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FOURTH-ORDER BLOCK METHODS FOR THE NUMERICAL SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS

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Abstract. Block methods of order two and three for the numerical solution of initial value problems are extended to four order. The proposed two fourth order block methods might be efficient for implementation in multiprocessor computers. The matrix coefficients like block methods of order two and three of these methods are chosen so that lower powers of blocksize appear in the principle local truncation errors. The stability polynomial is shown to be a perturbation of the $(p+1)^{th}$ order explicit Runge-Kutta method, scaled according to block size. In order to show the linear stability properties of the block predictor corrector methods, the maximum absolute errors using Type I and Type II methods with blocksize $k = 10$ and various step sizes are investigated numerically.

1. Introduction

Numerical methods for parallel solution of the initial value problem (IVP) are well established techniques in literature. In the last two decades a number of papers have appeared on this topic (see, for example [1]-[14] and references there in). One such technique is the block method which by means of a single application of a calculation unit, yields a sequence of new estimates for y in the differentiation equation:

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

with $y, f \in R^s$.

If the block size $k \geq 1$, then in simple cases the values of t for which solutions are computed would be evenly separated. In other words, each basic cycle of calculation has the potential to advance the solution by k new points in the t direction. Each such block can therefore, be considered as a unit of calculation. Let y_n denote the approximation to the exact solution

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$y(t_n)$ at $t = t_n$ and let f_n denote the value of $f(t_n, y_n)$, the approximation for $y'(t_n)$. For $n = mk$, a block of solutions can be represented by the vector $Y_m = (y_{n+1}, y_{n+2}, \ldots, y_{n+k})^T$ with $y_{n+j}\,(1 \leq j \leq k)$, the generated solution for $t_{n+j} = t_n + jh$, where t_n is the right-hand end point of the preceding block and h is the uniform spacing between solution values. Adopting the notation of [7], the formula for the block method can be expressed as

$$Y_m = ey_n + hdf_n + hBF(Y_m) \tag{2}$$

where e and d are $k-$vectors, B is a $k \times k$ matrix, and F is a $k-$vector whose j^{th} entry is $f_{n+j} = f(t_{n+j}, y_{n+j}), 1 \leq j \leq k$. Eq. (2) is implicit in Y_m, it has to be solved by iteratively using, in the first instance, predicted solution values. A predictor equation for Y can be expressed in the form

$$Y_m^{(0)} = ey_n + h\widetilde{d}f_n \tag{3}$$

where e and $\widetilde{d}$ are $k-$vectors. Substitution of $Y_m^{(0)}$ into the right-hand side of Eq. (2) yields the block predictor-corrector (BPC) method

$$Y_m = ey_n + hdf_n + hBF\left(ey_n + h\widetilde{d}f_n\right). \tag{4}$$

In accordance with the terminology used in the linear multi-step case, this application is called the PEC mode. One can continue this process by substituting the result of Eq. (4) into the right-hand side of (2) arriving at $P(EC)^{\nu}E^{1-\gamma}$ mode, in which $\gamma = 0$ indicates that a final evaluation is done before proceeding to the next block. Voss and Abbas [4] consider this approach using an explicit Euler predictor and then correcting it four times by a Simpson corrector applied in the composition. This method can be computed in the following steps for each equidistant step point $r = 1, \ldots, k$ as

$$\begin{aligned}
y_{n+r}^{(0)} &= y_n + rhf(t_n, y_n),\\
y_{n+1}^{(1)} &= y_n + \frac{h}{24}\Big[9f(t_n, y_n) + 19f(t_{n+1}, y_{n+1}^{(0)}) - 5f(t_{n+2}, y_{n+2}^{(0)})\\
&\qquad\qquad\qquad\qquad\qquad\qquad + f(t_{n+3}, y_{n+3}^{(0)})\Big],\\
y_{n+r}^{(i)} &= y_n + \frac{h}{3}\left[f(t_n, y_n) + 4\sum_{h=1}^{r} f(t_{n+j}, y_{n+j}^{(0)}) + 2\sum_{j=1}^{r-1} f\left(t_{n+2j}, y_{n+2j}^{(0)}\right)\right.\\
&\qquad\qquad\left. + f(t_{n+2r}, y_{n+2r}^{(0)})\right], \quad r = 2, 4, 6, \ldots \text{(even)}
\end{aligned} \tag{5}$$

$$y_{n+3}^{(i)} = y_n + \frac{3h}{8}\Big[f(t_n, y_n) + 3f(t_{n+1}, y_{n+1}^{(0)}) + 3f(t_{n+2}, y_{n+2}^{(0)}) + f(t_{n+3}, y_{n+3}^{(0)})\Big],$$

$$
\begin{aligned}
y_{n+r}^{(i)} = y_n + \frac{h}{3}\left[f(t_n, y_n) + 4\sum_{j=1}^{r} f(t_{n+j}, y_{n+j}^{(0)}) + 2\sum_{j=1}^{r-1} f(t_{n+2j}, y_{n+2j}^{(0)})\right] \\
+ \frac{17h}{24} f(t_{n+(r-3)}, y_{n+(r-2)}^{(0)}) + \frac{3h}{8}\Big[3f(t_{n+(r-2)}, y_{n+(r-2)}^{(0)}) \\
+ 3f(t_{n+(r-1)}, y_{n+(r-1)}^{(0)})\Big], \quad r = 5, 7, \ldots \text{(odd)};\ i = 1, 2, 3, 4.
\end{aligned} \tag{6}
$$

With $Y_m^{(0)}$ given by Eq. (3), method (5) has $P(EC)^4$ form $Y_m^{(0)} = ey_n + h\tilde{d}f_n$,

$$Y_m^{1+1} = ey_n + hdf_n + hBF(Y_m^l),\ \ l = 0, 1, 2 \text{ and } 3$$

$$
\begin{aligned}
e^T &= (1, 1, 1, \ldots, 1) \\
\tilde{d}^T &= (1, 2, 3, \ldots, k) \\
d^T &= (\frac{3}{8}, \frac{1}{3}, \frac{3}{8} \ldots, \frac{1}{3})
\end{aligned}
$$

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{19}{24} & \frac{-5}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 \\
\frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\
\frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 \\
\frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3}
\end{pmatrix}. \tag{7}
$$

The local error of Eq. (6) has the form $\frac{-kh^5}{180}\, y^{(5)}(t_n) + O\left(h^6\right)$. (7)

The main purpose of this paper is to form fourth order block methods for solving IVP's which are appropriate for an arbitrary blocksize $k \geq 1$. Additionally, the knowledge of the stability properties gives us the opportunity to choose well-organized methods which have large stability intervals. Furthermore, we prefer methods where the principle error term includes a small power of the blocksize. It is very useful to carry on this idea to create different methods in which the principal error term does not depend on the bloksize or a power that appears in the denominator of the error term. It is suggested in the conclusion that the scheme can be used to solve a large number of systems of differential equations for some models in the field of quantum optics.

2. A block predictor-corrector method

In this section, we consider a block method in which the order is fixed but the block length may be any value at the cost of increasing the truncation error. The method developed uses Eq. (3) to predict solutions to the problem and then applies a corrector in $P(EC)^{\nu}E$ mode. The method can be written as

$$\begin{aligned} Y_{m+1}^{(0)} &= e \otimes y_n + h(A_p \otimes I)F\left(Y_m^{(v)}\right), \\ Y_{m+1}^{(1+1)} &= e \otimes y_n + h(A \otimes I)F\left(Y_{m+1}^{(1)}\right), \;\; l = 0, \dots, v-1, \\ y_{m+1} &= e_{k+1}^T \, Y_{m+1}^{(v)}, \end{aligned} \tag{8}$$

where

$$A_p = \begin{bmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & 1 \\ 0 & 0 & \dots & \dots & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & k \end{bmatrix}$$

is the $(k+1)(k+1)$ prediction matrix and $A = (\alpha_{i,j})0 \leq i\,, j \leq k$ is the $(k+1)(k+1)$ Runge-Kutta matrix of coefficients. For starting purposes, we also take $Y_0^{(v)}$ to be the vector wherein, each $(k+1)$ component is y_0. With

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{8} & \frac{19}{24} & \frac{-5}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{17}{24} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} \end{pmatrix} \tag{9}$$

the matrix Eq. (9) is based on the Simpson rule and in the sequel we refer to it as $SBPC$ when applied in $P(EC)^4E$ mode.

Our main objective is to produce a fourth-order block method which might be very efficient for implementing in multiprocessor computers. In order to satisfy the condition that these methods have order four, the matrix of coefficients A in Eq. (9) must satisfy the following equations for $r =$

$1, 2, \ldots, k$:

$$\sum_{j=0}^{k} \alpha_{r,j} = r, \qquad \sum_{j=0}^{k} j\alpha_{r,j} = \frac{r^2}{2},$$

$$\sum_{j=0}^{k} j^2\alpha_{r,j} = \frac{r^3}{3}, \qquad \sum_{j=0}^{k} j^3\alpha_{r,j} = \frac{r^4}{4}.$$

3. Types of block methods

In the present section, we give two block methods of fourth order that are applicable for arbitrary blocksizes. These methods are developed by using predictor Eq. (3) to predict solutions to the problem and then a corrector in $P(EC)^4E$ mode is applied.

The local truncation errors for these methods are approximately equal to $\frac{-k^3h^5}{120}y^{(5)}(t_n)$ and $\frac{-k^2h^5}{720}y^5(t_n)$ respectively.

(a) TYPE I block method

The coefficients of the matrix A are given by

$$A=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{9}{24} & \frac{19}{24} & \frac{-5}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{8}{24} & \frac{32}{24} & \frac{8}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{9}{24} & \frac{27}{24} & \frac{27}{24} & \frac{9}{24} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{8}{3} & \frac{-4}{3} & \frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-55}{24} & \frac{275}{24} & \frac{-325}{24} & \frac{75}{8} & 0 & 0 & 0 & 0 & 0 & 0\\
-9 & 36 & -45 & 24 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-189}{8} & \frac{2107}{24} & \frac{-2597}{24} & \frac{1225}{24} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-152}{3} & \frac{544}{3} & \frac{-656}{3} & 96 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-765}{8} & \frac{2673}{8} & \frac{-3159}{8} & \frac{1323}{8} & 0 & 0 & 0 & 0 & 0 & 0\\
-165 & \frac{1700}{3} & \frac{-1975}{3} & \frac{800}{3} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The next term of the error of the corrector depends on $O(h^6)$. In order to obtain good results and better accuracy, it is recommended that the number of corrections be increased.

The linear stability properties of the block corrector Eq. (8) are determined through application of the test equation

$$y' = \lambda y \quad \lambda < 0$$

and by setting $z = \lambda h$, in the case $k = 10$, Eq. (8) can be written in the form

$$Q\underline{y} = \underline{b}y_n$$

where

$$Q=\begin{bmatrix} 1-\frac{19}{24}z & \frac{5}{24}z & -\frac{1}{24}z & 0 & 0 & . & . & . & . & . \\ -\frac{32}{24}z & 1-\frac{1}{3}z & 0 & 0 & . & . & . & . & . & . \\ -\frac{9}{8}z & -\frac{9}{8}z & 1-\frac{3}{8}z & 0 & . & . & . & . & . & . \\ -\frac{8}{3}z & \frac{4}{3}z & -\frac{8}{3}z & 1 & 0 & . & . & . & . & . \\ -\frac{275}{24}z & +\frac{325}{24}z & \frac{-75}{8}z & 0 & 1 & 0 & . & . & . & . \\ -36z & 45z & -24z & 0 & 0 & 1 & 0 & . & . & . \\ -\frac{2107}{24}z & -\frac{2597}{24}z & -\frac{1225}{24}z & 0 & 0 & 0 & 1 & 0 & . & . \\ -\frac{544}{3}z & \frac{656}{3}z & -96z & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{2673}{3}z & \frac{3159}{8}z & \frac{1323}{8}z & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1700}{3}z & +\frac{1975}{3}z & -\frac{800}{3}z & 0 & . & . & . & . & . & 1 \end{bmatrix},$$

$$\underline{y}=\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \\ y_{n+9} \\ y_{n+10} \end{bmatrix} \qquad \underline{b}_= \begin{bmatrix} 1+\frac{3}{8}z \\ 1+\frac{1}{3}z \\ 1+\frac{3}{8}z \\ 1 \\ 1-\frac{55}{24}z \\ 1-9z \\ 1-\frac{189}{8}z \\ 1-\frac{152}{3}z \\ 1-\frac{765}{8}z \\ 1-65z \end{bmatrix}.$$

(b) TYPE II block method

The coefficients of the matrix A are given by

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{50} & 0 & 0 & . & . & . & . & 0 & \frac{-1}{100} & \frac{21}{50} & \frac{29}{50} \\ -\frac{1}{25} & 0 & 0 & . & . & . & . & 0 & \frac{3}{25} & \frac{137}{100} & \frac{14}{25} \\ -\frac{29}{100} & 0 & 0 & . & . & . & . & 0 & \frac{11}{20} & \frac{49}{20} & \frac{29}{100} \\ -\frac{79}{100} & 0 & 0 & . & . & . & . & 0 & \frac{7}{5} & \frac{167}{50} & \frac{1}{20} \\ -\frac{153}{100} & 0 & 0 & . & . & . & . & 0 & \frac{267}{100} & \frac{381}{100} & \frac{1}{20} \\ -\frac{243}{100} & 0 & 0 & . & . & . & . & 0 & \frac{429}{100} & \frac{371}{100} & \frac{43}{100} \\ -\frac{67}{20} & 0 & 0 & . & . & . & . & 0 & \frac{153}{25} & \frac{74}{25} & \frac{32}{25} \\ -\frac{407}{100} & 0 & 0 & . & . & . & . & 0 & \frac{791}{100} & \frac{157}{100} & \frac{259}{100} \\ -\frac{43}{100} & 0 & 0 & . & . & . & . & 0 & \frac{934}{100} & \frac{-34}{100} & \frac{432}{100} \\ -\frac{376}{100} & 0 & 0 & . & . & . & . & 0 & \frac{501}{50} & \frac{-131}{50} & \frac{127}{20} \end{bmatrix}$$

The next term of the error of the corrector also depends on $O(h^6)$.

To analyze the stability, we apply the block corrector formula Eq. (8) to the test problem Eq. (10), and setting $z=\lambda h$, for $k=10$, yields Eq. (8):

$$Q\underline{y} = \underline{b}y_n \tag{10}$$

$$\text{with } Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{100}z & -\frac{21}{50}z & \frac{29}{50}z \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{25}z & -\frac{137}{100}z & -\frac{14}{25}z \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{11}{20}z & -\frac{49}{20}z & -\frac{29}{100}z \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{7}{5} & -\frac{167}{50}z & -\frac{1}{20}z \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{267}{100}z & -\frac{381}{100}z & -\frac{1}{20}z \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{429}{100}z & -\frac{371}{100}z & -\frac{43}{100}z \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{153}{25}z & -\frac{74}{25}z & -\frac{32}{25}z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{791}{100}z & -\frac{157}{100}z & -\frac{259}{100}z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{934}{100}z & 1+\frac{34}{100}z & -\frac{432}{100}z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{501}{50}z & +\frac{131}{50}z & 1-\frac{137}{20}z \end{bmatrix} \text{ and}$$

$$\underline{b} = \begin{bmatrix} 1+\frac{1}{50}z \\ 1-\frac{1}{25}z \\ 1-\frac{29}{100}z \\ 1-\frac{79}{100}z \\ 1-\frac{153}{100}z \\ 1-\frac{243}{100}z \\ 1-\frac{67}{20}z \\ 1-\frac{407}{100}z \\ 1-\frac{43}{100}z \\ 1-\frac{376}{100}z \end{bmatrix}.$$

4. Stability analysis

Using Cramer's rule, we find that

$$y_{n+r} = \frac{D_r(z)}{D(z)}y_n \quad r = 1, 2, \ldots, 10 \tag{11}$$

where $D(z) = \det(Q)$ and $D_r(z) = \det(Q_r)$ and Q_r is obtained from Q by replacing the r^{th} column by the vector $\underline{b}$. Absolute stability requires

$$\left| \frac{D_r(z)}{D(z)} \right| < 1.$$

In general, absolute stability properties depend on the predictor and the mode of implementation. Applying Eq. (8) with corrections to the standard linear test problem Eq. (10), yields $Y_{m+1}^v = T^m(z)Y_0^v$, with

$$E = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix},$$

and the stability matrix $T(z)$ is given by [3]

$$T(z) = E + \sum_{j=1}^{v} A^j E z^j + z^{v+1} A^v A_p. \tag{12}$$

Consequently, the stability boundary is the largest number α such that if $z \in (-\alpha, 0)$, then $\rho(T(z)) < 1$, where $\rho(T(z))$ denotes the spectral radius of $T(z)$.

Since $A^\nu A_p = ue_{k+1}^T$, it follows that

$$\rho(T(z)) = \left| 1 + k\,z + \cdots + \frac{(k\,z)^v}{v\,!} + u_{k+1}(k\,z)^{v+1} \right|. \tag{13}$$

Taking $v = p$, where p is the order of the block method, it is easy to show that

$$\rho(T(z)) = \left| 1 + kz + \cdots + \frac{(kz)^p}{p\,!} + \left(\frac{1}{(p+1)\,!} - \frac{C_{p+1}^k}{k^{p+1}} \right)(kz)^{p+1} \right|, \tag{14}$$

where C_{p+1}^k is the error constant of the method at the k^{th} point in the block. In this case, the stability polynomial is simply a perturbation of that corresponding to the $(p+1)^{th}$ order explicit Runge-Kutta method scaled according to blocksize.

Table 1. Stability boundaries

k	v	α_{TYPEI}	α_{TYPEII}
10	2	0.2501	0.2513
	4	0.3027	0.3116
	6	0.3812	0.3843
	8	0.4318	0.4655
	∞	3.0011	3.0024

Table 1 contains the values of α_{TYPEI} and α_{TYPEII} block predictor-corrector methods in $P(EC)^vE$ mode, based on TYPE I and TYPE II block methods respectively. Corresponding to $v = \infty$, Table 1 also contains the stability boundaries of the correctors obtained from Eq. (14).

The error constants for TYPE I and TYPE II are $C_5^k = -\frac{k^3}{120}$ and $C_5^k = -\frac{k^2}{720}$ respectively.

5. Numerical results

We first consider the linear IVP

$$y' = -\frac{y^3}{2}, \quad y(0) = 1, \tag{15}$$

for $t \in [0, 4]$ with exact solution $y = 1/\sqrt{t+1}$. We use TYPE I and TYPE II methods with block size $k = 10$ and various step sizes. We set e_1 and e_2 to be the maximum absolute errors at $t = 4$, we use step sizes h_1 and h_2, and we assume that $e_i = Ch_i^p$.

Table 2 also includes the observed rates of convergence calculated using $p = (lne_1/e_2)/\ (lnh_1/he_2)$.

Table 2. Approximate rate of convergence

k	h	TYPE I	p	TYPE II	p
10	0.2	0.9412 E-01	-	0.8346 E-01	-
	0.1	0.1269E-01	2.89	0.8659 E-03	3.27
	0.05	0.1129E-02	3.49	0.5964E-03	3.86
	0.04	0.5144E-03	4.12	0.2147E-03	4.56
	0.02	0.1979E-04	4.70	0.6005E-05	5.16

To numerically investigate the linear stability properties of the block predictor-corrector methods we consider the linear IVP

$$y' = -100(y - \sin(t)) + \cos(t), \quad y(0) = 0 \tag{16}$$

for $t \in [0, 1]$ with exact solution $y = \sin t$. Table 3 contains the maximum absolute errors using the proposed method with block size $k = 10$.

6. Comparison with other methods

In this section, we compare our 4th order block methods (Type I & Type II) which are mentioned in Section 3 and the fourth order Adams predictor-Corrector method [15] regarding the behavior of the global error.

The numerical results presented here are those produced by the above methods when applied to the following test problems:

(i) $y' = (20y - y^2)/80,\qquad y'(0) = 1\qquad 0 \le x \le 20$

(ii) $y' = y\cos x,\qquad y'(0) = 1\qquad 0 \le x \le 1$

Table 3. Maximum absolute error

h	TYPE I	TYPE II
$\frac{1}{400}$	0.6049E-04	0.2410E-04
$\frac{1}{360}$	0.1256E-03	0.5821E-04
$\frac{1}{320}$	0.7829E-03	0.9849E-03
$\frac{1}{300}$	0.2456E-02	0.1123E-03
$\frac{1}{280}$	0.7913E+02	0.2789E+02

Table 4. Test Problem (I)

		Global Error		
Number of Steps	Stepsize	Adams p-c	BM Type I	BM Type II
100	0.2	4.73×10^{-7}	2.14×10^{-5}	6.64×10^{-6}
200	0.1	4.18×10^{-8}	1.38×10^{-6}	2.21×10^{-6}
300	0.06667	9.14×10^{-9}	2.74×10^{-7}	8.89×10^{-7}
400	0.05	3.03×10^{-9}	8.72×10^{-8}	4.19×10^{-7}
500	0.04	1.28×10^{-9}	3.58×10^{-8}	1.22×10^{-7}
600	0.03333	6.28×10^{-10}	1.73×10^{-8}	7.76×10^{-8}
700	0.02857	3.43×10^{-10}	9.34×10^{-9}	3.99×10^{-8}
800	0.025	2.03×10^{-10}	5.48×10^{-9}	6.98×10^{-9}
900	0.0222	1.28×10^{-10}	3.42×10^{-9}	2.25×10^{-9}
1000	0.02	8.44×10^{-11}	2.25×10^{-9}	8.67×10^{-10}

Table (5). Test Problem (II)

Number of Steps	Stepsize	Global Error		
		Adams p-c	BM Type I	BM Type II
100	0.01	6.35×10^{-10}	1.55×10^{-8}	8.21×10^{-9}
200	0.005	3.79×10^{-11}	9.57×10^{-10}	3.41×10^{-9}
300	0.00333	7.37×10^{-12}	1.89×10^{-10}	6.24×10^{-10}
400	0.0025	2.31×10^{-12}	5.96×10^{-11}	1.32×10^{-10}
500	0.002	9.41×10^{-13}	2.44×10^{-11}	7.53×10^{-11}
600	0.001667	4.53×10^{-13}	1.18×10^{-11}	2.24×10^{-11}
700	0.0014286	2.44×10^{-13}	6.33×10^{-12}	6.44×10^{-12}
800	0.00125	1.46×10^{-13}	3.72×10^{-12}	3.21×10^{-12}
900	0.001111	8.84×10^{-14}	2.32×10^{-12}	8.78×10^{-13}
1000	0.001	5.51×10^{-14}	1.52×10^{-12}	4.21×10^{-13}

We observe from the above tables that the numerical results are satisfactory since there is no real difference in accuracy between Adams PC and the presented block methods.

7. Conclusion

In this paper we have presented two fourth order block methods for solving ordinary differential equations with their principal error having a small power of blocksize. These methods appear to have a rich behavior very similar to the Simpson block method ($SBPC$) [4]. Moreover, it would be interesting to implement these methods on multiprocessor computers in the future.

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