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ON UPPER DINI'S SYSTEMS AND U.S.C. FUNCTIONS WITH CONVEX LIMIT SETS

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ABSTRACT. We give an answer to the question in [HN] as to which upper Dini's systems of functions induces a Hausdorff metric topology on $U_0(X)$. We show that if X is a locally connected metric space then the Hausdorff metric topology on $U_0(X)$ induces as an upper Dini's system of functions the set of all bounded upper semicontinuous functions vanishing at infinity with convex limit sets.

1. INTRODUCTION

For a topological space X denote by $C(X)$, $U(X)$, the space of all continuous, upper semicontinuous (u.s.c.) real functions on X, respectively. A function $f : X \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of all real numbers) is said to be vanishing at infinity, if for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that $|f(x)| < \epsilon$ for every $x \notin K_\epsilon$ [HN]. By $C_0(X)$ and $U_0(X)$ we denote the space of all continuous functions vanishing at infinity and of all upper semicontinuous functions vanishing at infinity, respectively. The Dini's theorem [R] says that if X is a compact space and $\{f_n;\ n \in \mathbb{Z}^+\}$ (where $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$) a decreasing sequence of functions belonging to $U(X)$ pointwise converges to $f \in C(X)$ then the sequence $\{f_n;\ n \in \mathbb{Z}^+\}$ is uniformly convergent to f.

Definition 1.1. [Be1] *Let X be a compact metric space and $\Omega \subset U(X)$. If τ is a topology on $U(X)$, then Ω is called a Dini's class of functions induced by τ if*

(1) Ω *is τ-closed,*
(2) $C(X) \subset \Omega$,

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(3) *for each* $f \in \Omega$ *whenever* $\{f_n;\ n \in \mathbb{Z}^+\}$ *is a decreasing sequence of functions belonging to* $U(X)$ *and pointwise converges to* f *then* $\{f_n;\ n \in \mathbb{Z}^+\}$ τ*-converges to* f.

In this sense, if X is a compact space then $C(X)$ as a subclass of $U(X)$ is a Dini's class of functions induced by the topology of the uniform convergence on $U(X)$.

Beer in [Be1] showed that if X is a locally connected compact metric space then the Hausdorff metric topology on $U(X)$ obtained by identifying each u.s.c. function with the closure of its graph induces as a Dini's class of functions the set of all bounded u.s.c. functions with convex limit sets. The limit set for a function f from X to $\mathbb{R}$ at $x \in X$ is the set of all real r such that (x, r) belongs to the closure of the graph of f. There exists at most one Dini's class Ω induced by τ [Be1].

In [HN] a type of Dini's theorem for spaces that are not necessarily compact is presented.

Theorem A. [HN] *Let* X *be a topological space. Let* $\{f_n;\ n \in \mathbb{Z}^+\}$ *be a decreasing sequence of functions belonging to* $U_0(X)$ *and converging pointwise to a function* $f \in C_0(X)$. *Then the convergence is uniform.*

In [HN] is presented a more general definition of Dini's class (Dini's system).

Definition 1.2. [HN] *Let* X *be a topological space. Let* $\mathcal{F} \subset U(X)$ *and* τ *be a topology on* $\mathcal{F}$. *A collection* $\Omega \subset \mathcal{F}$ *is called an upper Dini's system for* $\mathcal{F}$ *induced by the topology* τ, *if the following are satisfied:*

(1) Ω *is* τ*-closed,*
(2) $\Omega \subset \mathcal{F}$,
(3) *If* $\{f_n;\ n \in \mathbb{Z}^+\}$ *is a decreasing sequence of functions belonging to* $\mathcal{F}$ *and pointwise converging to* $f \in \Omega$ *then* $\{f_n;\ n \in \mathbb{Z}^+\}$ τ*-converges to* f.

In [HN] the question is posed as to which upper Dini's system of functions induces a Hausdorff metric topology on $U_0(X)$. We show that if X is a locally connected metric space, then the Hausdorff metric topology on $U_0(X)$ induces as an upper Dini's system of functions the set of all bounded u.s.c. functions vanishing at infinity with convex limit sets.

If X is a perfectly normal T_1-space and $\mathcal{F} \subset U(X)$ and τ is a topology on $\mathcal{F}$, then there exists at most one upper Dini's system Ω for $\mathcal{F}$ induced by the topology τ such that $\Omega \supset C(X)$ [HN]. Similarly for functions vanishing at infinity we have [HN]: If X is a perfectly normal locally compact Hausdorff space, $\mathcal{F} \subset U_0(X)$ and τ is a topology on $\mathcal{F}$, then there exists at most one upper Dini's system Ω for $\mathcal{F}$ induced by the topology τ such that $\Omega \supset C_0(X)$.

2. Preliminaries

(Z, d) will denote a metrizable space Z with a compatible metric d. The open d–ball with the center $z_0 \in Z$ and a radius $\epsilon > 0$ will be denoted by $S_\epsilon(z_0)$ and the ϵ–parallel body $\cup\{S_\epsilon(a);\ a \in A\}$ for a subset A of Z will be denoted by $S_\epsilon(A)$. We denote by 2^Z the space of all closed subsets of Z and by $CL(Z)$ the space of all nonempty closed subsets of Z. The symbol $\overline{B}$ will stand for the closure of $B \subset Z$. If $A \in CL(Z)$, the distance functional

$$d(., A)\ :\ Z \mapsto\ [0, \infty)$$

is described by the familiar formula:

$$d(z, A) = \inf\{d(z, a);\ a \in A\}.$$

The Hausdorff metric H_d on 2^Z is defined as follows:

$$H_d(A, B) = \max\{\sup\{d(a, B);\, a \in A\},\, \sup\{d(b, A);\, b \in B\}\}$$

if A and B are nonempty. If $A \neq \emptyset$ we take $H_d(A, \emptyset) = H_d(\emptyset, A) = \infty$. On $CL(Z)$ we can use for Hausdorff distance the following equality:

$$H_d(A, B) = \inf\{\epsilon > 0;\ A \subset S_\epsilon(B) \text{ and } B \subset S_\epsilon(A)\}.$$

H_d defines an (extended-valued) metric on 2^Z. The generated topology is called the Hausdorff metric topology.

Now let (X, d) be a metric space, consider the product $X \times \mathbb{R}$ metrized in the following way (box metric).

$$\rho((x_1, r_1), (x_2, r_2)) = \max\{d(x_1, x_2), |r_1 - r_2|\}.$$

Denote the closure of the graph of the function f: $X \mapsto \mathbb{R}$ by $\overline{Grf}$ and the restriction of f to a subset A of X by $f \restriction A$.

If f and g are in $U(X)$ denote the Hausdorff distance generated by ρ from $\overline{Grf}$ to $\overline{Grg}$ by $h(f, g)$. Thus h is an extended valued metric on $U(X)$ [Be1].

In what follows X will be a metric space with a compatible metric d. Let f be a function from X to $\mathbb{R}$. For each $x \in X$ let:

$$L(f, x) = \{r \in \mathbb{R};\ (x, r) \in \overline{Grf}\}$$

(see in [Be1]).

Notice that $f(x) \in L(f, x)$, it is the largest element of $L(f, x)$ if f is u.s.c. and the smallest if f is lower semicontinuous (l.s.c.).

Denote by Ω the space of all bounded u.s.c. functions from X to $\mathbb{R}$ with convex $L(f, x)$ for all $x \in X$. If $f \in \Omega$ and $x \in X$ then $L(f, x)$ is the closed line segment $[\liminf_{y \mapsto x} f(y), f(x)]$.

Let F be a multifunction from X to $\mathbb{R}$. We say that F is upper semicontinuous at $x_0 \in X$ if whenever V is an open subset of $\mathbb{R}$ containing $F(x_0)$, then V contains $F(x)$ for each x in some neighborhood of x_0. We say that

F is bounded on $A \subset X$ provided that the set $F(A) = \cup\{F(x);\ x \in A\}$ is a bounded subset of $\mathbb{R}$. Then F is locally bounded provided that each point of X has a neighborhood on which F is bounded [Ho]. We say that a multifunction F from X to $\mathbb{R}$ has a closed graph if the set $\{(x, y);\ y \in F(x)\}$ is a closed set in $X \times \mathbb{R}$.

Denote by $\mathcal{A}$ the space of all nonempty valued locally bounded multifunction from X to $\mathbb{R}$ with closed graphs.

Let $F \in \mathcal{A}$. We define the functions $\alpha_F(x), \beta_F(x)$ as follows:

$$\alpha_F(x) = \max\{F(x)\},\ \beta_F(x) = \min\{F(x)\}.$$

Remark 2.1. It is easy to verify that if $F \in \mathcal{A}$ then the function α_F is u.s.c. and the function β_F is l.s.c..

Let $f \in \Omega$. The closure of the graph of f in $X \times \mathbb{R}$ can be considered as a multifunction from X to $\mathbb{R}$, which maps x to $\{y \in \mathbb{R};\ (x, y) \in \overline{Grf}\}$. We denote this multifunction by $\overline{f}$. It is clear that if $f \in \Omega$ then $\overline{f} \in \mathcal{A}$ and $\alpha_{\overline{f}}(x) = f(x)$. For every $x \in X$ we have that

$$\overline{f}(x) = [\liminf_{y \mapsto x} f(y), f(x)] = [\beta_{\overline{f}}(x), \alpha_{\overline{f}}(x)] = L(f, x).$$

3. Main results

Remark 3.1. It is easy to see that if a function $f : X \to \mathbb{R}$ is locally bounded then $\overline{f}$ is upper semicontinuous multifunction. In [Be2] (Proposition 6.2.12) it is shown that if multifunction F from X to $\mathbb{R}$ is upper semicontinuous with connected values then for each connected subset C of X the image set $F(C)$ is connected.

The proof of the next lemma follows from Remark 3.1.

Lemma 3.2. *Let (X, d) be a connected metric space. Let f be a locally bounded u.s.c. function from X to $\mathbb{R}$, such that $L(f, x)$ is convex for all $x \in X$ and let there exist $x_1, x_2 \in X$ such that $\alpha_{\overline{f}}(x_1) < \beta_{\overline{f}}(x_2)$. Let $\alpha_{\overline{f}}(x_1) < a < \beta_{\overline{f}}(x_2)$. Then there exists $x_3 \in X$ such that $a \in \overline{f}(x_3)$.*

Immediately from Lemma 3.2. we have the following two results.

Proposition 3.3. *Let (X, d) be a connected metric space. Let f be a locally bounded u.s.c. function from X to $\mathbb{R}$, such that $L(f, x)$ is convex for all $x \in X$. Then $\overline{f}(X)$ is a convex set in $\mathbb{R}$.*

Proposition 3.4. *Let (X, d) be a connected metric space. Let $f \in \Omega$. Then $\overline{f}(X)$ is a bounded, convex set in $\mathbb{R}$.*

Proposition 3.5. *Let (X, d) be a connected compact metric space. Let $f \in \Omega$. Then $\overline{f}(X)$ is a compact, convex set in $\mathbb{R}$.*

Proof. By Proposition 3.4. it is sufficient to prove that $\overline{f}(X)$ is a closed set in $\mathbb{R}$. Let $\{y_n;\ n \in Z^+\}$ be a sequence in $\overline{f}(X)$ convergent to y_0 in $\mathbb{R}$. We claim that $y_0 \in \overline{f}(X)$. For every y_n there is $x_n \in X$ such that $\beta_{\overline{f}}(x_n) \leq y_n \leq \alpha_{\overline{f}(x_n)}$. Since X is a compact by passing to a subsequence we can assume $\{x_n;\ n \in Z^+\}$ converges to a some point x_0. Since $\overline{f} \in \mathcal{A}$ by Remark 2.1. the function $\alpha_{\overline{f}}$ is u.s.c. and the function $\beta_{\overline{f}}$ is l.s.c. and thus $\beta_{\overline{f}}(x_0) \leq y_0 \leq \alpha_{\overline{f}}(x_0)$ and thus $y_0 \in \overline{f}(X)$. □

Theorem 3.6. *Let (X, d) be a locally connected metric space. Then Ω is a closed subspace of $(U(X), h)$.*

Proof. Let f be in the closure of Ω in $(U(X), h)$. It is easy to show that f is bounded. Fix $x \in X$. We claim that $L(f, x)$ is convex. If not, there exists $a \in \mathbb{R}$ where $\liminf_{y \mapsto x} f(y) < a < f(x)$ such that $(x, a) \notin \overline{Grf}$. Since $\overline{Grf}$ is closed set in $(X \times \mathbb{R}, \rho)$ then there exist $\epsilon > 0$ such that $S_\epsilon(x, a) \cap \overline{Grf} = \emptyset$. Let C be a connected neighborhood of x in $S_{\frac{\epsilon}{2}}(x)$. Let $\delta > 0$ be such that $S_\delta(x) \subset C$. Consider the open neighborhood $S_\delta(f)$ of f in $(U(X), h)$. There is $g \in \Omega$ such that $g \in S_\delta(f)$. Since $h(f, g) < \delta$ and since Hausdorff distance is generated by ρ there exist points $(x_1, y_1) \in \overline{Grg}$ and $(x_2, y_2) \in \overline{Grg}$ such that $\rho((x, \beta_{\overline{f}}(x)), (x_1, y_1)) < \delta$ and $\rho((x, \alpha_{\overline{f}}(x)), (x_2, y_2)) < \delta$. Thus $d(x, x_1) < \delta$ and $d(x, x_2) < \delta$. Thus $x_1 \in C$ and $x_2 \in C$. It is easy to see that $\alpha_{\overline{g}}(x_1) < a$ and $\beta_{\overline{g}}(x_2) > a$. Otherwise we get a contradiction of the fact that $g \in S_\delta(f)$ is in $(U(X), h)$. By Lemma 3.2. there exists $x_3 \in X$ such that $a \in \overline{g}(x_3)$. It follows that $\rho((x_3, a), \overline{Grf}) > \frac{\epsilon}{2}$ where $\rho((x_3, a), \overline{Grf}) = \inf\{\rho((x_3, a), (x, y));\ (x, y) \in \overline{Grf}\}$. Since $\delta < \frac{\epsilon}{2}$ we have a contradiction of the fact that $h(f, g) < \delta$. □

Denote by Ω_0 the set of all functions belonging to Ω and vanishing at infinity. By using of the previous Theorem we have the following Proposition.

Proposition 3.7. *Let (X, d) be a locally connected metric space. Then Ω_0 is a closed subspace of $(U_0(X), h)$.*

Remark 3.8. It is easy to show that if A is a subset of X and f, g are functions from X to $\mathbb{R}$ then the inequalities $h(f \restriction A, g \restriction A) < \epsilon$ and $h(f \restriction (X \setminus A), g \restriction (X \setminus A)) < \epsilon$ implies the inequality $h(f, g) < \epsilon$.

Theorem 3.9. *Let (X, d) be a metric space. Let $f \in \Omega_0$ and let $\{f_n;\ n \in \mathbb{Z}^+\}$ be a decreasing sequence of functions belonging to $U_0(X)$ convergent pointwise to f. Then $\{f_n;\ n \in \mathbb{Z}^+\}$ h-converges to f.*

Proof. We start the proof similarly as the proof of Theorem 1 in [HN]. The functions f and f_1 are vanishing at infinity. Then for $\epsilon > 0$ there is a compact set $K \subset X$ such that $\mid f(x) \mid < \frac{\epsilon}{2}$ if $x \notin K$ and $\mid f_1(x) \mid < \frac{\epsilon}{2}$ if

$x \notin K$. Since $\{f_n;\ n \in \mathbb{Z}^+\}$ is decreasing we have for each $n \in \mathbb{Z}^+$ and each $x \in X \setminus K$

$$| f_n(x) - f(x) | \leq | f_1(x) - f(x) | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then by Theorem A in [Be1] $h(f_n \restriction (X \setminus K), f \restriction (X \setminus K)) < \epsilon$ for each $n \in \mathbb{Z}^+$. By Theorem 1 in [Be1] there is n_0 such that $h(f_n \restriction K, f \restriction K) < \epsilon$ for all $n > n_0$. Then by using Remark 3.8. we have $h(f_n, f) < \epsilon$ for all $n > n_0$. □

Proposition 3.7. and Theorem 3.9. show that Ω_0 is a Dini's system of functions induced by Hausdorff metric topology on $U_0(X)$.

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