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ON THE NON-EXISTENCE OF CERTAIN TYPES OF WEAKLY SYMMETRIC MANIFOLD

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Abstract. An expression for the curvature tensor of a weakly symmetric manifold is obtained. Next it is shown that an Einstein weakly symmetric manifold of dimension > 2 does not exist. Further it is proved that a conformally flat weakly symmetric manifold of dimension > 3 is a quasi Einstein manifold. Finally a couple of results on conformally flat weakly symmetric manifold are presented.

1. Introduction

In [1] Chaki introduces a type of non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ whose curvature tensor R satisfies the condition

$$\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W \\ + A(W)R(Y,Z)X + g[R(Y,Z)W,X]\rho \quad (1.1)$$

where A is a non zero 1-form defined by $g(X,\rho) = A(X)$ for any vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such a manifold is called a pseudo symmetric manifold and is denoted by $(PS)_n$. Generalizing the notion of $(PS)_n$, the authors in [7] introduce a non flat Riemannian manifold $(M^n, g), (n \geq 2)$ whose curvature tensor satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W + D(Z)R(Y,X)W \\ + E(W)R(Y,Z)X + g[R(Y,Z)W,X]\mu \quad (1.2)$$

where A, B, D and E are 1-forms and μ is a vector field associated to a certain 1-form. Such a manifold is called weakly symmetric manifold and is denoted by $(WS)_n$. Recently in [5] and [6] it has been shown that the

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defining condition (1.2) of a $(WS)_n$ can always be expressed in the following form:

$$g((\nabla_X R)(Y,Z)W,U) = A(X)g(R(Y,Z)W,U) + B(Y)g(R(X,Z)W,U) + B(Z)g(R(Y,X)W,U) + E(W)g(R(Y,Z)X,U) + E(U)g(R(Y,Z)W,X). \tag{1.3}$$

It may be mentioned in this connection that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo symmetric space studied by Chaki [2], the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo symmetric space. A reduction in generalized pseudo symmetric space has been obtained in [4] and a reduction in $(WS)_n$is obtained in [5] and [6]. In this paper we have studied a weakly symmetric manifold whose defining condition satisfies (1.3). In the study of $(WS)_n$ an important role is played by the 1-form δ defined by

$$g(X,\nu) = \delta(X) = A(X) - 2B(X) \neq 0. \tag{1.4}$$

It is shown that if $\delta \neq 0$, then the curvature tensor of a $(WS)_n$ is determined by the Ricci tensor S and the non-zero 1-form T associated to a unit vector field α defined by

$$T(X) = g(X,\alpha) = \frac{\delta(X)}{\sqrt{\delta(\nu)}}. \tag{1.5}$$

Next we have considered an Einstein $(WS)_n$, and have shown that such manifold does not exist. In the last section we have studied conformally flat $(WS)_n$. First we have proved that a conformally flat $(WS)_n$ is a quasi Einstein (see[3]). Further it is shown that a conformally flat $(WS)_n$ does not exist if either its scalar curvature is constant or the unit vector field α (defined above) is geodesic.

2. Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively and Q denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, i.e.

$$S(X,Y) = g(QX,Y). \tag{2.1}$$

Let the vector fields ρ, λ and μ associated to the 1-forms A, B and E be defined by

$$g(X,\rho) = A(X) \tag{2.2}$$

$$g(X,\lambda) = B(X) \tag{2.3}$$

$$g(X,\mu) = E(X) \tag{2.4}$$

for all $X \in M$. Contracting (1.3) at Z, W and writing Z in place of U we get

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + B(R(X,Y)Z) + E(R(X,Z)Y) + E(Z)S(X,Y). \quad (2.5)$$

Permuting cyclically (1.3) twice over X, Y, Z and adding these permuted equations with (1.3) we obtain

$$[A(X) - 2B(X)]g(R(Y,Z)W,U) + [A(Y) - 2B(Y)]g(R(Z,X)W,U) + [A(Z) - 2B(Z)]g(R(X,Y)W,U) = 0. \quad (2.6)$$

Contracting at Z and W and writing Z in place of U, (2.6) reduces to

$$A(R(X,Y)Z) - 2B(R(X,Y)Z) = [A(X) - 2B(X)]S(Y,Z) - [A(Y) - 2B(Y)]S(X,Z). \quad (2.7)$$

Further contracting (2.7) over Y and Z we get

$$S(X,\rho) - 2S(X,\lambda) = \frac{r}{2}[A(X) - 2B(X)]. \quad (2.8)$$

As $(\nabla_X S)(Y,Z) = (\nabla_X S)(Z,Y)$ equation (2.2) provides

$$B(R(X,Y)Z) - E(R(X,Y)Z) = [B(X) - E(X)]S(Y,Z) - [B(Y) - E(Y)]S(X,Z)). \quad (2.9)$$

Contracting (2.9) over Y and Z yields

$$S(X,\lambda) - S(X,\mu) = \frac{r}{2}[B(X) - E(X)]. \quad (2.10)$$

An example of weakly symmetric manifold exists. For instance the metric g in the coordinate space $R^n (n \geq 4)$ defined by the formula

$$ds^2 = \varphi(dx^1)^2 + K_{ab}dx^a dx^b + 2dx^1 dx^n; \; a, b = 2, 3, \ldots (n-1)$$

where K_{ab} is a symmetric, non-singular matrix consisting of constants and φ is independent of x^n is a weakly symmetric manifold. For details we refer to [5]. Now we give the definition of Weyl conformal curvature tensor. The Weyl conformal curvature tensor C on an n-dimensional $(n > 3)$ Riemannian manifold is defined by (see [8])

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \quad (2.11)$$

For dimension $(n > 3)$, conformal flatness implies $C = 0$. Moreover for dimension $(n > 3)$, the condition $C = 0$ implies div $C = 0$, which is equivalent to

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(n-1)}\{(X.r)g(Y, Z) - (Y.r)g(X, Z)\}. \tag{2.12}$$

3. Curvature tensor of $(WS)_n$

In this article we have obtained an expression of the curvature tensor of a $(WS)_n$ which we have shown in the following theorem.

Theorem 1. *The curvature tensor of a* $(WS)_n$ *can be expressed as*

$$\begin{aligned} g(R(X, Y)W, U) = T(Y)[T(W)S(X, U) - T(U)S(X, W)] \\ - T(X)[T(W)S(Y, U) - T(U)S(Y, W)]. \end{aligned} \tag{3.1}$$

Proof. Since by (1.4) $\delta(X) \neq 0$, (2.7) assumes the form

$$\delta(R(X, Y)Z) = \delta(X)S(Y, Z) - \delta(Y)S(X, Z). \tag{3.2}$$

Contracting (3.1) over Y and Z we get

$$S(X, \nu) = \frac{r}{2}\,\delta(X). \tag{3.3}$$

This shows that $\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigen vector ν. Further through (1.4), (2.6) can be written as

$$\delta(X)g(R(Y, Z)W, U) + \delta(Y)g(R(Z, X)W, U) + \delta(Z)g(R(X, Y)W, U) = 0. \tag{3.4}$$

Setting $Z = \nu$ in (3.3) and using (3.1), (1.5) we get

$$\begin{aligned} g(R(X, Y)W, U) = T(Y)[T(W)S(X, U) - T(U)S(X, W)] \\ - T(X)[T(W)S(Y, U) - T(U)S(Y, W)]. \end{aligned} \tag{3.5}$$

□

4. Einstein $(WS)_n$

In this section we have considered Einstein $(WS)_n$ and prove the following

Theorem 2. *An Einstein* $(WS)_n$, $(n > 2)$ *does not exist.*

Proof. By hypothesis

$$S(X, Y) = \frac{r}{n}\,g(X, Y). \tag{4.1}$$

Using this in (2.8) and (2.10) respectively, we get

$$r[A(X) - 2B(X)] = 0, \;\; r[B(X) - E(X)] = 0. \tag{4.2}$$

If $r = 0$, then by (4.1) we see that $S = 0$ and hence by the Theorem 1, $R = 0$ which is inadmissible by definition. So we assume $r \neq 0$ in some open neighborhood N of $(WS)_n$. Thus from (4.2) we see that $A(X) = 2B(X)$ and $E(X) = B(X)$, for all X in $(WS)_n$. Using these, (1.4) takes the form

$$\begin{aligned} g((\nabla_X R)(Y,Z)W,U) = 2B(X)g(R(Y,Z)W,U) + B(Y)g(R(X,Z)W,X) \\ +B(Z)g(R(Y,X)W,U) + B(W)g(R(Y,Z)X,U) \\ +B(U)g(R(Y,Z)W,X). \end{aligned} \tag{4.3}$$

This shows that N is a $(PS)_n$ with B as its associated 1-form. Contracting (4.3) over Z and W we get

$$\begin{aligned} (\nabla_X S)(Y,U) = 2B(X)S(Y,U) + B(Y)S(X,U) \\ + B(U)S(X,Y) + B(R(X,Y)U) + B(R(X,U)Y). \end{aligned}$$

Further contracting over Y and U the last equation reduces to

$$X.r = 2rB(X) + 4S(X,\lambda). \tag{4.4}$$

Now since N is Einstein the scalar curvature r is constant and hence (4.4) shows that $rB(X) = 0$. But by the definition of a $(PS)_n$ (see[1]) the 1-form B is non-zero and hence $r = 0$. Consequently we arrive at a contradiction. □

5. Conformally flat $(WS)_n$

The notion of a quasi Einstein manifold was introduced by Chaki and Maity in [3]. According to them a non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a quasi Einstein manifold if there exists a non zero 1-form associated to a unit vector field such that its Ricci tensor is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bp(X)p(Y)$$

where a, b are scalars and $b \neq 0$.

In this section we prove the following theorems:

Theorem 3. *A conformally flat* $(WS)_n$ $(n > 3)$ *with* $\delta \neq 0$ *is a quasi Einstein manifold of non zero scalar curvature.*

Theorem 4. *A conformally flat* $(WS)_n$ $(n > 3)$ *with* $\delta \neq 0$ *does not exist if the scalar curvature is constant.*

Theorem 5. *A conformally flat* $(WS)_n$ $(n > 3)$ *with* $\delta \neq 0$ *does not exist if the vector field* α *defined by* (1.5) *is geodesic.*

Proof of Theorem 3. As C=0, we have by (2.11)

$$g(R(X,Y)Z,W) = \frac{1}{(n-2)}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\} - \frac{r}{(n-2)(n-1)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \quad (5.1)$$

Setting $W = \rho$ and $W = \lambda$ in (5.1) respectively we get

$$A(R(X,Y)Z) = \frac{1}{(n-2)}\{A(X)S(Y,Z) - A(Y)S(X,Z) + S(X,\rho)g(Y,Z) - S(Y,\rho)g(X,Z)\} - \frac{r}{(n-1)(n-2)}\{A(X)g(Y,Z) - A(Y)g(X,Z)\} \quad (5.2)$$

and

$$B(R(X,Y)Z) = \frac{1}{(n-2)}\{B(X)S(Y,Z) - B(Y)S(X,Z) + S(X,\lambda)g(Y,Z) - S(Y,\lambda)g(X,Z)\} - \frac{r}{(n-1)(n-2)}\{B(X)g(Y,Z) - B(Y)g(X,Z)\}. \quad (5.3)$$

Taking into account (5.2), (5.3) and (2.7) we obtain

$$(n-3)[\{A(X) - 2B(X)\}S(Y,Z) - \{A(Y) - 2B(Y)\}S(X,Z)] = [S(X,\rho) - 2S(X,\lambda)]g(Y,Z) - [S(Y,\rho) - 2S(Y,\lambda)]g(X,Z) - \frac{r}{(n-1)}\{A(X) - 2B(X)\}g(Y,Z) + \frac{r}{(n-1)}\{A(Y) - 2B(Y)\}g(X,Z). \quad (5.4)$$

Recalling (2.8) and since $(n > 3)$, (5.4) provides

$$[A(X) - 2B(X)][S(Y,Z) - \frac{r}{2(n-1)}g(Y,Z)] - [A(Y) - 2B(Y)][S(X,Z) - \frac{r}{2(n-1)}g(X,Z)] = 0. \quad (5.5)$$

Since $\delta(X) = A(X) - 2B(X) \neq 0$, (5.5) can be written as

$$\delta(X)[S(Y,Z) - \frac{r}{2(n-1)}g(Y,Z)] - \delta(Y)[S(X,Z) - \frac{r}{2(n-1)}g(X,Z)] = 0. \quad (5.6)$$

Putting $X = \nu$ in (5.6) and using (3.2) we get

$$S(Y,Z) = \frac{r}{2(n-1)}g(Y,Z) + \frac{(n-2)r}{2(n-1)}T(Y)T(Z) \quad (5.7)$$

where $T(X) = g(X, \alpha)$ and α is unit. This shows that the manifold is quasi Einstein. If possible, let $r = 0$, then from (5.7) we see that $S = 0$ and hence by Theorem 1, $R = 0$, which is inadmissible by the definition. □

Proof of Theorem 4. Differentiating covariently (5.7) along an arbitrary vector field X, we get

$$(\nabla_X S)(Y,Z) = \frac{(X.r)}{2(n-1)}\{g(Y,Z) + (n-2)T(Y)T(Z)\} + \frac{(n-2)r}{2(n-1)}\{T(Z)(\nabla_X T)Y + T(Y)(\nabla_X T)Z\}. \quad (5.8)$$

Since $C = 0$, div $C = 0$ and hence using (5.8) in (2.12) we obtain

$$\{(X.r)T(Y)T(Z) - (Y.r)T(X)T(Z)\} + r[T(Z)\{(\nabla_X T)Y - (\nabla_Y T)X\} + T(Y)(\nabla_X T)Z - T(X)(\nabla_Y T)Z] = 0. \quad (5.9)$$

Putting $Y = Z = e_i$ in (5.9) and summing over i, $1 \le i \le n$, we find

$$[(X.r) - (\alpha.r)] - r[(\nabla_\alpha T)X + T(X)(\nabla_{e_i} T)e_i] = 0. \quad (5.10)$$

Setting $X = \alpha$ in (5.10) and since α is unit, we have

$$r(\nabla_{e_i} T)e_i = 0. \quad (5.11)$$

Thus in view of (5.11), (5.10) assumes the form

$$r(\nabla_\alpha T)X = (X.r) - (\alpha.r)T(X). \quad (5.12)$$

Further taking $Y = \alpha$ in (5.9) and recalling (5.12) we at once obtain

$$r(\nabla_X T)Z = (Z.r)T(X) - (\alpha.r)T(X)T(Z). \quad (5.13)$$

Now we assume that r is constant. If $r = 0$ the manifold becomes flat. So we assume that r is non-zero in some open neighborhood N of $(WS)_n$. Hence on N, r is a non-zero constant. Thus from (5.13) we see that $\nabla_X \alpha = 0$, which implies that $R(X,Y)\alpha = 0$ and we obtain $S(X,\alpha) = 0$. By using this in (5.7) it shows that $r = 0$, and consequently it follows that N becomes flat. Therefore we arrive at a contradiction. □

Proof of Theorem 5. By hypothesis the unit vector field α is geodesic i.e. $\nabla_\alpha \alpha = 0$. Hence by (5.12) we have

$$X.r = (\alpha.r)T(X). \quad (5.14)$$

Therefore taking into account (5.14) and (5.13) we see that

$$r(\nabla_X T)Z = 0. \quad (5.15)$$

If $r = 0$, then the manifold becomes flat and which is inadmissible by definition. So we assume that $r \ne 0$ in a neighborhood N of $(WS)_n$. Thus (5.15) shows that $\nabla_X \alpha = 0$ and as before it follows that $r = 0$ in N, which is a contradiction. □

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