ON THE NON-EXISTENCE OF CERTAIN TYPES OF WEAKLY SYMMETRIC MANIFOLD

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Abstract. An expression for the curvature tensor of a weakly symmetric manifold is obtained. Next it is shown that an Einstein weakly symmetric manifold of dimension $> 2$ does not exist. Further it is proved that a conformally flat weakly symmetric manifold of dimension $> 3$ is a quasi Einstein manifold. Finally a couple of results on conformally flat weakly symmetric manifold are presented.

1. Introduction

In [1] Chaki introduces a type of non-flat Riemannian manifold $(M^n, g)$ $(n \geq 2)$ whose curvature tensor $R$ satisfies the condition

$$\nabla_X R(Y, Z)W = 2A(X)R(Y, Z)W + A(Y)R(X, Z)W + A(Z)R(Y, X)W + A(W)R(Y, Z)X + g[R(Y, Z)W, X] \rho$$  (1.1)

where $A$ is a non zero 1-form defined by $g(X, \rho) = A(X)$ for any vector field $X$ and $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. Such a manifold is called a pseudo symmetric manifold and is denoted by $(PS)_n$. Generalizing the notion of $(PS)_n$, the authors in [7] introduce a non flat Riemannian manifold $(M^n, g), (n \geq 2)$ whose curvature tensor satisfies the condition


where $A, B, D$ and $E$ are 1-forms and $\mu$ is a vector field associated to a certain 1-form. Such a manifold is called weakly symmetric manifold and is denoted by $(WS)_n$. Recently in [5] and [6] it has been shown that the

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defining condition (1.2) of a \( (WS)_n \) can always be expressed in the following form:

\[
g((\nabla \nabla R)(Y, Z)W, U) = A(X)g(R(Y, Z)W, U) + B(Y)g(R(X, Z)W, U) \\
+ B(Z)g(R(Y, X)W, U) + E(W)g(R(Y, Z)X, U) + E(U)g(R(Y, Z)W, X). \\
(1.3)
\]

It may be mentioned in this connection that although the definition of a \( (WS)_n \) is similar to that of a generalized pseudo symmetric space studied by Chaki [2], the defining condition of a \( (WS)_n \) is weaker than that of a generalized pseudo symmetric space. A reduction in generalized pseudo symmetric space has been obtained in [4] and a reduction in \( (WS)_n \) is obtained in [5] and [6]. In this paper we have studied a weakly symmetric manifold whose defining condition satisfies (1.3). In the study of \( (WS)_n \) an important role is played by the 1-form \( \delta \) defined by

\[
g(X, \nu) = \delta(X) = A(X) - 2B(X) \neq 0. \\
(1.4)
\]

It is shown that if \( \delta \neq 0 \), then the curvature tensor of a \( (WS)_n \) is determined by the Ricci tensor \( S \) and the non-zero 1-form \( T \) associated to a unit vector field \( \alpha \) defined by

\[
T(X) = g(X, \alpha) = \frac{\delta(X)}{\sqrt{\delta(\nu)}}. \\
(1.5)
\]

Next we have considered an Einstein \( (WS)_n \), and have shown that such manifold does not exist. In the last section we have studied conformally flat \( (WS)_n \). First we have proved that a conformally flat \( (WS)_n \) is a quasi Einstein (see[3]). Further it is shown that a conformally flat \( (WS)_n \) does not exist if either its scalar curvature is constant or the unit vector field \( \alpha \) (defined above) is geodesic.

2. Preliminaries

Let \( S \) and \( r \) denote the Ricci tensor of type \((0,2)\) and the scalar curvature respectively and \( Q \) denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, i.e.

\[
S(X, Y) = g(QX, Y). \\
(2.1)
\]

Let the vector fields \( \rho, \lambda \) and \( \mu \) associated to the 1-forms \( A, B \) and \( E \) be defined by

\[
g(X, \rho) = A(X) \\
g(X, \lambda) = B(X) \\
g(X, \mu) = E(X) \\
(2.4)
\]
for all $X \in M$. Contracting (1.3) at $Z, W$ and writing $Z$ in place of $U$ we get

$$
(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) \\
+ B(R(X, Y)Z) + E(R(X, Z)Y) + E(Z)S(X, Y). \tag{2.5}
$$

Permuting cyclically (1.3) twice over $X, Y, Z$ and adding these permuted equations with (1.3) we obtain

$$
\begin{align*}
[A(X) - 2B(X)]g(R(Y, Z)W, U) + [A(Y) - 2B(Y)]g(R(Z, X)W, U) \\
+ [A(Z) - 2B(Z)]g(R(X, Y)W, U) &= 0. \tag{2.6}
\end{align*}
$$

Contracting at $Z$ and $W$ and writing $Z$ in place of $U$, (2.6) reduces to

$$
\begin{align*}
A(R(X, Y)Z) - 2B(R(X, Y)Z) &= [A(X) - 2B(X)]S(Y, Z) \\
- [A(Y) - 2B(Y)]S(X, Z). \tag{2.7}
\end{align*}
$$

Further contracting (2.7) over $Y$ and $Z$ we get

$$
S(X, \rho) - 2S(X, \lambda) = \frac{r}{2}[A(X) - 2B(X)]. \tag{2.8}
$$

As $(\nabla_X S)(Y, Z) = (\nabla_X S)(Z, Y)$ equation (2.2) provides

$$
\begin{align*}
B(R(X, Y)Z) - E(R(X, Y)Z) &= [B(X) - E(X)]S(Y, Z) \\
- [B(Y) - E(Y)]S(X, Z). \tag{2.9}
\end{align*}
$$

Contracting (2.9) over $Y$ and $Z$ yields

$$
S(X, \lambda) - S(X, \mu) = \frac{r}{2}[B(X) - E(X)]. \tag{2.10}
$$

An example of weakly symmetric manifold exists. For instance the metric $g$ in the coordinate space $R^n (n \geq 4)$ defined by the formula

$$
ds^2 = \varphi(dx^1)^2 + K_{ab}dx^a dx^b + 2dx^1 dx^n; \quad a, b = 2, 3, \ldots (n-1)
$$

where $K_{ab}$ is a symmetric, non-singular matrix consisting of constants and $\varphi$ is independent of $x^n$ is a weakly symmetric manifold. For details we refer to [5]. Now we give the definition of Weyl conformal curvature tensor. The Weyl conformal curvature tensor $C$ on an $n$-dimensional ($n > 3$) Riemannian manifold is defined by (see [8])

$$
C(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\
- g(X, Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}. \tag{2.11}
$$
For dimension \((n > 3)\), conformal flatness implies \(C = 0\). Moreover for dimension \((n > 3)\), the condition \(C = 0\) implies \(\text{div } C = 0\), which is equivalent to
\[
(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(n - 1)} \{(X.r)g(Y, Z) - (Y.r)g(X, Z)\}.
\]
\[(2.12)\]

\[3.\] Curvature tensor of \((WS)_n\)

In this article we have obtained an expression of the curvature tensor of a \((WS)_n\) which we have shown in the following theorem.

**Theorem 1.** The curvature tensor of a \((WS)_n\) can be expressed as
\[
g(R(X, Y)W, U) = T(Y)[T(W)S(X, U) - T(U)S(X, W)] - T(X)[T(W)S(Y, U) - T(U)S(Y, W)]. \tag{3.1}
\]

**Proof.** Since by (1.4) \(\delta(X) \neq 0\), (2.7) assumes the form
\[
\delta(R(X, Y)Z) = \delta(X)S(Y, Z) - \delta(Y)S(X, Z). \tag{3.2}
\]
Contracting (3.1) over \(Y\) and \(Z\) we get
\[
S(X, \nu) = \frac{r}{2} \delta(X). \tag{3.3}
\]
This shows that \(\frac{r}{2}\) is an eigenvalue of the Ricci tensor \(S\) corresponding to the eigen vector \(\nu\). Further through (1.4), (2.6) can be written as
\[
\delta(X)g(R(Y, Z)W, U) + \delta(Y)g(R(Z, X)W, U) + \delta(Z)g(R(X, Y)W, U) = 0. \tag{3.4}
\]
Setting \(Z = \nu\) in (3.3) and using (3.1), (1.5) we get
\[
g(R(X, Y)W, U) = T(Y)[T(W)S(X, U) - T(U)S(X, W)] - T(X)[T(W)S(Y, U) - T(U)S(Y, W)]. \tag{3.5}
\]
\[
\Box
\]

\[4.\] **Einstein** \((WS)_n\)

In this section we have considered Einstein \((WS)_n\) and prove the following

**Theorem 2.** An Einstein \((WS)_n\), \((n > 2)\) does not exist.

**Proof.** By hypothesis
\[
S(X, Y) = \frac{r}{n} g(X, Y). \tag{4.1}
\]
Using this in (2.8) and (2.10) respectively, we get
\[
r[A(X) - 2B(X)] = 0, \quad r[B(X) - E(X)] = 0. \tag{4.2}
\]
If \( r = 0 \), then by (4.1) we see that \( S = 0 \) and hence by the Theorem 1, \( R = 0 \) which is inadmissible by definition. So we assume \( r \neq 0 \) in some open neighborhood \( N \) of \((WS)_{n}\). Thus from (4.2) we see that 
\[
A(X) = 2B(X) \quad \text{and} \quad E(X) = B(X), \quad \text{for all} \quad X \in (WS)_{n}.
\]
Using these, (1.4) takes the form
\[
g((\nabla X)R(Y,Z)W,U) = 2B(X)g(R(Y,Z)W,U) + B(Y)g(R(X,Z)W,X) \\
+ B(Z)g(R(Y,X)W,U) + B(W)g(R(Y,Z)X,U) \\
+ B(U)g(R(Y,Z)W,X).
\]
(4.3)
This shows that \( N \) is a \((PS)_{n}\) with \( B \) as its associated 1-form. Contracting (4.3) over \( Z \) and \( W \) we get
\[
(\nabla X)S(Y,U) = 2B(X)S(Y,U) + B(Y)S(X,U) \\
+ B(U)S(X,Y) + B(R(X,Y)U) + B(R(X,U)Y).
\]
Further contracting over \( Y \) and \( U \) the last equation reduces to
\[
X.r = 2rB(X) + 4S(X,\lambda).
\]
(4.4)
Now since \( N \) is Einstein the scalar curvature \( r \) is constant and hence (4.4) shows that \( rB(X) = 0 \). But by the definition of a \((PS)_{n}\) (see[1]) the 1-form \( B \) is non-zero and hence \( r = 0 \). Consequently we arrive at a contradiction. □

5. Conformally flat \((WS)_{n}\)

The notion of a quasi Einstein manifold was introduced by Chaki and Maity in [3]. According to them a non-flat Riemannian manifold \((M^n,g)\) \((n \geq 3)\) is said to be a quasi Einstein manifold if there exists a non zero 1-form associated to a unit vector field such that its Ricci tensor is not identically zero and satisfies the condition
\[
S(X,Y) = ag(X,Y) + bp(X)p(Y)
\]
where \(a,b\) are scalars and \(b \neq 0\).

In this section we prove the following theorems:

**Theorem 3.** A conformally flat \((WS)_{n}\) \((n > 3)\) with \(\delta \neq 0\) is a quasi Einstein manifold of non zero scalar curvature.

**Theorem 4.** A conformally flat \((WS)_{n}\) \((n > 3)\) with \(\delta \neq 0\) does not exist if the scalar curvature is constant.

**Theorem 5.** A conformally flat \((WS)_{n}\) \((n > 3)\) with \(\delta \neq 0\) does not exist if the vector field \(\alpha\) defined by (1.5) is geodesic.
Proof of Theorem 3. As $C=0$, we have by (2.11)

$$g(R(X, Y)Z, W) = \frac{1}{(n-2)} \{S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + S(X, W)g(Y, Z) - S(Y, W)g(X, Z)\}$$

$$- \frac{r}{(n-2)(n-1)} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}. \tag{5.1}$$

Setting $W = \rho$ and $W = \lambda$ in (5.1) respectively we get

$$A(R(X, Y)Z) = \frac{1}{(n-2)} \{A(X)S(Y, Z) - A(Y)S(X, Z) + S(X, \rho)g(Y, Z) - S(Y, \rho)g(X, Z)\}$$

$$- \frac{r}{(n-1)(n-2)} \{A(X)g(Y, Z) - A(Y)g(X, Z)\} \tag{5.2}$$

and

$$B(R(X, Y)Z) = \frac{1}{(n-2)} \{B(X)S(Y, Z) - B(Y)S(X, Z) + S(X, \lambda)g(Y, Z) - S(Y, \lambda)g(X, Z)\}$$

$$- \frac{r}{(n-1)(n-2)} \{B(X)g(Y, Z) - B(Y)g(X, Z)\}. \tag{5.3}$$

Taking into account (5.2), (5.3) and (2.7) we obtain

$$(n - 3)\{[A(X) - 2B(X)]S(Y, Z) - [A(Y) - 2B(Y)]S(X, Z)\} =$$

$$[S(X, \rho) - 2S(X, \lambda)]g(Y, Z) - [S(Y, \rho) - 2S(Y, \lambda)]g(X, Z)$$

$$- \frac{r}{(n-1)} \{A(X) - 2B(X)\}g(Y, Z) + \frac{r}{(n-1)} \{A(Y) - 2B(Y)\}g(X, Z). \tag{5.4}$$

Recalling (2.8) and since $(n > 3)$, (5.4) provides

$$[A(X) - 2B(X)][S(Y, Z) - \frac{r}{2(n-1)} g(Y, Z)]$$

$$- [A(Y) - 2B(Y)][S(X, Z) - \frac{r}{2(n-1)} g(X, Z)] = 0. \tag{5.5}$$

Since $\delta(X) = A(X) - 2B(X) \neq 0$, (5.5) can be written as

$$\delta(X)[S(Y, Z) - \frac{r}{2(n-1)} g(Y, Z)]$$

$$- \delta(Y)[S(X, Z) - \frac{r}{2(n-1)} g(X, Z)] = 0. \tag{5.6}$$

Putting $X = \nu$ in (5.6) and using (3.2) we get

$$S(Y, Z) = \frac{r}{2(n-1)}g(Y, Z) + \frac{(n-2)r}{2(n-1)}T(Y)T(Z) \tag{5.7}$$
where $T(X) = g(X, \alpha)$ and $\alpha$ is unit. This shows that the manifold is quasi Einstein. If possible, let $r = 0$, then from (5.7) we see that $S = 0$ and hence by Theorem 1, $R = 0$, which is inadmissible by the definition. □

Proof of Theorem 4. Differentiating covariantly (5.7) along an arbitrary vector field $X$, we get

$$(\nabla_X S)(Y, Z) = \frac{(X.r)}{2(n-1)} \{ g(Y, Z) + (n-2)T(Y)T(Z) \} + \frac{(n-2)r}{2(n-1)} \{ T(Z)(\nabla_X T)Y + T(Y)(\nabla_X T)Z \}. \quad (5.8)$$

Since $C = 0$, $\text{div} \ C = 0$ and hence using (5.8) in (2.12) we obtain

$$\{(X.r)T(Y)T(Z) - (Y.r)T(X)T(Z)\} + r[T(Z) \{ (\nabla_X T)Y - (\nabla_Y T)X \} + T(Y)(\nabla_X T)Z - T(X)(\nabla_Y T)Z] = 0. \quad (5.9)$$

Putting $Y = Z = e_i$ in (5.9) and summing over $i$, $1 \leq i \leq n$, we find

$$[(X.r) - (\alpha.r)] - r[\nabla_\alpha X + T(X)(\nabla_{e_i} T)e_i] = 0. \quad (5.10)$$

Setting $X = \alpha$ in (5.10) and since $\alpha$ is unit, we have

$$r(\nabla_{e_i} T)e_i = 0. \quad (5.11)$$

Thus in view of (5.11), (5.10) assumes the form

$$r(\nabla_{\alpha} T)X = (X.r) - (\alpha.r)T(X). \quad (5.12)$$

Further taking $Y = \alpha$ in (5.9) and recalling (5.12) we at once obtain

$$r(\nabla_X T)Z = (Z.r)T(X) - (\alpha.r)T(X)T(Z). \quad (5.13)$$

Now we assume that $r$ is constant. If $r = 0$ the manifold becomes flat. So we assume that $r$ is non-zero in some open neighborhood $N$ of $(WS)_n$. Hence on $N$, $r$ is a non-zero constant. Thus from (5.13) we see that $\nabla_X \alpha = 0$, which implies that $R(X, Y)\alpha = 0$ and we obtain $S(X, \alpha) = 0$. By using this in (5.7) it shows that $r = 0$, and consequently it follows that $N$ becomes flat. Therefore we arrive at a contradiction. □

Proof of Theorem 5. By hypothesis the unit vector field $\alpha$ is geodesic i.e. $\nabla_\alpha \alpha = 0$. Hence by (5.12) we have

$$X.r = (\alpha.r)T(X). \quad (5.14)$$

Therefore taking into account (5.14) and (5.13) we see that

$$r(\nabla_X T)Z = 0. \quad (5.15)$$

If $r = 0$, then the manifold becomes flat and which is inadmissible by definition. So we assume that $r \neq 0$ in a neighborhood $N$ of $(WS)_n$. Thus (5.15)
shows that $\nabla_X \alpha = 0$ and as before it follows that $r = 0$ in $N$, which is a contradiction. □

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References


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