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SOME COMMUTATIVE NEUTRIX CONVOLUTIONS INVOLVING THE FRESNEL INTEGRALS

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ABSTRACT. The Fresnel cosine integral $\mathrm{C}(x)$, the Fresnel sine integral $\mathrm{S}(x)$ and the associated functions $\mathrm{C}_+(x)$, $\mathrm{C}_-(x)$, $\mathrm{S}_+(x)$ and $\mathrm{S}_-(x)$ are defined as locally summable functions on the real line. Some convolutions and commutative neutrix convolutions of the Fresnel sine integral and its associated functions with x^r are evaluated.

The *Fresnel sine integral* $\mathrm{S}(x)$ is defined by

$$\mathrm{S}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 \, du,$$

see [7] and the associated functions $\mathrm{S}_+(x)$ and $\mathrm{S}_-(x)$ are defined by

$$\mathrm{S}_+(x) = H(x)\,\mathrm{S}(x), \;\; Ss_-(x) = H(-x)\,\mathrm{S}(x).$$

The *Fresnel cosine integral* $\mathrm{C}(x)$ is defined by

$$\mathrm{C}(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 \, du,$$

see [7] and the associated functions $\mathrm{C}_+(x)$ and $\mathrm{C}_-(x)$ are defined by

$$\mathrm{C}_+(x) = H(x)\,\mathrm{C}(x), \;\; Cc_-(x) = H(-x)\,\mathrm{C}(x),$$

where H denotes Heaviside's function.

We define the function $I_r(x)$ by

$$I_r(x) = \int_0^x u^r \sin u^2 \, du$$

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for $r = 0, 1, 2, \dots$. In particular

$$I_0(x) = \sqrt{\frac{\pi}{2}}\,\mathrm{S}(x), \;\; I_1(x) = \frac{1}{2}(1 - \cos x^2), \;\; I_2(x) = -\frac{1}{2} x \cos x^2 + \frac{\sqrt{\pi}}{2\sqrt{2}}\,\mathrm{C}(x).$$

We define the functions $\cos_+ x$, $\cos_- x$, $\sin_+ x$ and $\sin_- x$ by

$$\sin_+ x = H(x) \sin x, \qquad \sin_- x = H(-x) \sin x,$$
$$\cos_+ x = H(x) \cos x, \qquad \cos_- x = H(-x) \cos x.$$

If the classical convolution $f * g$ of two functions f and g exists then $g * f$ exists and

$$f * g = g * f. \tag{1}$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{2}$$

The classical definition of the convolution can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with following definition, see [6].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$. Then the convolution $f * g$ is defined by the equation*

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle$$

for arbitrary φ in $\mathcal{D}'$, provided f and g satisfy either of the conditions:

(a) *either f or g has bounded support,*
(b) *the supports of f and g are bounded on the same side.*

It follows that if the convolution $f * g$ exists by this definition then equations (1) and (2) are satisfied.

The following convolutions were proved in [5].

$$(\sin_+ x^2) * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} I_{r-i}(x) x_+^i,$$

$$(\sin_- x^2) * x_-^r = -\sum_{i=0}^{r} \binom{r}{i} I_{r-i}(x) x_-^i,$$

$$\mathrm{S}_+(x) * x_+^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1}(x) x_+^i,$$

$$\mathrm{S}_-(x) * x_-^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} I_{r-i+1}(x) x_-^i$$

for $r = 0, 1, 2, \dots$.

Definition 1 was extended in [2] with the next definition but first of all we let τ be a function in $\mathcal{D}$ having the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$, for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$, for $|x| \ge 1$.

The function τ_ν is now defined for $\nu > 0$ by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \le \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu. \end{cases}$$

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. The neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided the limit h exists in the sense that*

$$\operatorname*{N-lim}_{\nu\to\infty} \langle f_\nu * g, \varphi\rangle = \langle h, \varphi\rangle,$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1]*, having domain N' the positive real numbers, with negligible functions finite linear sums of the functions*

$$\nu^\lambda \ln^{r-1}\nu,\ \ \ln^r \nu,\ \ (\lambda \ne 0,\ r = 1, 2, \dots)$$

and all functions which converge to zero in the normal sense as ν tends to infinity.

Note that in this definition the convolution product $f_\nu * g$ is defined in Gel'fand and Shilov's sense, the distribution f_ν having bounded support.

It was proved in [2] that if $f * g$ exists in the classical sense or by Definition 1, then $f \circledast g$ exists and

$$f \circledast g = f * g.$$

The above definition of the neutrix convolution is in general non-commutative. The next definition gives a commutative neutrix convolution and was given in [3].

Definition 3. *Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ and $g_\nu = g\tau_\nu$ for $\nu > 0$. The commuative neutrix convolution product $f \boxed{*}\, g$ is defined as the neutrix limit of the sequence $\{f_\nu * g_\nu\}$, provided the limit h exists in the sense that*

$$\operatorname*{N-lim}_{\nu\to\infty} \langle f_\nu * g_\nu, \varphi\rangle = \langle h, \varphi\rangle,$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1]*, having domain N' the positive real numbers, with negligible functions finite linear sums of the functions*

$$\nu^\lambda \ln^{r-1}\nu,\ \ \ln^r \nu,\ \ (\lambda \ne 0,\ r = 1, 2, \dots)$$

and all functions which converge to zero in the normal sense as ν tends to infinity.

The following theorem, proved in [3] shows that the neutrix convolution is a generalization of the convolution.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \boxast g$ exists and*

$$f \boxast g = f * g.$$

Note that equation (1) holds for the neutrix convolution product but $(f \boxast g)'$ is not necessarily equal to $f' \boxast g$, but we do have the following theorem which was proved in [4].

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \boxast g$ exists. If* $\text{N-lim}_{\nu\to\infty}\langle (f\tau_\nu')*g_\nu, \varphi\rangle$ *exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathcal{D}$, then $f' \boxast g$ exists and*

$$(f \boxast g)' = f' \boxast g + h. \tag{3}$$

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions

$$\nu^r \cos\nu^2,\ \ \nu^r \sin\nu^2\ \ (r = 1, 2, \ldots).$$

We also need the following lemma, which was proved in [5]:

Lemma 1. *If $I_r = \text{N-lim}_{\nu\to\infty} I_r(\nu)$, then*

$$I_{4r} = \frac{(-1)^r(4r)!\sqrt{\pi}}{2^{4r+1}(2r)!\sqrt{2}} \tag{4}$$

$$I_{4r+1} = \frac{(-1)^r(2r)!}{2}, \tag{5}$$

$$I_{4r+2} = \frac{(-1)^r(4r+1)!\sqrt{\pi}}{2^{4r+2}(2r)!\sqrt{2}}, \tag{6}$$

$$I_{4r+3} = 0 \tag{7}$$

for $r = 0, 1, 2, \ldots$.

We now prove

Theorem 3. *The commutative neutrix convolution* $(\sin_+ x^2) \boxast x^r$ *exists and*

$$(\sin_+ x^2) \boxast x^r = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}I_{r-i}x^i \tag{8}$$

for $r = 0, 1, 2, \ldots$. *In particular,*

$$(\sin_+ x^2) \boxed{*}\, 1 = \frac{\sqrt{\pi}}{2\sqrt{2}}, \tag{9}$$

$$(\sin_+ x^2) \boxed{*}\, x = -\frac{1}{2} + \frac{\sqrt{\pi}}{2\sqrt{2}} x. \tag{10}$$

Proof. We put $(\sin_+ x^2)_\nu = (\sin_+ x^2)\tau_\nu(x)$ and $(x^r)_\nu = x^r \tau_\nu(x)$. Then the convolution $(\sin_+ x^2)_\nu * (x^r)_\nu$ exists and

$$(\sin_+ x^2)_\nu * (x^r)_\nu = \int_0^\nu \sin t^2 (x-t)^r \tau_\nu(x-t)\, dt + \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(t)\tau_\nu(x-t)\, dt. \tag{11}$$

If $0 \le |x| \le \nu$, then

$$\int_0^\nu \sin t^2 (x-t)^r \tau_\nu(x-t)\, dt = \sum_{i=0}^r \binom{r}{i} \int_0^\nu x^i (-t)^{r-i} \sin t^2\, dt = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i}(\nu) x^i$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_0^\nu \sin t^2 (x-t)^r \tau_\nu(x-t)\, dt = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} I_{r-i} x^i, \tag{12}$$

on using Lemma 1.

Further,

$$\left| \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(t)\tau_\nu(x-t)\, dt \right| \le \int_\nu^{\nu+\nu^{-\nu}} (t-x)^r\, dt \le (\nu+\nu^{-\nu})\nu^{-\nu}$$

and it follows that for each fixed x,

$$\lim_{\nu\to\infty} \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(t)\tau_\nu(x-t)\, dt = 0. \tag{13}$$

Equation (8) follows from equations (11), (12) and (13). Equations (9) and (10) follow immediately on using Lemma 1. □

Corollary 3.1. *The commutative neutrix convolution* $\sin_- x^2 \boxed{*}\, x^r$ *exists and*

$$(\sin_- x^2) \boxed{*}\, x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i+1} I_{r-i} x^i \tag{14}$$

for $r = 0, 1, 2, \ldots$. *In particular,*

$$(\sin_- x^2) \boxed{*}\, 1 = -\frac{\sqrt{\pi}}{2\sqrt{2}}, \tag{15}$$

$$(\sin_- x^2) \boxed{*}\, x = \frac{1}{2} - \frac{\sqrt{\pi}}{2\sqrt{2}} x. \tag{16}$$

Proof. Equation (14) follows on replacing x by $-x$ in equation (8) and noting that I_r must be replaced by

$$\operatorname*{N-lim}_{\nu \to \infty} I_r(-\nu) = (-1)^{r-1} \operatorname*{N-lim}_{\nu \to \infty} I_r(\nu) = (-1)^{r-1} I_r. \tag{17}$$

Equations (15) and (16) follow from equation (14) on using equations (4) and (5). $\square$

Corollary 3.2. *The commutative neutrix convolution* $(\sin x^2) \boxed{*}\, x^r$ *exists and*

$$(\sin x^2) \boxed{*}\, x^r = 0 \tag{18}$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (18) follows from equations (8) and (14) on noting that $\sin x^2 = \sin_+ x^2 + \sin_- x^2$. $\square$

Theorem 4. *The commutative neutrix convolution* $(x \cos_+ x^2) \boxed{*}\, x^r$ *exists and*

$$(x \cos_+ x^2) \boxed{*}\, x^r = \frac{r}{2} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^{r-i+1} I_{r-i} x^i \tag{19}$$

for $r = 1, 2, \ldots$. *In particular,*

$$(x \cos_+ x^2) \boxed{*}\, 1 = 0, \tag{20}$$

$$(x \cos_+ x^2) \boxed{*}\, x = \frac{1}{4}, \tag{21}$$

$$(x \cos_+ x^2) \boxed{*}\, x^2 = -\frac{\sqrt{\pi}}{4\sqrt{2}} + \frac{x}{2}. \tag{22}$$

Proof. We have

$$
\begin{aligned}
[(\sin_+ x^2)\tau_\nu'(x)] * (x^r)_\nu &= \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(x-t)\, d\tau_\nu(t) \\
&= -\sin \nu^2 (x-\nu)^r \tau_\nu(x-\nu) \\
&- 2\int_\nu^{\nu+\nu^{-\nu}} t\cos t^2 (x-t)^r \tau_\nu(t)\tau_\nu(x-t)\, dt \\
&+ r\int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^{r-1} \tau_\nu(t)\tau_\nu(x-t)\, dt \\
&+ \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(t)\tau_\nu'(x-t)\, dt. \qquad (23)
\end{aligned}
$$

Now $\tau_\nu(x-\nu)$ is either 0 or 1 for large enough ν and so

$$\operatorname*{N-lim}_{\nu\to\infty} \sin \nu^2 (x-\nu)^r \tau_\nu(x-\nu) = 0. \qquad (24)$$

Next we have

$$\left| \int_\nu^{\nu+\nu^{-\nu}} t\cos t^2 (x-t)^r \tau_\nu(t)\tau_\nu(x-t)\, dt \right| \le \int_\nu^{\nu+\nu^{-\nu}} t(t-x)^r\, dt \le (\nu+\nu^{-\nu})^{r+1}\nu^{-\nu}$$

and it follows that

$$\lim_{\nu\to\infty} \left| \int_\nu^{\nu+\nu^{-\nu}} t\cos t^2 (x-t)^r \tau_\nu(t)\tau_\nu(x-t)\, dt \right| = 0. \qquad (25)$$

Similarly,

$$\lim_{\nu\to\infty} \left| \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^{r-1} \tau_\nu(t)\tau_\nu(x-t)\, dt \right| = 0. \qquad (26)$$

Noting that $\tau_\nu'(x-t) = 0$ for large enough ν and $x \neq 0$, it follows that

$$\lim_{\nu\to\infty} \int_\nu^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(t)\tau_\nu'(x-t)\, dt = 0. \qquad (27)$$

If $x = 0$, then

$$\int_{\nu}^{\nu+\nu^{-\nu}} \sin t^2 (x-t)^r \tau_\nu(t)\tau_\nu'(x-t)\,dt = \int_{\nu}^{\nu+\nu^{-\nu}} (-t)^r \sin t^2 \tau_\nu(t)\tau_\nu'(-t)\,dt$$

$$= -\frac{1}{2}\int_{\nu}^{\nu+\nu^{-\nu}} (-t)^r \sin t^2\, d\tau_\nu^2(t)$$

$$= \frac{(-\nu)^r \sin \nu^2}{2} + \frac{(-1)^r}{2}\int_{\nu}^{\nu+\nu^{-\nu}} [rt^{r-1}\sin t^2 + 2t^{r+1}\cos t^2]\tau_\nu^2(t)\,dt$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_{\nu}^{\nu+\nu^{-\nu}} (-t)^r \sin t^2 \tau_\nu(t)\tau_\nu'(-t)\,dt = 0. \tag{28}$$

It now follows from equations (23) to (28) that

$$\operatorname*{N-lim}_{\nu\to\infty}[(\sin_+ x^2)\tau_\nu'(x)] * (x^r)_\nu = 0 \tag{29}$$

and then from Theorem 2 and equation (8) that

$$-r\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^{r-i}I_{r-i}x^i = [(\sin_+ x^2) \boxed{*}\, x^r]'$$
$$= (\sin_+ x^2)' \boxed{*}\, x^r$$
$$= 2(x\cos_+ x^2) \boxed{*}\, x^r.$$

Equations (19) to (22) follow. □

Corollary 4.1. *The commutative neutrix convolution* $(x\sin_- x^2)\boxed{*}x^r$ *exists and*

$$(x\cos_- x^2) \boxed{*}\, x^r = \frac{r}{2}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^{r-i}I_{r-i}x^i \tag{30}$$

for $r = 1, 2, \ldots$. *In particular,*

$$(x\cos_- x^2) \boxed{*}\, 1 = 0, \tag{31}$$

$$(x\cos_- x^2) \boxed{*}\, x = -\frac{1}{4}, \tag{32}$$

$$(x\cos_- x^2) \boxed{*}\, x^2 = \frac{\sqrt{\pi}}{4\sqrt{2}} - \frac{x}{2}. \tag{33}$$

Proof. Equations (30) to (33) follow on replacing x by $-x$ in equations (19) to (22) respectively and using equation (17). □

Corollary 4.2. *The commutative neutrix convolution* $(x\cos x^2) \boxed{*}\, x^r$ *exists and*

$$(x\cos x^2) \boxed{*}\, x^r = 0 \tag{34}$$

for $r = 1, 2, 3, \ldots$.

Proof. Equation (34) follows from equations (19) and (30) on noting that $\cos x^2 = \cos_+ x^2 + \cos_- x^2$. □

Theorem 5. *The commutative neutrix convolution* $\mathrm{S}_+(x) \boxed{*}\, x^r$ *exists and*

$$\mathrm{S}_+(x) \boxed{*}\, x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i+1} I_{r-i+1} x^i \tag{35}$$

for $r = 0, 1, 2, \ldots$. *In particular*

$$\mathrm{S}_+(x) \boxed{*}\, 1 = -\frac{1}{\sqrt{2\pi}}, \tag{36}$$

$$\mathrm{S}_+(x) \boxed{*}\, x = \frac{1}{8} - \frac{1}{\sqrt{2\pi}}\, x. \tag{37}$$

Proof. We put $[\mathrm{S}_+(x)]_\nu = \mathrm{S}_+(x)\tau_\nu(x)$ and $(x^r)_\nu = x^r\tau_\nu(x)$. Then the convolution $[\mathrm{S}_+(x)]_\nu * (x^r)_\nu$ exists and

$$[\mathrm{S}_+(x)]_\nu * (x^r)_\nu = \int_0^\nu \mathrm{S}(t)(x-t)^r\tau_\nu(x-t)\,dt + \int_\nu^{\nu+\nu^{-\nu}} \mathrm{S}(t)(x-t)^r\tau_\nu(t)\tau_\nu(x-t)\,dt. \tag{38}$$

If $0 \le |x| \le \nu$, then

$$\begin{aligned}
\sqrt{\frac{\pi}{2}} \int_0^\nu \mathrm{S}(t)(x-t)^r\tau_\nu(x-t)\,dt &= \int_0^\nu (x-t)^r \int_0^t \sin u^2\,du\,dt \\
&= \int_0^\nu \sin u^2 \int_u^\nu (x-t)^r\,dt\,du \\
&= -\frac{1}{r+1} \int_0^\nu \sin u^2[(x-\nu)^{r+1} - (x-u)^{r+1}]\,du \\
&= -\frac{1}{r+1} \int_0^\nu \sum_{i=0}^{r} \binom{r+1}{i} x^i[(-\nu)^{r-i+1} - (-u)^{r-i+1}] \sin u^2\,du
\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty}\int_0^{\nu} \mathrm{S}(t)(x-t)^r\tau_\nu(x-t)\,dt = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i+1}I_{r-i+1}x^i. \tag{39}$$

Further, for each fixed x

$$\sqrt{\frac{\pi}{2}}\left|\int_{\nu}^{\nu+\nu^{-\nu}} \mathrm{S}(t)(x-t)^r\tau_\nu(t)\tau_\nu(x-t)\,dt\right| \le \int_{\nu}^{\nu+\nu^{-\nu}} |x-t|^r\int_0^t |\sin u^2|\,du\,dt$$
$$\le \int_{\nu}^{\nu+\nu^{-\nu}} t(t-x)^r\,dt \le (\nu+\nu^{-\nu})^{r+1}\nu^{-\nu}$$

and it follows that

$$\lim_{\nu\to\infty}\int_{\nu}^{\nu+\nu^{-\nu}} \mathrm{S}(t)(x-t)^r\tau_\nu(t)\tau_\nu(x-t)\,dt = 0. \tag{40}$$

Equation (35) now follows immediately from equations (38), (39) and (40). □

Corollary 5.1. *The commutative neutrix convolution* $\mathrm{S}_-(x)\boxed{*}x^r$ *exists and*

$$\mathrm{S}_-(x)\boxed{*}x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}I_{r-i+1}x^i \tag{41}$$

for $r=0,1,2,\ldots$. *In particular*

$$\mathrm{S}_-(x)\boxed{*}1 = \frac{1}{\sqrt{2\pi}}, \tag{42}$$

$$\mathrm{S}_-(x)\circledast x = -\frac{1}{8}+\frac{1}{\sqrt{2\pi}}\,x. \tag{43}$$

Proof. Equations (41), (42) and (43) follow on replacing x by $-x$ in equations (35), (36) and (37) respectively and using equation (17). □

Corollary 5.2. *The commutative neutrix convolution* $\mathrm{S}(x)\boxed{*}x^r$ *exists and*

$$\mathrm{S}(x)\boxed{*}x^r = 0 \tag{44}$$

for $r=0,1,2,\ldots$.

Proof. Equation (44) follows from equations (35) and (41) on noting that $\mathrm{S}(x)=\mathrm{S}_+(x)+\mathrm{S}_-(x)$. □

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