RIGHT $\pi$-REGULAR SEMIRINGS

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Abstract. We prove the following results (1) If $R$ is a right and left $\pi$-regular semiring then $R$ is a $\pi$-regular semiring. (2) If $R$ is an additive cancellative semiprime, right Artinian or right $\pi$-regular right Noetherian semiring then $R$ is semisimple. (3) Let $I$ be a partitioning ideal of a semiring $R$ such that $Q = (R - I) \cup \{0\}$. If $I$ is a right regular ideal and the quotient semiring $R/I$ is right $\pi$-regular then $R$ is a right $\pi$-regular semiring.

For the definition of semiring we refer the readers to [2]. $\mathbb{Z}^+$ will denote the set of all non negative integers. The following lemma is easy to prove.

Lemma 1. Let $I$ be an ideal of a semiring $R$ and let $a, x \in R$ such that $a + I \subseteq x + I$. Then $ar + I \subseteq xr + I, ra + I \subseteq rx + I, a + r + I \subseteq x + r + I$ for all $r \in R$.

An ideal $I$ of a semiring $R$ is called subtractive if $a, a + b \in I, b \in R$ implies $b \in I$. An ideal $I$ of semiring $R$ is called a partitioning ideal if there exists a subset $Q$ of $R$ such that:

1. $R = \cup\{q + I : q \in Q\}$.
2. If $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$.

Lemma 2. If $I$ is a partitioning ideal of a semiring $R$ then $I$ is a subtractive ideal of $R$.

Proof. See Corollary 7.19 of [8], which holds true for semirings not necessarily with an identity element.

Let $I$ be a partitioning ideal of a semiring $R$ and let $R/I = \{q + I : q \in Q\}$. Then $R/I$ forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$$(q_1 + I) \oplus (q_2 + I) = q_3 + I$$

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where \( q_3 \in Q \) is the unique element such that \( q_1 + q_2 + I \subseteq q_3 + I \).

\[
(q_1 + I) \circ (q_2 + I) = q_4 + I
\]

where \( q_4 \in Q \) is the unique element such that \( q_1q_2 + I \subseteq q_4 + I \).

This semiring \( R/I \) is called the quotient semiring of \( R \) by \( I \). By definition of partitioning ideal, there exists a unique \( q \in Q \) such that \( 0 + I \subseteq q + 1 \). Then \( q + I \) is a zero element of \( R/I \).

An element \( a \) of a semiring \( R \) is called right \( \pi \)-regular (resp. \( \pi \)-regular) if there exist \( x, y \in R \) and a positive integer \( n \) such that \( a^n + a^{n+1}x = a^{n+1}y \) (resp. \( a^n + a^nxa^n = a^nya^n \)). A semiring \( R \) is called right \( \pi \)-regular (resp. \( \pi \)-regular) if every element of \( R \) is right \( \pi \)-regular (resp. \( \pi \)-regular). If \( n = 1 \) then \( a \) is called a right regular (resp. regular) element of \( R \). If every element of a semiring \( R \) is right regular (resp. regular) then \( R \) is called a right regular (resp. regular) semiring.

\[\Box\]

**Example 3.** (i) Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \{0, 1\} \right\} \). Then \( R \) is a right \( \pi \)-regular, \( \pi \)-regular semiring but not right regular or regular. The element \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) is not right regular or regular.

(ii) Let \( R = \{ (a_{ij})_{2 \times 2} : a_{ij} \in \{0, 1\} \} \). Then \( R \) is a regular semiring but not right, left regular.

(iii) Let \( R = (\mathbb{Z}^+ \cup \{-\infty\}, \oplus, \odot) \) where \( a \oplus b = \max\{a, b\} \) and \( a \odot b = a + b \). Then \( R \) is a commutative regular semiring with \( -\infty \) as a zero element and 0 as an identity element. Let \( a \in R \) where \( a \neq 0, -\infty \). Then there does not exist any \( y \in R \) such that \( a = a \oplus y \oplus a \).

A right ideal \( I \) of a semiring \( R \) is called right semiregular if for every \( a_1, a_2 \in I \) there exist \( r_1, r_2 \in I \) such that \( a_1 + r_1 + a_1r_1 + a_2r_2 = a_2 + r_2 + a_1r_2 + a_2r_1 \). The sum of all right semiregular right ideals of a semiring \( R \) is called the right Jacobson radical of \( R \). The right Jacobson radical of \( R \) is equal to the left Jacobson radical of \( R \). It is called the Jacobson-Bourne radical of \( R \). We denote it by \( J(R) \). It is a right semiregular right ideal of \( R \). It is a two sided ideal of \( R \). A semiring \( R \) is called semisimple if \( J(R) = 0 \) (see [4]). Let \( R = (\mathbb{Z}^+, +, \cdot) \). Then by Theorem 9 of [4] and Theorem 4 of [9], the matrix semiring \( R_{nxn} \) is semisimple. A semiring \( R \) is called semiprime if it has no non-zero nilpotent ideals. Every semisimple semiring is semiprime ([4], Theorem 7). A semiring \( R \) is called right Noetherian (resp. right Artinian) if there exists no infinite properly ascending (resp. descending) sequence of right ideals \( I_1 \subset I_2 \subset I_3 \subset \cdots \) (resp. \( I_1 \supset I_2 \supset I_3 \supset \cdots \)) of \( R \).

**Theorem 4.** If \( R \) is a right and left \( \pi \)-regular semiring then \( R \) is a \( \pi \)-regular semiring.
Proof. Let $a \in R$. Then there exist $x, y, x', y' \in R$ and positive integers $m, n$ such that
\[ a^m + a^{m+1} x = a^{m+1} y \]  
(1) \[ a^n + x'a^{n+1} = y'a^{n+1}. \]  
(2) Multiplying (1) by a from left, by $y$ from right and using (1), we get
\[ a^m + a^{m+1} x + a^{m+2} xy = a^{m+2} y^2. \]  
(3) Now multiplying (3) by $a$ from left, by $x$ from right and then adding $a^m$, we get
\[ a^m + a^{m+1} x + a^{m+2} x^2 + a^{m+3} xyx = a^m + a^{m+3} y^2 x. \]  
(4) By using (1), we can write (4) as
\[ a^m + a^{m+1} z_1 = a^{m+1} z_2 \]  
where $z_1 = y^2 x$, $z_2 = y + ax^2 + a^2 xyx$.  
(5) Multiplying (5) by $a$ from left and by $z_2$ from right, we get
\[ a^{m+1} z_2 + a^{m+2} z_1 z_2 = a^{m+2} z_2^2. \]  
(6) By using (5), this can be written as
\[ a^m + a^{m+2} w_1 = a^{m+2} w_2 \]  
where $w_1 = a z_1 + a^2 z_2 z_2$, $w_2 = z_2^2$.  
(7) Repeat the same process. After $n - 1$ steps, we get
\[ a^m + a^{m+n} u_1 = a^{m+n} u_2 \]  
for some $u_1, u_2 \in R$.  
(8) By using (2) and the similar argument as above, we get
\[ a^n + v_1 a^{m+n} = v_2 a^{m+n} \]  
for some $v_1, v_2 \in R$.  
(9) From (8) and (9), we get
\[ a^{m+n} + a^m v_1 a^{m+n} + a^m u_1 a^n + a^{m+n} u_1 v_1 a^{m+n} = a^{m+n} u_2 v_2 a^{m+n}. \]  
(10) Adding $a^{m+n} u_1 v_1 a^{m+n}$ in both sides of (10), we get
\[ a^{m+n} + (a^m + a^{m+n} u_1) v_1 a^{m+n} + a^{m+n} u_1 (a^n + v_1 a^{m+n}) = a^{m+n} (u_2 v_2 + u_1 v_1) a^{m+n}. \]  
(11) By using (8) and (9), the equation (11) can be written as
\[ a^{m+n} + a^{m+n} (u_1 v_2 + u_2 v_1) a^{m+n} = a^{m+n} (u_1 v_1 + u_2 v_2) a^{m+n}. \]  
Therefore $R$ is a $\pi$-regular semiring.

A semiring $R$ is called duo if every one sided ideal of $R$ is a two sided ideal of $R$.

**Corollary 5.** Let $R$ be a duo semiring. Then $R$ is a right $\pi$-regular semiring if and only if it is $\pi$-regular semiring.
Let $a \in R$. As in the above theorem, there exist $x, y \in R$ and a positive integer $m$ such that $a^m + a^{m+2}x = a^{m+2}y$. Let $\langle a^{m+1} \rangle_1$ be the left ideal of $R$ generated by $a^{m+1}$. It is a two-sided ideal of $R$. So $a^{m+1}x, a^{m+1}y \in \langle a^{m+1} \rangle_1$. Hence $a^{m+1}x = sa^{m+1} + ja^{m+1}$, $a^{m+1}y = ta^{m+1} + ka^{m+1}$ for some $s, t \in R$ and $j, k \in \mathbb{Z}^+$. Hence $a^m + (sa^{m+1} + ja^{m+1}) = a(ta^{m+1} + ka^{m+1})$. So $a^m + ua^{m+1} = va^{m+1}$ for some $u, v \in R$. Therefore $R$ is a left $\pi$-semiring. So $R$ is a $\pi$-regular semiring. Conversely, let $a \in R$. Then there exist $x, y \in R$ and a positive integer $m$ such that $a^m + a^m xa^m = a^m ya^m$. Let $\langle a^m \rangle$, be the right ideal of $R$ generated by $a^m$. It is a two-sided ideal of $R$. So $xa^m, ya^m \in \langle a^m \rangle$. Hence $xa^m = a^m s + a^m j$, $ya^m = a^m t + a^m k$ for some $s, t \in R$ and $j, k \in \mathbb{Z}^+$. Now $a^m + a^m (a^m s + a^m j) = a^m (a^m t + a^m k)$. So $a^m + a^m (s + j) = a^{2m} (t + k)$. If $m > 1$ then $a^m + a^{m+1} w = a^{m+1} z$ for some $w, z \in R$. If $m = 1$ then $a + a^2 (s + j) = a^2 (t + k)$. As in the above theorem, it is easy to see that $a + a^2 u = a^2 v$ for some $u, v \in R$. Therefore $R$ is a right $\pi$-regular semiring.

\textbf{Lemma 6.} Let $R$ be a right $\pi$-regular semiring and $a \in J(R)$. Then $a^n + w = w$ for some $w \in J(R)$ and positive integer $n$.

\textbf{Proof.} Let $a \in J(R)$. Then there exist $x, y \in R$ and a positive integer $n$ such that

$$a^n + a^{n+1}x = a^{n+1}y.$$  \hspace{1cm} (1)

Since $ax, ay \in J(R)$, there exist $s_1, s_2 \in J(R)$ such that $ax + s_1 + axs_1 + ays_2 = ay + s_2 + ays_1 + axs_2$. Now multiplying this by $a^n$ from left, we get $a^{n+1}x + a^n s_1 + a^{n+1}xs_1 + a^{n+1}ys_2 = a^{n+1}y + a^n s_2 + a^{n+1}ys_1 + a^{n+1}xs_2$. Using (1), it can be written as $a^{n+1}x + a^{n} s_1 + a^{n+1}xs_1 + (a^n + a^{n+1}x)s_2 = a^n + a^n x + a^n s_2 + (a^n + a_{n+1}x)s_1 + a^{n+1}xs_2$. Now $a^n + w = w$ where $w = a^{n+1}x + a^n s_1 + a^{n+1}xs_1 + a^n s_2 + a^{n+1}xs_2 \in J(R)$. \hfill $\Box$

\textbf{Theorem 7.} If $R$ is an additive cancellative semiprime, right Artinian or right $\pi$-regular right Noetherian semiring then $R$ is semisimple.

\textbf{Proof.} Let $R$ be right Artinian. Easily, $R$ is right $\pi$-regular. By Lemma 6, $J(R)$ is a nil ideal. It is easy to see that $J(R)$ is a nilpotent ideal of $R$. So $J(R) = 0$. Let $R$ be right $\pi$-regular right Noetherian semiring. Hence $J(R)$ is a nil ideal. We can see that $R$ has no non-zero nil ideal. So $J(R) = 0$. \hfill $\Box$

The condition that $R$ is an additive cancellative semiring is essential.

\textbf{Example 8.} Let $R = \{\{0, 1, 2, \ldots, n\}, \text{max}, \text{min}\}$. Then $R$ is a commutative semiprime Noetherian $\pi$-regular semiring with an identity element $n$. It is not an additive cancellative semiring. Let $I$ be an ideal of $R$ and let $a_1, a_2 \in I$. Then we can choose $r_1 = r_2 = \max\{a_1, a_2\} \in I$ such that $a_1 + r_1 + a_1r_1 + a_2r_2 = a_2 + r_2 + a_1r_2 + a_2r_1$ holds. Hence every ideal of $R$ is semiregular. Thus $J(R) = R$. So $R$ is not semisimple.
Example 9. Let $R = (\mathbb{Z}^+ \cup \{\infty\}, \max, \min)$. Then $R$ is a commutative semiprime Artinian semiring with an identity element $\infty$. It is not an additive cancellative semiring. Every ideal of $R$ is semiregular. So $R$ is not semisimple.

Lemma 10. Let $R$ be a semiring and $a, x \in R$. If $a^n + a^{n+1}x$ is right regular element where $n > 0$ then $a$ is right $\pi$-regular element.

Proof. Let $a^n + a^{n+1}x$ be right regular element. Then $a^n + a^{n+1}x + (a^n + a^{n+1}x)^2w = (a^n + a^{n+1}x)^2z$ for some $w, z \in R$. Hence $a^n + a^{n+1}(x + a^{n-1}w + a^n xw + x a^n w + x a^{n+1}xw) = a^{n+1}(a^{n-1} + a^n x + xa^n + xa^{n+1}x)z$. Therefore $a$ is right $\pi$-regular.

Proposition 11. Every ideal of a right $\pi$-regular semiring is right $\pi$-regular.

Proof. Let $I$ be an ideal of a right $\pi$-regular semiring $R$. Let $a \in I$. Then there exist $x, y \in R$ and a positive integer $n$ such that

$$a^n + a^{n+1}x = a^{n+1}y. \quad (1)$$

Multiplying this equation by $a$ from left, by $x$ from right and then adding $a^n$ in both sides, we get

$$a^{n+1}y + a^{n+2}x^2 = a^n + a^{n+1}x + a^{n+2}x^2 = a^n + a^{n+2}yx. \quad (2)$$

Now multiplying (1) by $a$ from left and by $y$ from right and then adding $a^{n+2}x^2$ in both sides, we get $a^{n+2}x^2 + a^{n+1}y + a^{n+2}xy = a^{n+2}x^2 + a^{n+2}y^2$.

Using (2) we have $a^n + a^{n+2}yx + a^{n+2}xy = a^{n+2}x^2 + a^{n+2}y^2$. So $a^n + a^{n+1}u_1 = a^{n+1}u_2$ where $u_1 = ayx + axy, u_2 = ax^2 + ay^2 \in I$. Hence $I$ is a right $\pi$-regular ideal.

Proposition 12. A semiring $R$ is right $\pi$-regular if and only if $R/I$ is right $\pi$-regular for every partitioning ideal $I$ of $R$.

Proof. Define $f : R \rightarrow R/I$ by $f(a) = q + I$ where $q \in Q$ is the unique element such that $a + I \subseteq q + 1$. Then $f$ is an onto homomorphism. If $R$ is right $\pi$-regular semiring then so is $R/I$. Conversely, since $I = 0$ is a partitioning ideal of $R$ with $Q = R$, we see that $R \cong R/0$ is right $\pi$-regular semiring.

Theorem 13. Let $I$ be a partitioning ideal of a semiring $R$ such that $Q = (R - I) \cup \{0\}$. If $I$ is a right regular ideal and the quotient semiring $R/I$ is right $\pi$-regular then $R$ is a right $\pi$-regular semiring.

Proof. Let $q \in R$. If $q \in I$ then $q$ is right $\pi$-regular. Suppose $q \notin I$. Then $q \in Q$ and $q \neq 0$. Since $R/I$ is right $\pi$-regular, there exist $q' + I$, $q'' + I \in R/I$ and a positive integer $n$ such that $(q + I)^n \oplus (q + I)^{n+1} \circ (q + I) = (q + I)^{n+1} \circ (q'' + I) = q'' + I$ where $q'' \in Q$ is the unique element such that
1. \( q^{n+1}q'' + I \subseteq q^* + I \).
2. \( q^n + q^{n+1}q' + I \subseteq q^* + I \)

which follows by Lemma 1.

(i) Suppose \( q^{n+1}q'' \in I \). We have \( q^{n+1}q'' = q^* + x \) for some \( x \in I \). Since by Lemma 2, \( I \) is a subtractive ideal of \( R \), \( q^* \in I \). We have \( q^n + q^{n+1}q' \subseteq q^n + q^{n+1}q' + I \subseteq q^* + I \subseteq I \). By Lemma 10, \( q \) is right \( \pi \)-regular. (ii) If \( q^n + q^{n+1}q' \notin I \) then \( q^n + q^{n+1}q' \in Q \). So using (1) and the definition of partitioning ideal, we get \( q^n + q^{n+1}q' = q^* \). Thus \( q^n + q^{n+1}q' + I \subseteq q^* + I = q^{n+1}q'' + I \). If \( q^n + q^{n+1}q' \in Q \) then \( q^n + q^{n+1}q' = q^{n+1}q'' \). Hence \( q \) is right \( \pi \)-regular. If \( q^n + q^{n+1}q' \notin \), then \( q^n + q^{n+1}q' \in I \). Hence by Lemma 10, \( q \) is right \( \pi \)-regular. Now \( R \) is right \( \pi \)-regular.

The following example satisfies all the hypotheses of the above theorem.

**Example 14.** Let \( R = (\mathbb{Z}^+ \cup \{\infty\}, \text{max, min}) \). Then \( R \) is a commutative semiring with an identity element \( \infty \). Let \( I = \{0, 1, 2, 3, 4, 5\} \). Then \( I \) is a partitioning ideal of \( R \) where \( Q = \{0, 6, 7, 8, \ldots \} \cup \{\infty\} \). Hence \( Q = (R - I) \cup \{0\} \). Clearly \( I, R/I \) and \( R \) are regular.

**References**

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