}\Big]\,dx,$$

$$\langle x^{-2s}\ln^m|x|, \varphi(x)\rangle = \int_0^\infty x^{-2s}\ln^m x\Big[\varphi(x) + \varphi(-x) - 2\sum_{i=0}^{s-1}\frac{\varphi^{(2i)}(0)}{(2i)!}x^{2i}\Big]\,dx$$

for $s = 1, 2, \dots$ and $m = 0, 1, 2, \dots$, where φ is an arbitrary function in $\mathcal{D}$, see Gel'fand and Shilov [4].

The next definition for the non-commutative neutrix product of two distributions was given in [3] and generalizes Definition 1.

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a, b) if*

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle f(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a, b), where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda\ln^{r-1}n, \quad \ln^r n: \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It is obvious that if the product fg exists, then the neutrix product $f \circ g$ exists and $fg = f \circ g$.

The following theorem is easily proved.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix product $f \circ g'$ (or $f' \circ g$) exists. Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists and*

$$(f \circ g)' = f \circ g' + f' \circ g. \tag{1}$$

Using the neutrix product, the next theorem was proved in [3].

Theorem 2. *The neutrix products $\delta^{(r)}(x) \circ x^{-s}$ and $x^{-s} \circ \delta^{(r)}(x)$ exist and*

$$\delta^{(r)}(x) \circ x^{-s} = 0, \tag{2}$$

$$x^{-s} \circ \delta^{(r)}(x) = \frac{(-1)^s r!}{(r+s)!}\delta^{(r+s)}(x) \tag{3}$$

for $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$.

We first of all prove the following theorem.

Theorem 3. *The neutrix product $\delta^{(r)}(x) \circ \ln^m |x|$ exists and*

$$\delta^{(r)}(x) \circ \ln^m |x| = 2c_m \delta^{(r)}(x), \tag{4}$$

for $r = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$, where

$$c_m = \int_0^1 \ln^m u \delta(u)\, du$$

for $m = 1, 2, \ldots$.

Proof. Putting

$$(\ln^m |x|)_n = \ln^m |x| * \delta_n(x) = \int_{-1/n}^{1/n} \ln^m |x - t| \delta_n(t)\, dt,$$

we have

$$\begin{aligned}
\langle \delta^{(r)}(x), x^k (\ln^m |x|)_n \rangle &= (-1)^r \langle \delta(x), [x^k (\ln^m |x|)_n]^{(r)} \rangle \\
&= (-1)^r \sum_{i=0}^{k} \binom{r}{i} \frac{k!}{(k-i)!} \langle \delta(x), x^{k-i} [(\ln^m |x|)_n]^{(r-i)} \rangle \\
&= (-1)^r k! \binom{r}{k} \int_{-1/n}^{1/n} \ln^m |t| \delta_n^{(r-k)}(t)\, dt \\
&= (-1)^k k! n^{r-k} \binom{r}{k} \int_{-1}^{1} \ln^m |u/n| \rho^{(r-k)}(u)\, du,
\end{aligned}$$

for $k = 0, 1, 2, \dots r-1$. It follows that

$$\operatorname*{N-lim}_{n\to\infty}\langle \delta^{(r)}(x), x^k(\ln^m|x|)_n\rangle = 0, \tag{5}$$

for $k = 0, 1, 2, \dots, r-1$.

When $k = r$, we have

$$\langle \delta^{(r)}(x), x^r(\ln^m|x|)_n\rangle = (-1)^r r! \int_{-1}^{1} \ln^m|u/n|\rho(u)\,du$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\langle \delta^{(r)}(x), x^r(\ln^m|x|)_n\rangle = (-1)^r r! \int_{-1}^{1} \ln^m|u|\rho(u)\,du = 2(-1)^r r! c_m. \tag{6}$$

When $k = r+1$, we have for an arbitrary infinitely differentiable function ψ,

$$\begin{aligned}\langle \delta^{(r)}(x), x^{r+1}(\ln^m|x|)_n\psi(x)\rangle &= (-1)^r\langle \delta(x), [x^{r+1}(\ln^m|x|)_n\psi(x)]^{(r)}\rangle \\ &= 0. \end{aligned} \tag{7}$$

If now φ is an arbitrary function in $\mathcal{D}$, we have

$$\varphi(x) = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!}x^k + \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!}x^{r+1},$$

where $0 < \xi < 1$. It follows that

$$\begin{aligned}\langle \delta^{(r)}(x)(\ln^m|x|)_n, \varphi(x)\rangle = &\sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!}\langle \delta^{(r)}(x), x^k(\ln^m|x|)_n\rangle \\ &+ \frac{1}{(r+1)!}\langle \delta^{(r)}(x), x^{r+1}(\ln^m|x|)_n\varphi^{(r+1)}(\xi x)\rangle\end{aligned}$$

and it now follows from equations (5) to (7) that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle \delta^{(r)}(x)(\ln^m|x|)_n, \varphi(x)\rangle &= 2(-1)^r c_m \varphi^{(r)}(0) \\ &= 2c_m\langle \delta^{(r)}(x), \varphi(x)\rangle,\end{aligned}$$

proving equation (4) for $r = 0, 1, 2, \dots$ and $m = 1, 2, \dots$. This completes the proof of the theorem. □

We now prove

Theorem 4. *The neutrix product $\delta^{(r)}(x) \circ (x^{-s}\ln^m|x|)$ exists and*

$$\delta^{(r)}(x) \circ (x^{-s}\ln^m|x|) = 0, \tag{8}$$

for $r = 0, 1, 2, \dots$ and $s, m = 1, 2, \dots$.

Proof. We first of all prove that

$$\delta^{(r)}(x) \circ (x^{-1} \ln^m |x|) = 0 \tag{9}$$

for $r, m = 0, 1, 2, \ldots$.

Differentiating the equation

$$\delta^{(r)}(x) \circ \ln^{m+1} |x| = 2c_{m+1}\delta^{(r)}(x)$$

and using Theorem 3, we have

$$\delta^{(r+1)}(x) \circ \ln^{m+1} |x| + (m+1)\delta^{(r)}(x) \circ (x^{-1} \ln^m |x|) = 2c_{m+1}\delta^{(r+1)}(x)$$

and equation (9) follows.

Next, suppose that $\delta^{(r)}(x) \circ (x^{-2} \ln^m |x|)$ exists and

$$\delta^{(r)}(x) \circ (x^{-2} \ln^m |x|) = 0 \tag{10}$$

for $r = 0, 1, 2, \ldots$ and some m. This is true when $m = 0$. Differentiating the equation

$$\delta^{(r)}(x) \circ (x^{-1} \ln^{m+1} |x|) = 0$$

we get

$$\delta^{(r+1)}(x) \circ (x^{-1} \ln^{m+1} |x|) - \delta^{(r)}(x) \circ (x^{-2} \ln^{m+1} |x|) + (m+1)\delta^{(r)}(x) \circ (x^{-2} \ln^m |x|) = -\delta^{(r)}(x) \circ (x^{-2} \ln^{m+1} |x|) = 0$$

on using equation (9) and our assumption. Equation (10) follows by induction for $r, m = 0, 1, 2, \ldots$.

We have therefore proved that

$$\delta^{(r)}(x) \circ (x^{-i} \ln^m |x|) = 0 \tag{11}$$

for $i = 1, 2$ and $r, m = 0, 1, 2, \ldots$.

We can now prove similarly that

$$\delta^{(r)}(x) \circ (x^{-3} \ln^m |x|) = 0$$

for $r, m = 0, 1, 2, \ldots$ and so on.

In general, suppose that equation (11) holds for $i = 1, 2, \ldots, s$ and some s and $r, m = 0, 1, 2, \ldots$. This is true when $s = 1$ or 2. Then suppose that

$$\delta^{(r)}(x) \circ (x^{-s-1} \ln^m |x|) = 0, \tag{12}$$

for $r = 0, 1, 2, \ldots$ and some m. This is true when $m = 0$. From our assumption on equation (11) with $i = s$ and $m + 1$ for m, we have

$$\delta^{(r)}(x) \circ (x^{-s} \ln^{m+1} |x|) = 0. \tag{13}$$

Differentiating equation (13), we have

$$\delta^{(r+1)}(x) \circ (x^{-s} \ln^{m+1}|x|) - s\delta^{(r)}(x) \circ (x^{-s-1} \ln^{m+1}|x|) \\ + (m+1)\delta^{(r)}(x) \circ (x^{-s-1} \ln^{m}|x|) = 0$$

and it follows from our assumptions that

$$\delta^{(r)}(x) \circ (x^{-s-1} \ln^{m+1}|x|) = 0.$$

Equation (12) follows by induction for $r, m = 0, 1, 2, \dots$. Equation (11) therefore holds $i = 1, 2, \dots, s+1$ and $r, m = 0, 1, 2, \dots$ and so follows by induction for $r, m = 0, 1, 2, \dots$ and $s = 1, 2, \dots$. This completes the proof of the theorem. □

In the next theorem we put

$$c_{r,m} = \int_0^1 u^r \ln^m u \rho^{(r)}(u)\, du$$

for $r = 0, 1, 2, \dots$ and $m = 1, 2, \dots$.

Integrating by parts, we have

$$\begin{aligned} c_{r,m} &= -\int_0^1 [mu^{r-1} \ln^{m-1} u + ru^{r-1} \ln^m u]\rho^{(r-1)}(u)\, du \\ &= -mc_{r-1,m-1} - rc_{r-1,m}, \\ c_{1,m} &= -mc_{0,m-1} - c_{0,m}, \\ &= -mc_{m-1} - c_m. \end{aligned}$$

It follows by induction that

$$\begin{aligned} c_{r,m} &= -(-1)^r r! c_m - mr! \sum_{i=1}^{r} \frac{(-1)^i c_{r-i,m-1}}{(r-i+1)!} \\ &= (-1)^r r! c_m + (-1)^r mr! c_{m-1} + mr! \sum_{i=1}^{r-1} \frac{(-1)^i c_{r-i,m-1}}{(r-i+1)!} \end{aligned}$$

and so each $c_{r,m}$ can be expressed as a linear sum of $c_1, c_2, \dots, c_m$ for $r, m = 1, 2, \dots$.

Theorem 5. *The neutrix product* $\ln^m|x| \circ \delta^{(r)}(x)$ *exists and*

$$\ln^m|x| \circ \delta^{(r)}(x) = \frac{2(-1)^r c_{r,m}}{r!}\delta^{(r)}(x) \tag{14}$$

for $r = 0, 1, 2, \dots$ *and* $m = 1, 2, \dots$.

Proof. We have

$$\begin{aligned}\langle \ln^m |x|, x^k \delta_n^{(r)}(x)\rangle &= \int_{-1/n}^{1/n} \ln^m |x| x^k \delta_n^{(r)}(x)\, dx \\ &= n^{r-k} \int_{-1}^{1} \ln^m |u/n| u^k \rho^{(r)}(u)\, du\end{aligned}$$

for $k = 0, 1, 2, \ldots$. It follows that

$$\underset{n\to\infty}{\text{N}-\lim} \langle \ln^m |x|, x^k \delta_n^{(r)}(x)\rangle = 0, \tag{15}$$

for $k = 0, 1, 2, \ldots, r-1$.

When $k = r$, we have

$$\langle \ln^m |x|, x^r \delta_n^{(r)}(x)\rangle = \int_{-1}^{1} u^r \ln^m |u/n| \rho^{(r)}(u)\, du$$

and it follows that

$$\underset{n\to\infty}{\text{N}-\lim} \langle \ln^m |x|, x^r \delta_n^{(r)}(x)\rangle = \int_{-1}^{1} u^r \ln^m |u| \rho^{(r)}(u)\, du = 2c_{r,m}. \tag{16}$$

When $k = r+1$, we have for an arbitrary infinitely differentiable function ψ,

$$\begin{aligned}\langle \ln^m |x|, x^{r+1} \delta_n^{(r)}(x)\psi(x)\rangle &= n^{-1} \int_{-1}^{1} u^{r+1} \ln^m |u/n| \rho^{(r)}(u)\psi(u/n)\, du \\ &= O(n^{-1} \ln^m n).\end{aligned} \tag{17}$$

If now φ is an arbitrary function in $\mathcal{D}$, we have

$$\varphi(x) = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} x^{r+1},$$

where $0 < \xi < 1$. It follows that

$$\begin{aligned}\langle \ln^m |x|, \delta_n^{(r)}(x)\varphi(x)\rangle = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} \langle \ln^m |x|, x^k \delta_n^{(r)}(x)\rangle \\ + \frac{1}{(r+1)!} \langle \ln^m |x|, x^{r+1} \delta_n^{(r)}(x), \varphi^{(r+1)}(\xi x)\rangle\end{aligned}$$

and it now follows from equations (15) to (17) that

$$\begin{aligned}\underset{n\to\infty}{\text{N}-\lim} \langle \ln^m |x|, \delta_n^{(r)}(x)\varphi(x)\rangle &= \frac{2c_{r,m}}{r!} \varphi^{(r)}(0) \\ &= \frac{2(-1)^r c_{r,m}}{r!} \langle \delta^{(r)}(x), \varphi(x)\rangle,\end{aligned}$$

proving equation (14) for $r = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$. This completes the proof of the theorem. □

We finally prove the following generalization of equation (3).

Theorem 6. *The neutrix product* $(x^{-s} \ln^m |x|) \circ \delta^{(r)}(x)$ *exists for* $r = 0, 1, \ldots$ *and* $s, m = 1, 2, \ldots$.

In particular,

$$(x^{-1} \ln^m |x|) \circ \delta^{(r)}(x) = \frac{2(-1)^{r+1} c_{r,m}}{(r+1)!} \delta^{(r+1)}(x) \tag{18}$$

for $r = 0, 1, 2, \ldots$ *and* $m = 1, 2, \ldots$.

Proof. We first of all prove equation (18). We have

$$\begin{aligned}(m+1)\langle x^{-1} \ln^m |x|, x^k \delta_n^{(r)}(x)\rangle &= -\int_{-1/n}^{1/n} \ln^{m+1} |x| [x^k \delta_n^{(r)}(x)]' \, dx \\ &= -n^{r-k+1} \int_{-1}^{1} [\ln |u| - \ln n]^{m+1} [u^k \rho^{(r)}(u)]' \, du \end{aligned} \tag{19}$$

on making the substitution $nx = u$. It follows that

$$\underset{n\to\infty}{\text{N}-\lim} \langle x^{-1} \ln^m |x|, x^k \delta_n^{(r)}(x)\rangle = 0 \tag{20}$$

for $k = 0, 1, 2, \ldots, r$,

$$\begin{aligned}\underset{n\to\infty}{\text{N}-\lim} \langle x^{-1} \ln^m |x|, x^{r+1} \delta_n^{(r+1)}(x)\rangle &= -(m+1)^{-1} \int_{-1}^{1} \ln^{m+1} |u| [u^{r+1} \rho^{(r)}(u)]' \, du \\ &= \int_{-1}^{1} u^r \ln^m |u| \rho^{(r)}(u) \, du \\ &= 2c_{r,m} \end{aligned} \tag{21}$$

when $k = r + 1$ and

$$\underset{n\to\infty}{\text{N}-\lim} \langle x^{-1} \ln^m |x|, x^{r+1} \delta_n^{(r)}(x) \psi(x)\rangle = 0 \tag{22}$$

for any infinitely differentiable function $\psi(x)$, when $k = r + 2$.

If now φ is an arbitrary function in $\mathcal{D}$, we have

$$\varphi(x) = \sum_{k=0}^{r+1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(r+1)}(\xi x)}{(r+2)!} x^{r+2},$$

where $0 < \xi < 1$. It follows that

$$\langle x^{-1}\ln^m|x|, \delta_n^{(r)}(x)\varphi(x)\rangle = \sum_{k=0}^{r+1}\frac{\varphi^{(k)}(0)}{k!}\langle x^{-1}\ln^m|x|, x^k\delta_n^{(r)}(x)\rangle + \frac{1}{(r+2)!}\langle \ln^m|x|, x^{r+2}\delta_n^{(r+1)}(x), \varphi^{(r+2)}(\xi x)\rangle$$

and it now follows from equations (20) to (22) that

$$\operatorname*{N-lim}_{n\to\infty}\langle x^{-1}\ln^m|x|, \delta_n^{(r)}(x)\varphi(x)\rangle = \frac{2c_{r,m}}{(r+1)!}\varphi^{(r+1)}(0) = \frac{2(-1)^{r+1}c_{r,m}}{(r+1)!}\langle\delta^{(r+1)}(x), \varphi(x)\rangle,$$

proving equation (18) for $r = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$.

Next, suppose that $(x^{-2}\ln^m|x|)\circ\delta^{(r)}(x)$ exists and

$$(x^{-2}\ln^m|x|)\circ\delta^{(r)}(x) = c_{r,2,m}\delta^{(r+2)}(x) \tag{23}$$

for $r = 0, 1, 2, \ldots$ and some m. This is true when $m = 0$ with $c_{r,2,0} = r!/(r+2)!$. Differentiating the equation

$$(x^{-1}\ln^{m+1}|x|)\circ\delta^{(r)}(x) = \frac{2(-1)^{r+1}c_{r,m}}{(r+1)!}\delta^{(r+1)}(x)$$

we get

$$\begin{aligned}(x^{-1}\ln^{m+1}|x|)\circ\delta^{(r+1)}(x) &- (x^{-2}\ln^{m+1}|x|)\circ\delta^{(r)}(x) \\ &+ (m+1)(x^{-2}\ln^m|x|)\circ\delta^{(r)}(x) = \frac{2(-1)^{r+1}c_{r,m}}{(r+1)!}\delta^{(r+2)}(x) \\ = \frac{2(-1)^{r+2}c_{r+1,m}}{(r+2)!}\delta^{(r+2)}(x) &- (x^{-2}\ln^{m+1}|x|)\circ\delta^{(r)}(x) \\ &+ (m+1)c_{r,2,m}\delta^{(r+2)}(x)\end{aligned}$$

on using equation (18) and our assumption. This proves the existence of $(x^{-2}\ln^{m+1}|x|)\circ\delta^{(r)}(x)$ and

$$(x^{-2}\ln^{m+1}|x|)\circ\delta^{(r)}(x) = c_{r,2,m+1}\delta^{(r+2)}(x)$$

with

$$c_{r,2,m+1} = \frac{2(-1)^r c_{r,m}}{(r+1)!} + \frac{2(-1)^r c_{r+1,m}}{(r+2)!} + (m+1)c_{r,2,m}.$$

Therefore $(x^{-2}\ln^{m+1}|x|)\circ\delta^{(r)}(x)$ exists and equation (23) follows by induction for $r, m = 0, 1, 2, \ldots$.

We have therefore proved that

$$(x^{-i}\ln^m|x|)\circ\delta^{(r)}(x) = c_{r,i,m}\delta^{(r+i)}(x) \tag{24}$$

for $i = 1, 2$ and $r, m = 0, 1, 2, \ldots$.

We can now prove similarly that

$$(x^{-3}\ln^m|x|)\circ\delta^{(r)}(x) = c_{r,3,m}\delta^{(r+3)}(x)$$

for $r, m = 0, 1, 2, \ldots$ and so on.

In general, suppose that equation (24) holds for $i = 1, 2, \ldots, s$ and some s and $r, m = 0, 1, 2, \ldots$. This is true when $s = 1$ or 2. Then suppose that $(x^{-s-1}\ln^m|x|)\circ\delta^{(r)}(x)$ exists and

$$(x^{-s-1}\ln^m|x|)\circ\delta^{(r)}(x) = c_{r,s+1,m}\delta^{(r+s+1)}(x), \tag{25}$$

for $r = 0, 1, 2, \ldots$ and some m. This is true with

$$c_{r,s+1,0} = \frac{(-1)^{s+1}r!}{(r+s+1)!},$$

when $m = 0$. From our assumption on equation (24) with $i = s$ and $m+1$ for m, we have

$$(x^{-s}\ln^{m+1}|x|)\circ\delta^{(r)}(x) = c_{r,s,m+1}\delta^{(r+s)}(x). \tag{26}$$

Differentiating equation (26), we have

$$\begin{aligned}(x^{-s}\ln^{m+1}|x|)\circ\delta^{(r+1)}(x) &- s(x^{-s-1}\ln^{m+1}|x|)\circ\delta^{(r)}(x)\\ &+ (m+1)(x^{-s-1}\ln^m|x|)\circ\delta^{(r)}(x)\\ = [c_{r+1,s,m+1} + (m+1)c_{r,s+1,m}]\delta^{(r+s+1)}(x) &- s(x^{-s-1}\ln^{m+1}|x|)\circ\delta^{(r)}(x)\\ &= c_{r,s,m+1}\delta^{(r+s+1)}(x),\end{aligned}$$

from our assumptions and it follows that

$$\begin{aligned}(x^{-s-1}\ln^{m+1}|x|)\circ\delta^{(r)}(x) &= s^{-1}[c_{r+1,s,m+1} + (m+1)c_{r,s+1,m}\\ &\quad - c_{r,s,m+1}]\delta^{(r+s+1)}(x)\\ &= c_{r,s+1,m+1}\delta^{(r+s+1)}(x).\end{aligned}$$

Equation (25) therefore holds for $m+1$, with

$$c_{r,s+1,m+1} = s^{-1}[c_{r+1,s,m+1} + (m+1)c_{r,s+1,m} - c_{r,s+1,m}],$$

and so follows by induction for $r, m = 0, 1, 2, \ldots$. Then equation (24) holds for $i = 1, 2, \ldots, s+1$. Equation (24) follows by induction for $r, m = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$. This completes the proof of the theorem. □

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B. Fisher
Department of Mathematics
Leicester University
Leicester, LE1 7RH, England
E–mail: fbr@le.ac.uk

I. Ege and E. Özçag
Department of Mathematics
University of Hacettepe
06532-Beytepe, Ankara, Turkey
E–mail: incicankoy@hotmail.com
E–mail: ozcag1@hacettepe.edu.tr