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RIGHT π-REGULAR SEMIRINGS

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ABSTRACT. We prove the following results (1) If R is a right and left π-regular semiring then R is a π-regular semiring. (2) If R is an additive cancellative semiprime, right Artinian or right π-regular right Noetherian semiring then R is semisimple. (3) Let I be a partitioning ideal of a semiring R such that $Q = (R - I) \cup \{0\}$. If I is a right regular ideal and the quotient semiring R/I is right π-regular then R is a right π-regular semiring.

For the definition of semiring we refer the readers to [2]. Z^+ will denote the set of all non negative integers. The following lemma is easy to prove.

Lemma 1. *Let I be an ideal of a semiring R and let a, $x \in R$ such that $a + I \subseteq x + I$. Then $ar + I \subseteq xr + I$, $ra + I \subseteq rx + I$, $a + r + I \subseteq x + r + I$ for all $r \in R$.*

An ideal I of a semiring R is called subtractive if a, $a + b \in I$, $b \in R$ implies $b \in I$. An ideal I of semiring R is called a partitioning ideal if there exists a subset Q of R such that:

1. $R = \cup\{q + I : q \in Q\}$.
2. If $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$.

Lemma 2. *If I is a partitioning ideal of a semiring R then I is a subtractive ideal of R.*

Proof. See Corollary 7.19 of [8], which holds true for semirings not necessarily with an identity element.

Let I be a partitioning ideal of a semiring R and let $R/I = \{q+I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$$(q_1 + I) \oplus (q_2 + I) = q_3 + I$$

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where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$.

$$(q_1 + I) \odot (q_2 + I) = q_4 + I$$

where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$.

This semiring R/I is called the quotient semiring of R by I. By definition of partitioning ideal, there exists a unique $q \in Q$ such that $0 + I \subseteq q + 1$. Then $q + I$ is a zero element of R/I.

An element a of a semiring R is called right π-regular (resp. π-regular) if there exist $x, y \in R$ and a positive integer n such that $a^n + a^{n+1}x = a^{n+1}y$ (resp. $a^n + a^n x a^n = a^n y a^n$). A semiring R is called right π-regular (resp. π-regular) if every element of R is right π-regular (resp. π-regular). If $n = 1$ then a is called a right regular (resp. regular) element of R. If every element of a semiring R is right regular (resp. regular) then R is called a right regular (resp. regular) semiring. □

Example 3. (i) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in (\{0, 1\}, \max, \min) \right\}$. Then R is a right π-regular, π-regular semiring but not right regular or regular. The element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not right regular or regular.

(ii) Let $R = \{(a_{ij})_{2\times 2} : a_{ij} \in (\{0, 1\}, \max, \min)\}$. Then R is a regular semiring but not right, left regular.

(iii) Let $R = (Z^+ \cup \{-\infty\}, \oplus, \odot)$ where $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$. Then R is a commutative regular semiring with $-\infty$ as a zero element and 0 as an identity element. Let $a \in R$ where $a \neq 0, -\infty$. Then there does not exist any $y \in R$ such that $a = a \oplus y \oplus a$.

A right ideal I of a semiring R is called right semiregular if for every $a_1, a_2 \in I$ there exist $r_1, r_2 \in I$ such that $a_1 + r_1 + a_1 r_1 + a_2 r_2 = a_2 + r_2 + a_1 r_2 + a_2 r_1$. The sum of all right semiregular right ideals of a semiring R is called the right Jacobson radical of R. The right Jacobson radical of R is equal to the left Jacobson radical of R. It is called the Jacobson-Bourne radical of R. We denote it by $J(R)$. It is a right semiregular right ideal of R. It is a two sided ideal of R. A semiring R is called semisimple if $J(R) = 0$ (see [4]). Let $R = (Z^+, +, \cdot)$. Then by Theorem 9 of [4] and Theorem 4 of [9], the matrix semiring $R_{n\times n}$ is semisimple. A semiring R is called semiprime if it has no non-zero nilpotent ideals. Every semisimple semiring is semiprime ([4], Theorem 7). A semiring R is called right Noetherian (resp. right Artinian) if there exists no infinite properly ascending (resp. descending) sequence of right ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ (resp. $I_1 \supset I_2 \supset I_3 \supset \cdots$) of R.

Theorem 4. *If R is a right and left π-regular semiring then R is a π-regular semiring.*

Proof. Let $a \in R$. Then there exist $x, y, x', y' \in R$ and positive integers m, n such that

$$a^m + a^{m+1}x = a^{m+1}y \tag{1}$$

$$a^n + x'a^{n+1} = y'a^{n+1}. \tag{2}$$

Multiplying (1) by a from left, by y from right and using (1), we get

$$a^m + a^{m+1}x + a^{m+2}xy = a^{m+2}y^2\,. \tag{3}$$

Now multiplying (3) by a from left, by x from right and then adding a^m, we get

$$a^m + a^{m+1}x + a^{m+2}x^2 + a^{m+3}xyx = a^m + a^{m+3}y^2x. \tag{4}$$

By using (1), we can write (4) as

$$a^m + a^{m+3}z_1 = a^{m+1}z_2 \text{ where } z_1 = y^2x, z_2 = y + ax^2 + a^2xyx. \tag{5}$$

Multiplying (5) by a from left and by z_2 from right, we get

$$a^{m+1}z_2 + a^{m+4}z_1z_2 = a^{m+2}z_2^2. \tag{6}$$

By using (5), this can be written as

$$a^m + a^{m+2}w_1 = a^{m+2}w_2 \text{ where } w_1 = az_1 + a^2z_1z_2, w_2 = z_2^2. \tag{7}$$

Repeat the same process. After $n-1$ steps, we get

$$a^m + a^{m+n}u_1 = a^{m+n}u_2 \text{ for some } u_1, u_2 \in R. \tag{8}$$

By using (2) and the similar argument as above, we get

$$a^n + v_1a^{m+n} = v_2a^{m+n} \text{ for some } v_1, v_2 \in R. \tag{9}$$

From (8) and (9), we get

$$a^{m+n} + a^mv_1a^{m+n} + a^{m+n}u_1a^n + a^{m+n}u_1v_1a^{m+n} = a^{m+n}u_2v_2a^{m+n}. \tag{10}$$

Adding $a^{m+n}u_1v_1a^{m+n}$ in both sides of (10), we get

$$a^{m+n} + (a^m + a^{m+n}u_1)v_1a^{m+n} + a^{m+n}u_1(a^n + v_1a^{m+n}) \\ = a^{m+n}(u_2v_2 + u_1v_1)a^{m+n}. \tag{11}$$

By using (8) and (9), the equation (11) can be written as

$$a^{m+n} + a^{m+n}(u_1v_2 + u_2v_1)a^{m+n} = a^{m+n}(u_1v_1 + u_2v_2)a^{m+n}.$$

Therefore R is a π-regular semiring. □

A semiring R is called duo if every one sided ideal of R is a two sided ideal of R.

Corollary 5. *Let R be a duo semiring. Then R is a right π-regular semiring if and only if it is π-regular semiring.*

Proof. Let $a \in R$. As in the above theorem, there exist $x, y \in R$ and a positive integer m such that $a^m + a^{m+2}x = a^{m+2}y$. Let $\langle a^{m+1}\rangle_1$ be the left ideal of R generated by a^{m+1}. It is a two sided ideal of R. So $a^{m+1}x, a^{m+1}y \in \langle a^{m+1}\rangle_1$. Hence $a^{m+1}x = sa^{m+1} + ja^{m+1}$, $a^{m+1}y = ta^{m+1} + ka^{m+1}$ for some $s, t \in R$ and $j, k \in Z^+$. Hence $a^m + a(sa^{m+1} + ja^{m+1}) = a(ta^{m+1} + ka^{m+1})$. So $a^m + ua^{m+1} = va^{m+1}$ for some $u, v \in R$. Therefore R is a left π-regular semiring. So R is a π-regular semiring. Conversely, let $a \in R$. Then there exist $x, y \in R$ and a positive integer m such that $a^m + a^m xa^m = a^m ya^m$. Let $\langle a^m\rangle_r$ be the right ideal of R generated by a^m. It is a two sided ideal of R. So $xa^m, ya^m \in \langle a^m\rangle_r$. Hence $xa^m = a^m s + a^m j$, $ya^m = a^m t + a^m k$ for some $s, t \in R$ and $j, k \in Z^+$. Now $a^m + a^m(a^m s + a^m j) = a^m(a^m t + a^m k)$. So $a^m + a^{2m}(s + j) = a^{2m}(t + k)$. If $m > 1$ then $a^m + a^{m+1}w = a^{m+1}z$ for some $w, z \in R$. If $m = 1$ then $a + a^2(s + j) = a^2(t + k)$. As in the above theorem, it is easy to see that $a + a^2u = a^2v$ for some $u, v \in R$. Therefore R is a right π-regular semiring. □

Lemma 6. *Let R be a right π-regular semiring and $a \in J(R)$. Then $a^n + w = w$ for some $w \in J(R)$ and positive integer n.*

Proof. Let $a \in J(R)$. Then there exist $x, y \in R$ and a positive integer n such that

$$a^n + a^{n+1}x = a^{n+1}y. \tag{1}$$

Since $ax, ay \in J(R)$, there exist $s_1, s_2 \in J(R)$ such that $ax + s_1 + axs_1 + ays_2 = ay + s_2 + ays_1 + axs_2$. Now multiplying this by a^n from left, we get $a^{n+1}x + a^n s_1 + a^{n+1}xs_1 + a^{n+1}ys_2 = a^{n+1}y + a^n s_2 + a^{n+1}ys_1 + a^{n+1}xs_2$. Using (1), it can be written as $a^{n+1}x + a^n s_1 + a^{n+1}xs_1 + (a^n + a^{n+1}x)s_2 = a^n + a^{n+1}x + a^n s_2 + (a^n + a_{n+1}x)s_1 + a^{n+1}xs_2$. Now $a^n + w = w$ where $w = a^{n+1}x + a^n s_1 + a^{n+1}xs_1 + a^n s_2 + a^{n+1}xs_2 \in J(R)$. □

Theorem 7. *If R is an additive cancellative semiprime, right Artinian or right π-regular right Noetherian semiring then R is semisimple.*

Proof. Let R be right Artinian. Easily, R is right π-regular. By Lemma 6, $J(R)$ is a nil ideal. It is easy to see that $J(R)$ is a nilpotent ideal of R. So $J(R) = 0$. Let R be right π-regular right Noetherian semiring. Hence $J(R)$ is a nil ideal. We can see that R has no non-zero nil ideal. So $J(R) = 0$. □

The condition that R is an additive cancellative semiring is essential.

Example 8. Let $R = (\{0, 1, 2, \ldots, n\}, \max, \min)$. Then R is a commutative semiprime Noetherian π-regular semiring with an identity element n. It is not an additive cancellative semiring. Let I be an ideal of R and let $a_1, a_2 \in I$. Then we can choose $r_1 = r_2 = \max\{a_1, a_2\} \in I$ such that $a_1 + r_1 + a_1r_1 + a_2r_2 = a_2 + r_2 + a_1r_2 + a_2r_1$ holds. Hence every ideal of R is semiregular. Thus $J(R) = R$. So R is not semisimple.

Example 9. Let $R = (Z^+ \cup \{\infty\}, \max, \min)$. Then R is a commutative semiprime Artinian semiring with an identity element ∞. It is not an additive cancellative semiring. Every ideal of R is semiregular. So R is not semisimple.

Lemma 10. *Let R be a semiring and $a, x \in R$. If $a^n + a^{n+1}x$ is right regular element where $n > 0$ then a is right π-regular element.*

Proof. Let $a^n + a^{n+1}x$ be right regular element. Then $a^n + a^{n+1}x + (a^n + a^{n+1}x)^2 w = (a^n + a^{n+1}x)^2 z$ for some $w, z \in R$. Hence $a^n + a^{n+1}(x + a^{n-1}w + a^n xw + xa^n w + xa^{n+1}xw) = a^{n+1}(a^{n-1} + a^n x + xa^n + xa^{n+1}x)z$. Therefore a is right π-regular. □

Proposition 11. *Every ideal of a right π-regular semiring is right π-regular.*

Proof. Let I be an ideal of a right π-regular semiring R. Let $a \in I$. Then there exist $x, y \in R$ and a positive integer n such that

$$a^n + a^{n+1}x = a^{n+1}y. \tag{1}$$

Multiplying this equation by a from left, by x from right and then adding a^n in both sides, we get

$$a^{n+1}y + a^{n+2}x^2 = a^n + a^{n+1}x + a^{n+2}x^2 = a^n + a^{n+2}yx. \tag{2}$$

Now multiplying (1) by a from left and by y from right and then adding $a^{n+2}x^2$ in both sides, we get $a^{n+2}x^2 + a^{n+1}y + a^{n+2}xy = a^{n+2}x^2 + a^{n+2}y^2$. Using (2) we have $a^n + a^{n+2}yx + a^{n+2}xy = a^{n+2}x^2 + a^{n+2}y^2$. So $a^n + a^{n+1}u_1 = a^{n+1}u_2$ where $u_1 = ayx + axy$, $u_2 = ax^2 + ay^2 \in I$. Hence I is a right π-regular ideal. □

Proposition 12. *A semiring R is right π-regular if and only if R/I is right π-regular for every partitioning ideal I of R.*

Proof. Define $f : R \to R/I$ by $f(a) = q + I$ where $q \in Q$ is the unique element such that $a + I \subseteq q + 1$. Then f is an onto homomorphism. If R is right π-regular semirng then so is R/I. Conversely, since $I = 0$ is a partitioning ideal of R with $Q = R$, we see that $R \cong R/0$ is right π-regular semiring. □

Theorem 13. *Let I be a partitioning ideal of a semiring R such that $Q = (R - I) \cup \{0\}$. If I is a right regular ideal and the quotient semiring R/I is right π-regular then R is a right π-regular semiring.*

Proof. Let $q \in R$. If $q \in I$ then q is right π-regular. Suppose $q \notin I$. Then $q \in Q$ and $q \neq 0$. Since R/I is right π-regular, there exist $q' + I$, $q'' + I \in R/I$ and a positive integer n such that $(q+I)^n \oplus (q+I)^{n+1} \odot (q'+I) = (q+I)^{n+1} \odot (q''+I) = q^* + I$ where $q^* \in Q$ is the unique element such that

1. $q^{n+1}q'' + I \subseteq q^* + I$.
2. $q^n + q^{n+1}q' + I \subseteq q^* + I$

which follows by Lemma 1.
(i) Suppose $q^{n+1}q'' \in I$. We have $q^{n+1}q'' = q^* + x$ for some $x \in I$. Since by Lemma 2, I is a substractive ideal of R, $q^* \in I$. We have $q^n + q^{n+1}q' \in q^n + q^{n+1}q' + I \subseteq q^* + I \subseteq I$. By Lemma 10, q is right π-regular. (ii) If $q^{n+1}q'' \notin I$ then $q^{n+1}q'' \in Q$. So using (1) and the definition of partitioning ideal, we get $q^{n+1}q'' = q^*$. Thus $q^n + q^{n+1}q' + I \subseteq q^* + I = q^{n+1}q'' + I$. If $q^n + q^{n+1}q' \in Q$ then $q^n + q^{n+1}q' = q^{n+1}q''$. Hence q is right π-regular. If $q^n + q^{n+1}q' \notin Q$ then $q^n + q^{n+1}q' \in I$. Hence by Lemma 10, q is right π-regular. Now R is right π-regular. □

The following example satisfies all the hypotheses of the above theorem.

Example 14. Let $R = (Z^+ \cup \{\infty\}, \max, \min)$. Then R is a commutative semiring with an identity element ∞. Let $I = \{0, 1, 2, 3, 4, 5\}$. Then I is a partitioning ideal of R where $Q = \{0, 6, 7, 8, \ldots\} \cup \{\infty\}$. Hence $Q = (R - I) \cup \{0\}$. Clearly I, R/I and R are regular.

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