SARAJEVO JOURNAL OF MATHEMATICS
Vol.2 (15) (2006), 237–245

SOME MODIFICATIONS OF THE TRAPEZOIDAL RULE

ZLATKO UDOVIČIĆ

ABSTRACT. By using the quadratic interpolating spline a new class of the quadrature rules was obtained. Those formulas are modifications of the well known trapezoidal rule. The basic characteristic of those formulas is a free parameter. With appropriate choice of that parameter, accuracy of the trapezoidal rule can be improved up to $O(h^4)$. Besides this, by using this nonstandard techniques some well known quadrature rules were also obtained.

1. INTRODUCTION AND PRELIMINARIES

1.1. **Classical quadrature rules.** Classical quadrature rules are studied at basic courses in numerical analysis and they are here just used to provide an system of notation.

Let $a = x_0 < x_1 < \cdots < x_{2m} = b$, where $x_k = x_0 + kh$ and let $f_k = f(x_k)$, $0 \le k \le 2m$. Then

$$\int_a^b f(x)dx = Q_R[f,a,b,2m] + E_R[f,a,b,2m],$$

or

$$\int_a^b f(x)dx = Q_S[f,a,b,2m] + E_S[f,a,b,2m],$$

where

$$Q_R[f,a,b,2m] = 2h\sum_{k=1}^{m} f_{2k-1} \tag{1}$$

is a generalized rule for the central rectangles, while

$$Q_S[f,a,b,n] = \frac{h}{3}\left(f_0 + 2\sum_{k=1}^{m-1} f_{2k} + 4\sum_{k=1}^{m} f_{2k-1} + f_{2m}\right) \tag{2}$$

is a generalized Simpsons rule. Errors of those formulas are respectively:

$$E_R[f,a,b,2m] = \frac{(b-a)f''(\overline{\xi}_R)}{6}\,h^2, \tag{3}$$

2000 *Mathematics Subject Classification.* 65D32, 65D07.

for some $\overline{\xi}_R \in [a,b]$, supposing $f \in C^2[a,b]$ and

$$E_S[f,a,b,2m] = -\frac{(b-a)f^{iv}(\overline{\xi}_S)}{180}\,h^4, \tag{4}$$

for some $\overline{\xi}_S \in [a,b]$, supposing $f \in C^4[a,b]$. Furthermore, if $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\int_a^b f(x)dx = Q_T[f,a,b,n] + E_T[f,a,b,n],$$

where

$$Q_T[f,a,b,n] = \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{n-1} f_k + f_n\right)$$

is the generalized trapezoidal rule and

$$E_T[f,a,b,n] = -\frac{(b-a)f''(\overline{\xi}_T)}{12}\,h^2,$$

for some $\overline{\xi}_T \in [a,b]$, supposing $f \in C^2[a,b]$, is error for this rule.

1.2. **Quadratic interpolating spline.** Since that quadratic interpolating spline is usually avoided in the literature, basic details about it's construction are presented here (the cubic spline is most often considered, see [1, 3]).

Definition 1. *Let* $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *be partition of the interval* $[a,b]$. *The function* $S_\Delta : [a,b] \to \mathbb{R}$, *with the following properties:*

1. S_Δ *is a polynomial of the second degree in every subinterval* $[x_k, x_{k+1}]$, $0 \le k \le n-1$ *and*
2. $S_\Delta \in C^1[a,b]$

is called the quadratic spline in relation to the partition Δ. *If the interpolating properties*

- $S_\Delta(x_k) = f(x_k), 0 \le k \le n$

are added to the previous definition, the quadratic interpolating spline of the function f *will be obtained, usually denoted by* $S_{\Delta,f}$.

Let the values $f_k = f(x_k)$, $0 \le k \le n$, of the function f, be given. From the property 1. of the Definition 1, it follows that

$$x \in [x_k, x_{k+1}] \Rightarrow S_{\Delta,f}(x) = a_k x^2 + b_k x + c_k = s_k(x),\ 0 \le k \le n-1,$$

which means that there are $3n$ unknown coefficients. To determine those coefficients $2n$ equations will be obtained from the interpolating properties, while $n-1$ equations will be obtained from property 2. of the same definition.

So, there is one parameter which can be chosen freely. The mentioned equations are:

$$a_k x_k^2 + b_k x_k + c_k = f_k,\ 0 \le k \le n-1, \tag{5}$$

$$a_k x_{k+1}^2 + b_k x_{k+1} + c_k = f_{k+1},\ 0 \le k \le n-1, \tag{6}$$

$$2a_k x_{k+1} + b_k = 2a_{k+1} x_{k+1} + b_{k+1},\ 0 \le k \le n-2. \tag{7}$$

After subtracting equation (5) from equation (6) we obtain:

$$b_k = \frac{f_{k+1} - f_k}{x_{k+1} - x_k} - a_k(x_{k+1} + x_k).$$

The last equation, together with equation (7), after elementary calculation gives the recurrence formulas for calculating the coefficients $a_k, 1 \le k \le n-1$. Hence, the coefficients of the quadratic interpolating spline can be calculated in the following way:

$$\begin{aligned}
&a_0 - \text{arbitrary},\\
&a_{k+1} = \frac{f_{k+2} - f_{k+1}}{(x_{k+2} - x_{k+1})^2} - \frac{f_{k+1} - f_k}{(x_{k+2} - x_{k+1})(x_{k+1} - x_k)}\\
&\qquad\qquad - a_k \frac{x_{k+1} - x_k}{x_{k+2} - x_{k+1}},\ 0 \le k \le n-2,\\
&b_k = \frac{f_{k+1} - f_k}{x_{k+1} - x_k} - a_k(x_{k+1} + x_k),\ 0 \le k \le n-1,\\
&c_k = f_k - a_k x_k^2 - b_k x_k,\ 0 \le k \le n-1.
\end{aligned} \tag{8}$$

By using the formulas (8), again after elementary calculations, the basic modification of the trapezoidal rule will be obtained:

$$\begin{aligned}
\int_{x_k}^{x_{k+1}} f(x)dx &= \int_{x_k}^{x_{k+1}} s_k(x)dx + E[f, x_k, x_{k+1}]\\
&= (x_{k+1} - x_k)\left[\frac{1}{2}(f_{k+1} + f_k)\right.\\
&\qquad \left. - \frac{a_k}{6}(x_{k+1} - x_k)^2\right] + E[f, x_k, x_{k+1}].
\end{aligned}$$

If the partition of the interval $[a, b]$ is uniform, i.e. if $x_k = x_0 + kh, 0 \le k \le n$, where $x_0 = a$ and $h = \frac{b-a}{n}$, then the first relation in (8) and the previous relation becomes much more simple:

$$\begin{aligned}
a_k &= \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2} - a_{k-1}\\
&= \sum_{i=1}^{k} (-1)^{k-i} \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + (-1)^k a_0,\ 1 \le k \le n-1
\end{aligned} \tag{9}$$

and

$$\int_{x_k}^{x_{k+1}} f(x)dx = \frac{h}{2}(f_k + f_{k+1}) - \frac{h^3}{6}a_k + E[f, x_k, x_{k+1}],\ 0 \le k \le n-1. \quad (10)$$

Generalized formula (10) is

$$\int_a^b f(x)dx = Q_T[f, a, b, n] - \frac{h^3}{6}\sum_{k=0}^{n-1} a_k + E[f, a, b, n]. \quad (11)$$

2. Basic results

2.1. The case $n = 2m$.

2.1.1. *Simpsons rule.* From the relations (9) and (11) it follows

$$\int_a^b f(x)dx = Q[f, a, b, 2m] + E[f, a, b, 2m],$$

where

$$\begin{aligned} Q[f, a, b, 2m] &= \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{2m-1} f_k + f_{2m}\right) - \frac{h^3}{6}\sum_{k=0}^{2m-1} a_k \\ &= \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{2m-1} f_k + f_{2m}\right) \\ &\quad - \frac{h^3}{6}\sum_{k=1}^{m} \frac{f_{2k} - 2f_{2k-1} + f_{2k-2}}{h^2} \\ &= Q_S[f, a, b, 2m]. \end{aligned} \quad (12)$$

Now from (4) it follows that

$$E[f, a, b, 2m] = E_S[f, a, b, 2m] = -\frac{(b-a)f^{iv}(\overline{\xi}_S)}{180}h^4,$$

for some $\overline{\xi}_S \in [a, b]$, assuming $f \in C^4[a, b]$.

2.1.2. *The first modification.* If the equation

$$\frac{f_{2k} - 2f_{2k-1} + f_{2k-2}}{h^2} = f''_{2k-1} + \frac{f^{iv}(\tau_k)}{12}h^2,$$

(for some $\tau_k \in [x_{2k-2}, x_{2k}]$, assuming $f \in C^4[a, b]$) is put in (12), the first modification of the trapezoidal rule will be obtained

$$\int_a^b f(x)dx = Q_1[f, a, b, 2m] + E_1[f, a, b, 2m],$$

where

$$Q_1[f,a,b,2m] = \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{2m-1} f_k + f_{2m}\right) - \frac{h^3}{6}\sum_{k=1}^{m} f''_{2k-1} \tag{13}$$

and

$$\begin{aligned} E_1[f,a,b,2m] &= -\frac{(b-a)f^{iv}(\overline{\xi}_S)}{180}h^4 - \frac{h^3}{6}\sum_{k=1}^{m}\frac{h^2}{12}f^{iv}(\tau_k) \\ &= -\frac{(b-a)f^{iv}(\overline{\xi}_S)}{180}h^4 - \frac{(b-a)f^{iv}(\overline{\tau})}{144}h^4 \\ &= -\frac{(b-a)f^{iv}(\overline{\tau}_1)}{80}h^4, \end{aligned}$$

for some $\overline{\tau}_1 \in [a,b]$.

2.1.3. *The second modification.* Formulas (1) and (3) together give:

$$f'_{2m} - f'_0 = \int_{x_0}^{x_{2m}} f''(x)dx = 2h\sum_{k=1}^{m} f''_{2k-1} + \frac{(b-a)f^{iv}(\overline{\xi}_R)}{6}h^2,$$

and from this relation follows the second modification of the trapezoidal rule (well known Ermits quadrature rule, for example see [2, 3]). Hence:

$$\int_a^b f(x)dx = Q_2[f,a,b,2m] + E_2[f,a,b,2m],$$

where

$$Q_2[f,a,b,2m] = \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{2m-1} f_k + f_{2m}\right) - \frac{h^2}{12}(f'_{2m} - f'_0) \tag{14}$$

and

$$\begin{aligned} E_2[f,a,b,2m] &= \frac{(b-a)f^{iv}(\overline{\xi}_R)}{72}h^4 - \frac{(b-a)f^{iv}(\overline{\tau}_1)}{80}h^4 \\ &= \frac{b-a}{720}\big(10f^{iv}(\overline{\xi}_R) - 9f^{iv}(\overline{\tau}_1)\big)h^4, \end{aligned} \tag{15}$$

for some $\overline{\xi}_R, \overline{\tau}_1 \in [a,b]$.

The estimation of the error (15) is much less exact then the estimation which can be found in [2], but these estimations becomes equal if we assume that $10f^{iv}(\overline{\xi}_R) - 9f^{iv}(\overline{\tau}_1) \in [m_4, M_4]$, where $m_4 = \min_{x\in[a,b]} f^{iv}(x)$ and $M_4 = \max_{x\in[a,b]} f^{iv}(x)$.

One can check that both of formulas (13) and (14) are exact for all polynomials of the third degree.

2.2. **The case $n = 2m + 1$.** Combining the formulas (2) and (10) gives

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_{2m+1}} f(x)dx$$
$$= Q_T[f, x_0, x_1] - \frac{h^3}{6}a_0 + Q_S[f, x_1, b, 2m]$$
$$+ E[f, a, b, 2m+1], \qquad (16)$$

where $Q_T[f, x_0, x_1] = \frac{h}{2}(f_0 + f_1)$, and parameter a_0 can be chosen arbitrarily. In general, accuracy of the previous rule is $O(h^3)$, but it can be improved up to $O(h^4)$ with an appropriate choice of the parameter a_0. One of the possibilities is to choose the parameter a_0 to be equal to the main part of the reminder $E_T[f, x_0, x_1] = -\dfrac{f''(\xi_T)}{12}h^3, \xi_T \in [x_0, x_1]$.

2.2.1. *The first modification.* For every $\lambda \in [0, 1]$, there exists (at least one) $X \in [x_0, x_1]$ such that $f''(X) = \lambda f''_0 + (1-\lambda)f''_1$. Then

$$f''(\xi_T) = f''(X) + f'''(\theta)(\xi_T - X), \theta \in [\xi_T, X],$$

so that with the choice $a_0 = \dfrac{f''(X)}{2}$ we get:

$$\int_a^b f(x)dx = Q_1[f, a, b, 2m+1] + E_1[f, a, b, 2m+1],$$

where

$$Q_1[f, a, b, 2m+1] = \frac{h}{2}(f_0 + f_1) - \frac{h^3 f''(X)}{12} + Q_S[f, x_1, b, 2m] \qquad (17)$$

and

$$E_1[f, a, b, 2m+1] = -\frac{f'''(\theta)}{12}(\xi_T - X)h^3 - \frac{(b - x_1)f^{iv}(\overline{\xi}_S)}{180}h^4,$$

for some $\xi_T, X \in [x_0, x_1], \theta \in [\xi_T, X], \overline{\xi}_S \in [x_1, b]$.

The combination of the formulas (13) and (17) gives

$$\int_a^b f(x)dx = \widetilde{Q}_1[f, a, b, 2m+1] + \widetilde{E}_1[f, a, b, 2m+1],$$

where

$$\widetilde{Q}_1[f, a, b, 2m+1] = \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{2m} f_k + f_{2m+1}\right) - \frac{h^3}{6}\Big(\frac{f''(X)}{2} + \sum_{k=1}^{m} f''_{2k}\Big)$$

and

$$\widetilde{E}_1[f, a, b, 2m+1] = -\frac{f'''(\theta)}{12}(\xi_T - X)h^3 - \frac{(b - x_1)f^{iv}(\overline{\tau}_1)}{80}h^4$$

for some $\xi_T, X \in [x_0, x_1], \theta \in [\xi_T, X], \overline{\tau}_1 \in [x_1, b]$.

2.2.2. *The second modification.* Since we have

$$f''(\xi_T) = \frac{f_1' - f_0'}{h} - \frac{(x_1 - \xi_T)^2 f'''(\theta_1) - (x_0 - \xi_T)^2 f'''(\theta_0)}{2h},$$

for some $\theta_0 \in [x_0, \xi_T], \theta_1 \in [\xi_T, x_1]$, the choice $a_0 = \dfrac{f_1' - f_0'}{2h}$ gives

$$\int_a^b f(x)dx = Q_2[f, a, b, 2m+1] + E_2[f, a, b, 2m+1],$$

where

$$Q_2[f, a, b, 2m+1] = \frac{h}{2}(f_0 + f_1) - \frac{h^2}{12}(f_1' - f_0') + Q_S[f, x_1, b, 2m] \tag{18}$$

and

$$E_2[f, a, b, 2m+1] = \frac{(x_1 - \xi_T)^2 f'''(\theta_1) - (x_0 - \xi_T)^2 f'''(\theta_0)}{24} h^2 - \frac{(b - x_1) f^{iv}(\overline{\xi}_S)}{180} h^4,$$

for some $\theta_0 \in [x_0, \xi_T], \theta_1 \in [\xi_T, x_1], \overline{\xi}_S \in [x_1, b]$. Similarly as in the previous case, the formulas (14) and (18) together give

$$\int_a^b f(x)dx = \widetilde{Q}_2[f, a, b, 2m+1] + \widetilde{E}_2[f, a, b, 2m+1],$$

where

$$\widetilde{Q}_2[f, a, b, 2m+1] = \frac{h}{2}\left(f_0 + 2\sum_{k=1}^{2m} f_k + f_{2m+1}\right) - \frac{h^3}{12}(f_{2m+1}' - f_0')$$

(again Ermits quadrature rule) and

$$\widetilde{E}_2[f, a, b, 2m+1] = \frac{(x_1 - \xi_T)^2 f'''(\theta_1) - (x_0 - \xi_T)^2 f'''(\theta_0)}{24} h^2 - \frac{(b - x_1)h^4}{720}(10 f^{iv}(\overline{\xi}_R) - 9 f^{iv}(\overline{\tau}_1)),$$

for some $\theta_0 \in [x_0, \xi_T], \theta_1 \in [\xi_T, x_1], \overline{\xi}_R, \overline{\tau}_1 \in [x_1, b]$.

2.3. **Another possibility for the choice of the parameter a_0.** One can check that the parameter a_0 can not be chosen in a such way that formula (16) (without the last term) integrates exactly all polynomials with degree not greater than three. But, if $P(x) = Ax^3 + Bx^2 + Cx + D$, with the choice

$$a_0 = B + 3A(x_0 + \frac{h}{2}), \tag{19}$$

the formula (16) will be true for the polynomial P.

Furthermore, for every $X \in [x_0, x_1]$ one has

$$f(x) \approx f(X) + f'(X)(x - X) + \frac{1}{2} f''(X)(x - X)^2 + \frac{1}{6} f'''(X)(x - X)^3,$$

and in accordance with (19), with the choice

$$a_0 = \frac{1}{2}[f''(X) + f'''(X)(x_0 + \frac{h}{2} - X)],$$

the formula (16) (again without the last term) becomes exact for the "main part" of the function f. Hence,

$$\int_a^b f(x)dx = \overline{Q}_2[f, a, b, 2m + 1] + \overline{E}_2[f, a, b, 2m + 1].$$

In the last formula one has

$$\overline{Q}_2[f, a, b, 2m + 1] = Q_T[f, x_0, x_1] - \frac{h^3}{12}[f''(X) + f'''(X)(x_0 + \frac{h}{2} - X)] + Q_S[f, x_1, b, 2m],$$

while the order of accuracy is still $O(h^4)$, since

$$f''(\xi_T) = f''(X) + f'''(X)(\xi_T - X) + \frac{1}{2} f^{iv}(\theta)(\xi_T - X)^2,$$

and finally

$$\overline{E}_2 = -\frac{(\xi_T - x_0 - \frac{h}{2}) f'''(X) + \frac{1}{2}(\xi_T - X)^2 f^{iv}(\theta)}{12} h^3 - \frac{(b - x_1) f^{iv}(\overline{\xi}_S)}{180} h^4.$$

Acknowledgement. The author have to express his gratitude to the referee for very useful suggestions.

References

[1] K. C. Chui, *An Introduction to Wavelets*, Academic Press Inc., 1992.
[2] A. Quarteroni, R. Sacco and F. Saleri, *Numerical Mathematics*, Springer-Verlag, 2000.
[3] J. Stoer, *Numerische Mathematik 1*, Springer-Verlag, 1994.

(Received: March 9, 2006)
(Revised: July 2, 2006)

Faculty of Sciences
Department of Mathematics
Zmaja od Bosne 35
71000 Sarajevo
Bosnia and Herzegovina
E–mail: zzlatko@bih.net.ba