ON UPPER DINI’S SYSTEMS AND U.S.C. FUNCTIONS WITH CONVEX LIMIT SETS

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Abstract. We give an answer to the question in [HN] as to which upper Dini’s systems of functions induces a Hausdorff metric topology on $U_0(X)$. We show that if $X$ is a locally connected metric space then the Hausdorff metric topology on $U_0(X)$ induces as an upper Dini’s system of functions the set of all bounded upper semicontinuous functions vanishing at infinity with convex limit sets.

1. Introduction

For a topological space $X$ denote by $C(X)$, $U(X)$, the space of all continuous, upper semicontinuous (u.s.c.) real functions on $X$, respectively. A function $f : X \rightarrow \mathbb{R}$ (where $\mathbb{R}$ denotes the set of all real numbers) is said to be vanishing at infinity, if for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that $|f(x)| < \epsilon$ for every $x \notin K_\epsilon$ [HN]. By $C_0(X)$ and $U_0(X)$ we denote the space of all continuous functions vanishing at infinity and of all upper semicontinuous functions vanishing at infinity, respectively. The Dini’s theorem [R] says that if $X$ is a compact space and $\{f_n; n \in \mathbb{Z}^+\}$ (where $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$) a decreasing sequence of functions belonging to $U(X)$ pointwise converges to $f \in C(X)$ then the sequence $\{f_n; n \in \mathbb{Z}^+\}$ is uniformly convergent to $f$.

Definition 1.1. [Be1] Let $X$ be a compact metric space and $\Omega \subset U(X)$. If $\tau$ is a topology on $U(X)$, then $\Omega$ is called a Dini’s class of functions induced by $\tau$ if

1. $\Omega$ is $\tau$-closed,
2. $C(X) \subset \Omega$,

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(3) for each \( f \in \Omega \) whenever \( \{f_n; \ n \in \mathbb{Z}^+\} \) is a decreasing sequence of functions belonging to \( U(X) \) and pointwise converges to \( f \) then \( \{f_n; \ n \in \mathbb{Z}^+\} \) \( \tau \)-converges to \( f \).

In this sense, if \( X \) is a compact space then \( C(X) \) as a subclass of \( U(X) \) is a Dini’s class of functions induced by the topology of the uniform convergence on \( U(X) \).

Beer in [Be1] showed that if \( X \) is a locally connected compact metric space then the Hausdorff metric topology on \( U(X) \) obtained by identifying each u.s.c. function with the closure of its graph induces as a Dini’s class of functions the set of all bounded u.s.c. functions with convex limit sets. The limit set for a function \( f \) from \( X \) to \( \mathbb{R} \) at \( x \in X \) is the set of all real \( r \) such that \( (x, r) \) belongs to the closure of the graph of \( f \). There exists at most one Dini’s class \( \Omega \) induced by \( \tau \) [Be1].

In [HN] a type of Dini’s theorem for spaces that are not necessarily compact is presented.

**Theorem A.** [HN] Let \( X \) be a topological space. Let \( \{f_n; \ n \in \mathbb{Z}^+\} \) be a decreasing sequence of functions belonging to \( U_0(X) \) and converging pointwise to a function \( f \in C_0(X) \). Then the convergence is uniform.

In [HN] is presented a more general definition of Dini’s class (Dini’s system).

**Definition 1.2.** [HN] Let \( X \) be a topological space. Let \( \mathcal{F} \subset U(X) \) and \( \tau \) be a topology on \( \mathcal{F} \). A collection \( \Omega \subset \mathcal{F} \) is called an upper Dini’s system for \( \mathcal{F} \) induced by the topology \( \tau \), if the following are satisfied:

1. \( \Omega \) is \( \tau \)-closed,
2. \( \Omega \subset \mathcal{F} \),
3. If \( \{f_n; \ n \in \mathbb{Z}^+\} \) is a decreasing sequence of functions belonging to \( \mathcal{F} \) and pointwise converging to \( f \in \Omega \) then \( \{f_n; \ n \in \mathbb{Z}^+\} \) \( \tau \)-converges to \( f \).

In [HN] the question is posed as to which upper Dini’s system of functions induces a Hausdorff metric topology on \( U_0(X) \). We show that if \( X \) is a locally connected metric space, then the Haussdorff metric topology on \( U_0(X) \) induces as an upper Dini’s system of functions the set of all bounded u.s.c. functions vanishing at infinity with convex limit sets.

If \( X \) is a perfectly normal \( T_1 \)-space and \( \mathcal{F} \subset U(X) \) and \( \tau \) is a topology on \( \mathcal{F} \), then there exists at most one upper Dini’s system \( \Omega \) for \( \mathcal{F} \) induced by the topology \( \tau \) such that \( \Omega \supset C(X) \) [HN]. Similarly for functions vanishing at infinity we have [HN]: If \( X \) is a perfectly normal locally compact Hausdorff space, \( \mathcal{F} \subset U_0(X) \) and \( \tau \) is a topology on \( \mathcal{F} \), then there exists at most one upper Dini’s system \( \Omega \) for \( \mathcal{F} \) induced by the topology \( \tau \) such that \( \Omega \supset C_0(X) \).
2. Preliminaries

\((Z, d)\) will denote a metrizable space \(Z\) with a compatible metric \(d\). The open \(d\)-ball with the center \(z_0 \in Z\) and a radius \(\epsilon > 0\) will be denoted by \(S_\epsilon(z_0)\) and the \(\epsilon\)-parallel body \(\bigcup\{S_\epsilon(a) ; a \in A\}\) for a subset \(A\) of \(Z\) will be denoted by \(S_\epsilon(A)\). We denote by \(2^Z\) the space of all closed subsets of \(Z\) and by \(\text{CL}(Z)\) the space of all nonempty closed subsets of \(Z\). The symbol \(\overline{B}\) will stand for the closure of \(B \subset Z\). If \(A \in \text{CL}(Z)\), the distance functional
\[
d(\cdot, A) : Z \mapsto [0, \infty)
\]
is described by the familiar formula:
\[
d(z, A) = \inf\{d(z, a) ; a \in A\}.
\]
The Hausdorff metric \(H_d\) on \(2^Z\) is defined as follows:
\[
H_d(A, B) = \max\{\sup\{d(a, B) ; a \in A\}, \sup\{d(b, A) ; b \in B\}\}
\]
if \(A\) and \(B\) are nonempty. If \(A \neq \emptyset\) we take \(H_d(A, \emptyset) = H_d(\emptyset, A) = \infty\). On \(\text{CL}(Z)\) we can use for Hausdorff distance the following equality:
\[
H_d(A, B) = \inf\{\epsilon > 0 ; A \subset S_\epsilon(B) \text{ and } B \subset S_\epsilon(A)\}.
\]
\(H_d\) defines an (extended-valued) metric on \(2^Z\). The generated topology is called the Hausdorff metric topology.

Now let \((X, d)\) be a metric space, consider the product \(X \times \mathbb{R}\) metrized in the following way (box metric).
\[
\rho((x_1, r_1), (x_2, r_2)) = \max\{d(x_1, x_2), |r_1 - r_2|\}.
\]
Denote the closure of the graph of the function \(f : X \mapsto \mathbb{R}\) by \(\overline{\text{Gr}f}\) and the restriction of \(f\) to a subset \(A\) of \(X\) by \(f \upharpoonright A\).

If \(f\) and \(g\) are in \(U(X)\) denote the Hausdorff distance generated by \(\rho\) from \(\overline{\text{Gr}f}\) to \(\overline{\text{Gr}g}\) by \(h(f, g)\). Thus \(h\) is an extended valued metric on \(U(X)\) [Be1].

In what follows \(X\) will be a metric space with a compatible metric \(d\). Let \(f\) be a function from \(X\) to \(\mathbb{R}\). For each \(x \in X\) let:
\[
L(f, x) = \{r \in \mathbb{R} ; (x, r) \in \overline{\text{Gr}f}\}
\]
(see in [Be1]).

Notice that \(f(x) \in L(f, x)\), it is the largest element of \(L(f, x)\) if \(f\) is u.s.c. and the smallest if \(f\) is lower semicontinuous (l.s.c.).

Denote by \(\Omega\) the space of all bounded u.s.c. functions from \(X\) to \(\mathbb{R}\) with convex \(L(f, x)\) for all \(x \in X\). If \(f \in \Omega\) and \(x \in X\) then \(L(f, x)\) is the closed line segment \(\lim \inf_{y \to x} f(y), f(x)\).

Let \(F\) be a multifunction from \(X\) to \(\mathbb{R}\). We say that \(F\) is upper semicontinuous at \(x_0 \in X\) if whenever \(V\) is an open subset of \(\mathbb{R}\) containing \(F(x_0)\), then \(V\) contains \(F(x)\) for each \(x\) in some neighborhood of \(x_0\). We say that
$F$ is bounded on $A \subset X$ provided that the set $F(A) = \bigcup\{F(x) ; x \in A\}$ is a bounded subset of $\mathbb{R}$. Then $F$ is locally bounded provided that each point of $X$ has a neighborhood on which $F$ is bounded [Ho]. We say that a multifunction $F$ from $X$ to $\mathbb{R}$ has a closed graph if the set $\{(x, y) ; y \in F(x)\}$ is a closed set in $X \times \mathbb{R}$.

Denote by $A$ the space of all nonempty valued locally bounded multifunction from $X$ to $\mathbb{R}$ with closed graphs.

Let $F \in A$. We define the functions $\alpha_F(x), \beta_F(x)$ as follows:

$$
\alpha_F(x) = \max\{F(x)\}, \quad \beta_F(x) = \min\{F(x)\}.
$$

**Remark 2.1.** It is easy to verify that if $F \in A$ then the function $\alpha_F$ is u.s.c. and the function $\beta_F$ is l.s.c.

Let $f \in \Omega$. The closure of the graph of $f$ in $X \times \mathbb{R}$ can be considered as a multifunction from $X$ to $\mathbb{R}$, which maps $x$ to $\{y \in \mathbb{R} ; (x, y) \in \overline{Grf}\}$. We denote this multifunction by $\overline{f}$. It is clear that if $f \in \Omega$ then $\overline{f} \in A$ and $\alpha_{\overline{f}}(x) = f(x)$. For every $x \in X$ we have that

$$
\overline{f}(x) = [\liminf_{y \to x} f(y), f(x)] = [\beta_{\overline{f}}(x), \alpha_{\overline{f}}(x)] = L(f, x).
$$

### 3. Main results

**Remark 3.1.** It is easy to see that if a function $f : X \to \mathbb{R}$ is locally bounded then $\overline{f}$ is upper semicontinuous multifunction. In [Be2] (Proposition 6.2.12) it is shown that if multifunction $F$ from $X$ to $\mathbb{R}$ is upper semicontinuous with connected values then for each connected subset $C$ of $X$ the image set $F(C)$ is connected.

The proof of the next lemma follows from Remark 3.1.

**Lemma 3.2.** Let $(X, d)$ be a connected metric space. Let $f$ be a locally bounded u.s.c. function from $X$ to $\mathbb{R}$, such that $L(f, x)$ is convex for all $x \in X$ and let there exist $x_1, x_2 \in X$ such that $\alpha_{\overline{f}}(x_1) < \beta_{\overline{f}}(x_2)$. Let $\alpha_{\overline{f}}(x_1) < a < \beta_{\overline{f}}(x_2)$. Then there exists $x_3 \in X$ such that $a \in \overline{f}(x_3)$.

Immediately from Lemma 3.2, we have the following two results.

**Proposition 3.3.** Let $(X, d)$ be a connected metric space. Let $f$ be a locally bounded u.s.c. function from $X$ to $\mathbb{R}$, such that $L(f, x)$ is convex for all $x \in X$. Then $\overline{f}(X)$ is a convex set in $\mathbb{R}$.

**Proposition 3.4.** Let $(X, d)$ be a connected metric space. Let $f \in \Omega$. Then $\overline{f}(X)$ is a bounded, convex set in $\mathbb{R}$.

**Proposition 3.5.** Let $(X, d)$ be a connected compact metric space. Let $f \in \Omega$. Then $\overline{f}(X)$ is a compact, convex set in $\mathbb{R}$.
**Proof.** By Proposition 3.4. it is sufficient to prove that \( \overline{f}(X) \) is a closed set in \( \mathbb{R} \). Let \( \{y_n; \ n \in \mathbb{Z}^+\} \) be a sequence in \( \overline{f}(X) \) convergent to \( y_0 \) in \( \mathbb{R} \). We claim that \( y_0 \in \overline{f}(X) \). For every \( y_n \) there is \( x_n \in X \) such that \( \beta_f(x_n) \leq y_n \leq \alpha_f(x_n) \). Since \( X \) is a compact by passing to a subsequence we can assume \( \{x_n; \ n \in \mathbb{Z}^+\} \) converges to some point \( x_0 \). Since \( \overline{f} \in \mathcal{A} \) by Remark 2.1. the function \( \alpha_f \) is u.s.c. and the function \( \beta_f \) is l.s.c. and thus \( \beta_f(x_0) \leq y_0 \leq \alpha_f(x_0) \) and thus \( y_0 \in \overline{f}(X) \). \( \square \)

**Theorem 3.6.** Let \( (X, d) \) be a locally connected metric space. Then \( \Omega \) is a closed subspace of \( (U(X), h) \).

**Proof.** Let \( f \) be in the closure of \( \Omega \) in \( (U(X), h) \). It is easy to show that \( f \) is bounded. Fix \( x \in X \). We claim that \( L(f, x) \) is convex. If not, there exists \( a \in \mathbb{R} \) where \( \liminf_{y \to x} f(y) < a < f(x) \) such that \( (x, a) \notin \overline{Grf} \). Since \( \overline{Grf} \) is closed set in \( (X \times \mathbb{R}, \rho) \) then there exist \( \epsilon > 0 \) such that \( S_f(x, a) \cap \overline{Grf} = \emptyset \). Let \( C \) be a connected neighborhood of \( x \) in \( S_f(x, \epsilon) \). Let \( \delta > 0 \) be such that \( S_{\delta}(x) \subset C \). Consider the open neighborhood \( S_{\delta}(f) \) of \( f \) in \( (U(X), h) \). There is \( g \in \Omega \) such that \( g \in S_{\delta}(f) \). Since \( h(f, g) < \delta \) and since Hausdorff distance is generated by \( \rho \) there exist points \( (x_1, y_1) \in \overline{Grg} \) and \( (x_2, y_2) \in \overline{Grg} \) such that \( \rho((x, \beta_f(x)), (x_1, y_1)) < \delta \) and \( \rho((x, \alpha_f(x)), (x_2, y_2)) < \delta \). Thus \( d(x, x_1) < \delta \) and \( d(x, x_2) < \delta \). Thus \( x_1 \in C \) and \( x_2 \in C \). It is easy to see that \( \alpha_f(x_1) < a \) and \( \beta_f(x_2) > a \). Otherwise we get a contradiction of the fact that \( g \in S_{\delta}(f) \) is in \( (U(X), h) \). By Lemma 3.2. there exists \( x_3 \in X \) such that \( a \in \overline{g}(x_3) \). It follows that \( \rho((x_3, a), \overline{Grf}) > \frac{\epsilon}{2} \). Since \( \delta < \frac{\epsilon}{2} \) we have a contradiction of the fact that \( h(f, g) < \delta \). \( \square \)

Denote by \( \Omega_0 \) the set of all functions belonging to \( \Omega \) and vanishing at infinity. By using of the previous Theorem we have the following Proposition.

**Proposition 3.7.** Let \( (X, d) \) be a locally connected metric space. Then \( \Omega_0 \) is a closed subspace of \( (U_0(X), h) \).

**Remark 3.8.** It is easy to show that if \( A \) is a subset of \( X \) and \( f, g \) are functions from \( X \) to \( \mathbb{R} \) then the inequalities \( h(f \mid A, g \mid A) < \epsilon \) and \( h(f \mid (X \setminus A), g \mid (X \setminus A)) < \epsilon \) implies the inequality \( h(f, g) < \epsilon \).

**Theorem 3.9.** Let \( (X, d) \) be a metric space. Let \( f \in \Omega_0 \) and let \( \{f_n; \ n \in \mathbb{Z}^+\} \) be a decreasing sequence of functions belonging to \( U_0(X) \) convergent pointwise to \( f \). Then \( \{f_n; \ n \in \mathbb{Z}^+\} \) \( h \)-converges to \( f \).

**Proof.** We start the proof similarly as the proof of Theorem 1 in [HN]. The functions \( f \) and \( f_1 \) are vanishing at infinity. Then for \( \epsilon > 0 \) there is a compact set \( K \subset X \) such that \( |f(x)| < \frac{\epsilon}{2} \) if \( x \notin K \) and \( |f(x)| < \frac{\epsilon}{2} \) if
Since \( \{f_n; \, n \in \mathbb{Z}^+\} \) is decreasing we have for each \( n \in \mathbb{Z}^+ \) and each \( x \in X \setminus K \)
\[
|f_n(x) - f(x)| \leq |f_1(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Then by Theorem A in [Be1] \( h(f_n \upharpoonright (X \setminus K), f \upharpoonright (X \setminus K)) < \epsilon \) for each \( n \in \mathbb{Z}^+ \). By Theorem 1 in [Be1] there is \( n_0 \) such that \( h(f_n \upharpoonright K, f \upharpoonright K) < \epsilon \) for all \( n > n_0 \). Then by using Remark 3.8. we have \( h(f_n, f) < \epsilon \) for all \( n > n_0 \).

Proposition 3.7. and Theorem 3.9. show that \( \Omega_0 \) is a Dini’s system of functions induced by Hausdorff metric topology on \( U_0(X) \).

**References**


