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APPROXIMATION BY GENERALIZED FABER SERIES IN WEIGHTED BERGMAN SPACES ON INFINITE DOMAINS WITH A QUASICONFORMAL BOUNDARY

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Abstract. Using an integral representation on infinite domains with a quasiconformal boundary the generalized Faber series for the functions in the weighted Bergman space $A^2(G,\omega)$ are defined and its approximation properties are investigated.

1. Introduction and main results

Let G be a simply connected domain in the complex plane $\mathbb{C}$ an let ω be a weight function given on G. For functions f analytic in G we set

$$A^1(G) := \left\{ f : \iint\limits_G |f(z)|\, d\sigma_z < \infty \right\}$$

and

$$A^2(G,\omega) := \left\{ f : \iint\limits_G |f(z)|^2\, \omega(z) d\sigma_z < \infty \right\},$$

where $d\sigma_z$ denotes the Lebesgue measure in the complex plane $\mathbb{C}$.

If $\omega = 1$, we denote $A^2(G) := A^2(G,1)$. The space $A^2(G)$ is called the Bergman space on G. We refer to the spaces $A^2(G,\omega)$ as "weighted Bergman spaces". It becomes a normed spaces if we define

$$\|f\|_{A^2(G,\omega)} := \left(\iint\limits_G |f(z)|^2\, \omega(z) d\sigma_z \right)^{1/2}.$$

Now, let L be a finite quasiconformal curve in the complex plane $\mathbb{C}$. We recall that L is called a quasiconformal curve if there exists a quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto L. We denote by G_1 and G_2 the bounded and unbounded components of $\mathbb{C} \setminus L$,

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respectively. It is clear that if $f \in A^2(G_2)$, then it has a zero at ∞ of order at least two. As in the bounded case [7, p. 5], $A^2(G_2)$ is a Hilbert space with the inner product

$$\langle f, g\rangle := \iint\limits_{G_2} f(z)\overline{g(z)}d\sigma_z,$$

which can be easily verified. Moreover, the set of polynomials of $1/z$ are dense in $A^2(G_2)$ with respect to the norm

$$\|f\|_{A^2(G_2)} := (\langle f, f\rangle)^{1/2}.$$

Indeed, let $f \in A^2(G_2)$. If we substitute $z = 1/\zeta$ and define

$$f(z) = f(1/\zeta) =: f_*(\zeta),$$

then G_2 maps to a finite domain G_ζ, and $f_* \in A^2(G_\zeta)$, because

$$\iint\limits_{G_\zeta} |f_*(\zeta)|^2 d\sigma_\zeta = \iint\limits_{G_2} |f(z)|^2 \frac{d\sigma_z}{|z|^4} \le c \iint\limits_{G_2} |f(z)|^2 d\sigma_z < \infty,$$

with some constant $c > 0$. Since f has a zero at ∞ of order at least two, the point $\zeta = 0$ is the zero of f_* at least of second order and

$$\iint\limits_{G_\zeta} \left|\frac{f_*(\zeta)}{\zeta^2}\right|^2 d\sigma_\zeta = \iint\limits_{G_2} |f(z)|^2 d\sigma_z < \infty.$$

Hence $f_*(\zeta)/\zeta^2 \in A^2(G_\zeta)$. If $P_n(\zeta)$ is a polynomial of ζ, then we have

$$\begin{aligned}\iint\limits_{G_\zeta} \left|P_n(\zeta) - \frac{f_*(\zeta)}{\zeta^2}\right|^2 d\sigma_\zeta &= \iint\limits_{G_\zeta} \left|P_n(\zeta)\zeta^2 - f_*(\zeta)\right|^2 \frac{1}{|\zeta|^4} d\sigma_\zeta \\ &= \iint\limits_{G_2} \left|P_n(1/z)\frac{1}{z^2} - f(z)\right|^2 d\sigma_z.\end{aligned}$$

This implies that the set of polynomials of $1/z$ are dense in $A^2(G_2)$, since the set of polynomials $P_n(\zeta)$ are dense in $A^2(G_\zeta)$ with respect to the norm

$$\|f\|_{A^2(G_\zeta)} := (\langle f, f\rangle)^{1/2},$$

(see, for example: [7, Ch. 1]). Also, for $n = 1, 2, \dots$ there exists a polynomial $P_n^*(1/z)$ of $1/z$, of degree$\le n$, such that $E_n(f, G_2) = \|f - P_n^*\|_{A^2(G_2)}$ (see, for example: [6, p. 59, Theorem 1.1.]), where

$$E_n(f, G_2) := \inf\Big\{ \|f - P\|_{A^2(G_2)} : P \text{ is a polynomial of } 1/z, \text{ of degree } \le n\Big\}$$

denotes the minimal error of approximation of f by polynomials of $1/z$ of degree at most n. The polynomial $P_n^*(1/z)$ is called the best approximant polynomial of $1/z$ to $f \in A^2(G_2)$.

Let D be the open unit disc and $w = \varphi(z)$ the conformal mapping of G_1 onto $C\overline{D} := \mathbb{C} \setminus \overline{D}$, normalized by conditions

$$\varphi(0) = \infty \quad \text{and} \quad \lim_{z \to 0} z\varphi(z) > 0,$$

and let ψ be the inverse of φ. For an arbitrary fixed number $R > 1$ we put

$$L_R := \{z : |\varphi(z)| = R\}, \quad G_{2,R} := \{z : z \in G_1, 1 < |\varphi(z)| < R\} \cup \overline{G_2}.$$

If a function $g(z)$ is analytic in G_1 and having at $z = 0$ a zero of order $\nu \geq 2$, then for every natural number $m \geq 1$ the function $g(z)\varphi^{m+\nu}(z)$ has a pole of order m at the origin, i.e. the following expansion holds

$$g(z)\varphi^{m+\nu}(z) = F_m(1/z, g) + Q_m(z, g) \qquad \text{for } z \in G_1, \tag{1}$$

where $F_m(1/z, g)$ denotes the polynomial of negative powers of z and the term $Q_m(z, g)$ contains non-negative powers of z. Hence $Q_m(z, g)$ is a function analytic in the domain G_1. The polynomial $F_m(1/z, g)$ of negative powers of z is called the generalized Faber polynomial of order m for the domain G_2. If $z \in G_2$, then integrating in the positive direction along L, we have

$$\begin{aligned} F_m(1/z, g) &= -\frac{1}{2\pi i} \int_L \frac{g(\zeta)\left[\varphi(\zeta)\right]^{m+\nu}}{\zeta - z} d\zeta \\ &= -\frac{1}{2\pi i} \int_{|w|=1} \frac{w^{m+\nu} g\left[\psi(w)\right]\psi'(w)}{\psi(w) - z} dw. \end{aligned}$$

This formula implies that the functions $F_m(1/z, g)$, $m = 1, 2, \ldots$ are the Laurent coefficients in the expansion of the function

$$\frac{g\left[\psi(w)\right]\psi'(w)}{\psi(w) - z} \qquad z \in G_2, \quad w \in C\overline{D}$$

in the neighborhood of the point $w = \infty$, i. e. the following expansion holds

$$\frac{g\left[\psi(w)\right]\psi'(w)}{\psi(w) - z} = \sum_{m=1}^{\infty} F_m(1/z, g) \frac{1}{w^{m+\nu+1}},$$

which converges absolutely and uniformly on compact subsets of $G_2 \times C\overline{D}$. Differentiation of this equality with respect to z gives

$$\frac{g\left[\psi(w)\right]\psi'(w)}{\left[\psi(w) - z\right]^2} = \sum_{m=1}^{\infty} F'_m(1/z, g) \left(-\frac{1}{z^2}\right) \frac{1}{w^{m+\nu+1}}$$

or

$$\frac{z^2 g\left[\psi\left(w\right)\right]\psi'(w)}{\left[\psi\left(w\right)-z\right]^2}=\sum_{m=1}^{\infty}-F'_m\left(1/z,g\right)\frac{1}{w^{m+\nu+1}} \tag{2}$$

for every $(z,w)\in G_2\times C\overline{D}$, where the series converges absolutely and uniformly on compact subsets of $G_2\times C\overline{D}$. More information for Faber and generalized Faber polynomials can be found in [12, p. 44 and p. 255] and [7, p. 42].

In [4], V. I. Belyi gave the following integral representation for the functions f analytic and bounded in the domain G_1

$$f(z)=-\frac{1}{\pi}\iint\limits_{G_2}\frac{(f\circ y)(\zeta)}{(\zeta-z)^2}y_{\overline{\zeta}}(\zeta)d\sigma_\zeta, \qquad z\in G_1. \tag{3}$$

Here $y(z)$ is a K-quasiconformal reflection across the boundary L, i.e., a sense-reversing K-quasiconformal involution of the extended complex plane keeping every point of L fixed, such that $y(G_1)=G_2$, $y(G_2)=G_1$, $y(0)=\infty$ and $y(\infty)=0$. Such a mapping of the plane does exist [11, p. 99]. As follows from Ahlfors' theorem [1, p. 80] the reflection $y(z)$ can always be chosen canonical in the sense that it is differentiable on $\mathbb{C}$ almost everywhere, except possibly at the points of the curve L, and for any sufficiently small fixed $\delta>0$ it satisfies the relations

$$\begin{aligned} |y_\varsigma|+|y_{\overline{\varsigma}}|&\le c_1, \quad \text{for} \quad \delta<|\varsigma|<1/\delta \text{ and } \varsigma\notin L,\\ |y_\varsigma|+|y_{\overline{\varsigma}}|&\le c_2\left|\varsigma\right|^{-2}, \quad \text{for} \quad |\varsigma|\ge 1/\delta \text{ and } |\varsigma|\le\delta. \end{aligned} \tag{4}$$

with some constants c_1 and c_2, independent of ζ.

Let g be an analytic function in G_1, non-vanishing in $G_1\backslash\{0\}$ and having in $z=0$ a zero of order $\nu\ge 2$, and let

$$\iint\limits_{G_1}\left|g(z)\right|^2 d\sigma_z<\infty. \tag{5}$$

For every such g we define a weight function ω in the following manner.

$$\omega(z):=\frac{1}{\left|(g\circ y)\left(z\right)\right|^2}, \qquad z\in G_2,$$

where y is a canonical reflection across the boundary L. We denote by $W^2(G_2)$ the set all of weight functions ω defined as above.

In this work, for the first time, we obtain (Section 2, Lemma 1) an integral representation on the domain G_2 for a function $f\in A^1(G_2)$. By means of this integral representation in Section 2 we define a generalized Faber series of a function $f \in A^1(G_2)$ to be of the form

$$\sum_{m=1}^{\infty} a_m(f,g)F'_m(1/z,g),$$

with the generalized Faber coefficients $a_m(f,g)$, $m = 1, 2, \ldots$.

Our main results are presented in the following theorems, which are proved in Section 3.

Theorem 1. *Let $f \in A^2(G_2,\omega)$, $\omega \in W^2(G_2)$. If*

$$\sum_{m=1}^{\infty} a_m(f,g)F'_m(1/z,g)$$

is a generalized Faber series of f, then this series converges uniformly to f on the compact subsets of G_2.

A uniqueness theorem for the series

$$\sum_{m=1}^{\infty} a_m(f,g)F'_m(1/z,g)$$

which converges to $f \in A^2(G_2,\omega)$ with respect to the norm $\|\cdot\|_{A^2(G_2,\omega)}$ is given next.

Theorem 2. *Let g be an analytic function, bounded in G_1, non-vanishing in $G_1 \backslash \{0\}$ and having at $z = 0$ a zero of order $\nu \geq 2$, and let $\{b_m\}$ be a complex number sequence. If the series*

$$\sum_{m=1}^{\infty} b_m F'_m(1/z,g)$$

converges to a function $f \in A^2(G_2,\omega)$ in the norm $\|\cdot\|_{A^2(G_2,\omega)}$, then b_m, $m = 1, 2, \ldots$, *are the generalized Faber coefficients of f.*

Let y_R be K_R-quasiconformal reflection across the boundary L_R. The following theorem estimates the error of the approximation of $f \in A^2(G_{2,R})$ by the partial sums of the series

$$\sum_{m=1}^{\infty} a_m(f)F'_m(1/z)$$

in the norm $\|\cdot\|_{A^2(G_2,\omega)}$ with regard to $E_n(f, G_{2,R})$ for the special case $\omega(z) = 1/|z|^4$ of the weighted function ω given on G_2.

Theorem 3. *Let $R > 1$. If $f \in A^2(G_{2,R})$, $\omega(z) := 1/|z|^4$ and*

$$S_n(f, 1/z) = \sum_{m=1}^{n+1} a_m(f) F'_m(1/z)$$

is the nth partial sum of its generalized Faber series

$$\sum_{m=1}^{\infty} a_m(f) F'_m(1/z),$$

then

$$\|f - S_n(f,\cdot)\|_{A^2(G_2,\omega)} \leq \frac{c}{\sqrt{(1-k_R^2)(R^2-1)}} \frac{E_n(f, G_{2,R})}{R^{n+1}},$$

for all natural numbers n and with a constant c independent of n, where $k_R := (K_R - 1)/(K_R + 1)$.

For bounded domains the problems considered here were investigated in [8] and [10]. Similar results in the non-weighted case were stated and proved in [9] and [5], respectively.

We shall use c, c_1, $c_2 \dots$ to denote constants depending only on parameters that are not important for the problem under consideration.

2. Auxiliary results

Considering only the canonical quasiconformal reflections, I. M. Batchaev [3] generalized the integral representation (3) to functions $f \in A^1(G_1)$. An accurate proof of the Batchaev's result is given in [2, p. 110, Th. 4.4]. Here we prove an analog of this integral representation for unbounded domains. Namely, the following result holds.

Lemma 1. *Let $f \in A^1(G_2)$. If $y(z)$ is a canonical quasiconformal reflection with respect to L, then we have*

$$f(z) = -\frac{1}{\pi} \iint_{G_1} \frac{(f \circ y)(\zeta)}{(\zeta - z)^2 [y(\zeta)]^2} y_{\bar{\zeta}}(\zeta)\, d\sigma_\zeta, \qquad z \in G_2. \tag{6}$$

Proof. Let $y(z)$ be a canonical quasiconformal reflection and $f \in A^1(G_2)$. If we substitute $\zeta = 1/u$ for $\zeta \in G_2$ and define

$$f(\zeta) = f(1/u) =: f_*(u),$$

then G_2 maps to a finite domain G_u and $f_* \in A^1(G_u)$. If $y^*(t)$ is a canonical quasiconformal reflection with respect to ∂G_u, from Batchaev's result we have

$$f_*(t) = -\frac{1}{\pi}\iint\limits_{C\overline{G_u}} \frac{(f_* \circ y^*)(u)}{(u-t)^2} y^*_{\overline{u}}(u) d\sigma_u, \qquad t \in G_u,$$

where $C\overline{G_u} := \mathbb{C}\backslash\overline{G_u}$. Substituting $u = 1/\zeta$ in this integral representation we get

$$f(z) = f(1/t) = f_*(t) = -\frac{1}{\pi}\iint\limits_{G_1} \frac{(f_* \circ y^*)(1/\zeta)}{(1/\zeta - 1/z)^2} y^*_{\overline{u}}(1/\zeta)\, J d\sigma_\zeta$$

$$= \frac{1}{\pi}\iint\limits_{G_1} \frac{f\left[1/y^*(1/\zeta)\right] z^2}{(\zeta - z)^2} y^*_{\overline{\zeta}}(1/\zeta)\, d\sigma_\zeta, \qquad z \in G_2.$$

If we define

$$y(\zeta) := \frac{1}{y^*(1/\zeta)},$$

then $y(\zeta)$ becomes a canonical quasiconformal reflection with respect to L. Consequently, for $f \in A^1(G_2)$ we get

$$f(z) = -\frac{1}{\pi}\iint\limits_{G_1} \frac{(f \circ y)(\zeta) z^2}{(\zeta - z)^2 [y(\zeta)]^2}\, y_{\overline{\zeta}}(\zeta) d\sigma_\zeta, \qquad z \in G_2.$$

□

From now on, the reflection $y(z)$ assumed to be a canonical K-quasiconformal reflection with respect to L.

Let $f \in A^1(G_2)$. Substituting $\zeta = \psi(w)$ in (6), we get

$$f(z) = -\frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(f \circ y)\left[\psi(w)\right] \overline{\psi'}(w) y_{\overline{\zeta}}\left[\psi(w)\right]}{\left[(y \circ \psi)(w)\right]^2} \cdot \frac{z^2 \psi'(w)}{\left[\psi(w) - z\right]^2}\, d\sigma_w$$

$$= -\frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(f \circ y)\left[\psi(w)\right] \overline{\psi'}(w) y_{\overline{\zeta}}\left[\psi(w)\right]}{\left[(y \circ \psi)(w)\right]^2 g\left[\psi(w)\right]} \cdot \frac{g\left[\psi(w)\right] z^2 \psi'(w)}{\left[\psi(w) - z\right]^2}\, d\sigma_w, \qquad z \in G_2. \tag{7}$$

Thus, if we define the coefficients $a_m(f, g)$ by

$$a_m(f,g) := \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(f \circ y)\left[\psi(w)\right] \overline{\psi'}(w)}{w^{m+\nu+1} g\left[\psi(w)\right]\left[(y \circ \psi)(w)\right]^2} y_{\overline{\zeta}}\left[\psi(w)\right] d\sigma_w, \; m = 1, 2, \ldots \tag{8}$$

then, by (2) and (7), we can associate a formal series $\sum_{m=1}^{\infty} a_m(f,g)F'_m$ $(1/z,g)$ with the function $f \in A^1(G_2)$, i.e.,

$$f(z) \sim \sum_{m=1}^{\infty} a_m(f,g)F'_m\left(1/z,g\right).$$

We call this formal series a generalized Faber series of $f \in A^1(G_2)$, and the coefficients $a_m(f,g)$ are called generalized Faber coefficients of f.

For $R > 1$ we set

$$G_{2,R} := \{z : z \in G_1, 1 < |\varphi(z)| < R\} \cup \overline{G_2}.$$

Lemma 2. *Let g be an analytic function on G_1 and let for some fixed constant $R_0 \in (1,\infty)$*

$$\iint\limits_{G_{2,R_o}\setminus G_2} |g(z)|^2 \, d\sigma_z < \infty.$$

Then the series

$$\sum_{m=1}^{\infty} \frac{|F'_m\left(1/z,g\right)|}{m+1}$$

is convergent uniformly on compact subsets of G_2.

Proof. Let z be a fixed point in G_2. Then the power series

$$\sum_{m=1}^{\infty} \frac{F'_m\left(1/z,g\right)}{m+1} w^{m+1}$$

defines an analytic function

$$A(z,w) := \sum_{m=1}^{\infty} \frac{F'_m\left(1/z,g\right)}{m+1} w^{m+1}, \qquad w \in D \tag{9}$$

in D. By taking the derivative of (9) with respect to w and considering (2) we get

$$A'_w(z,w) := \sum_{m=1}^{\infty} F'_m\left(1/z,g\right) w^m = -\frac{z^2\psi'\left(1/w\right) g\left[\psi\left(1/w\right)\right]}{\left[\psi\left(1/w\right) - z\right]^2 w^2}, \qquad w \in D. \tag{10}$$

Let $0 < r < 1$. Since

$$\sum_{m=1}^{\infty} F'_m\left(1/z,g\right) w^m$$

is convergent uniformly and absolutely on the closed disc $\overline{D}(0,r)$, the relation (10) implies that

$$\iint\limits_{\overline{D}(0,r)} \left|A'_w(z,w)\right|^2 d\sigma_w = \pi \sum_{m=1}^{\infty} \frac{\left|F'_m\left(1/z,g\right)\right|^2}{m+1} r^{2m+2}. \tag{11}$$

Hence by (10) and (11) we have

$$\pi \sum_{m=1}^{\infty} \frac{\left|F'_m\left(1/z,g\right)\right|^2}{m+1} r^{2m+2} = \iint\limits_{\overline{D}(0,r)} \left|\frac{z^2\psi'\left(1/w\right) g\left[\psi\left(1/w\right)\right]}{\left[\psi\left(1/w\right)-z\right]^2 w}\right|^2 d\sigma_w. \tag{12}$$

On the other hand, for the fixed constant $R_0 \in (1,\infty)$ we get

$$\begin{aligned} S(z) :&= \iint\limits_{D} \left|\frac{z^2\psi'\left(1/w\right) g\left[\psi\left(1/w\right)\right]}{\left[\psi\left(1/w\right)-z\right]^2 w^2}\right|^2 d\sigma_w \\ &= \int_0^1\int_0^{2\pi} \left|\frac{z^2\psi'\left(e^{-i\theta}/r\right) g\left[\psi\left(e^{-i\theta}/r\right)\right]}{\left[\psi\left(e^{-i\theta}/r\right)-z\right]^2 r^2 e^{2i\theta}}\right|^2 r\,dr\,d\theta \\ &= \int_1^{\infty}\int_0^{2\pi} \left|\frac{z^2\psi'\left(Re^{-i\theta}\right) g\left[\psi\left(Re^{-i\theta}\right)\right]}{\left[\psi\left(Re^{-i\theta}\right)-z\right]^2 (1/R^2)e^{2i\theta}}\right|^2 \frac{1}{R^3}\,dR\,d\theta \\ &= \left(\int_1^{R_0}\int_0^{2\pi} + \int_{R_0}^{\infty}\int_0^{2\pi}\right)\cdots =: J_1 + J_2. \end{aligned} \tag{13}$$

and

$$\begin{aligned} J_1 &= \int_1^{R_0}\int_0^{2\pi} \frac{|z|^4\left|\psi'\left(Re^{-i\theta}\right)\right|^2\left|g\left[\psi\left(Re^{-i\theta}\right)\right]\right|^2}{\left|\left[\psi\left(Re^{-i\theta}\right)-z\right]\right|^4} R\,dR\,d\theta \\ &\le c_3 \int_1^{R_0}\int_0^{2\pi} \left|\psi'\left(Re^{-i\theta}\right)\right|^2\left|g\left[\psi\left(Re^{-i\theta}\right)\right]\right|^2 dR\,d\theta \\ &= c_3 \iint\limits_{G_{2,R_o}\setminus\overline{G_2}} |g(z)|^2\,d\sigma_z < \infty. \end{aligned}$$

Analogously one can establish the uniform boundedness of the integral J_2. Consequently, from (13) we have

$$S(z) < \infty.$$

On the other hand, letting $r \to 1$ in (12) we get

$$\pi \sum_{m=1}^{\infty} \frac{|F'_m(1/z, g)|^2}{m+1} = S(z).$$

Since $S(z)$ is continuous in G_2 with respect to z, the Dini's theorem implies that the series

$$\sum_{m=1}^{\infty} \frac{|F'_m(1/z, g)|^2}{m+1}$$

is convergent uniformly on compact subsets of G_2. □

Lemma 3. *If $f \in A^2(G_2, \omega)$ and $y(\zeta)$ a canonical K-quasiconformal reflection with respect to L, then*

$$\iint_{G_1} |(f \circ y)(\zeta)|^2 \omega[y(\zeta)] |y_{\bar{\zeta}}(\zeta)|^2 d\sigma_\zeta \le \frac{\|f\|^2_{A^2(G_2,\omega)}}{1-k^2},$$

where $k := (K-1)/(K+1)$.

Proof. Since $\overline{y}(\zeta)$ is a canonical K-quasiconformal mapping of the extended complex plane onto itself, we have $|\overline{y}_{\bar{\zeta}}|/|\overline{y}_\zeta| \le k$ and $|\overline{y}_\zeta|^2 - |\overline{y}_{\bar{\zeta}}|^2 > 0$. Also, it is known that $|\overline{y}_{\bar{\zeta}}| = |y_\zeta|$ and $|\overline{y}_\zeta| = |y_{\bar{\zeta}}|$. Therefore, $|y_\zeta|/|y_{\bar{\zeta}}| \le k$ and $|y_{\bar{\zeta}}|^2 - |y_\zeta|^2 > 0$. Hence

$$\begin{aligned}
&\iint_{G_1} |(f \circ y)(\zeta)|^2 \omega[y(\zeta)] |y_{\bar{\zeta}}(\zeta)|^2 d\sigma_\zeta \\
&= \iint_{G_1} |(f \circ y)(\zeta)|^2 \omega[y(\zeta)] \left(1 - |y_\zeta|^2 / |y_{\bar{\zeta}}|^2\right)^{-1} \left(|y_{\bar{\zeta}}|^2 - |y_\zeta|^2\right) d\sigma_\zeta \\
&\le \frac{1}{1-k^2} \iint_{G_1} |(f \circ y)(\zeta)|^2 \omega[y(\zeta)] \left(|y_{\bar{\zeta}}|^2 - |y_\zeta|^2\right) d\sigma_\zeta.
\end{aligned}$$

Since $\left(|y_\zeta|^2 - |y_{\bar{\zeta}}|^2\right)$ is the Jacobian of $y(\zeta)$, substituting ζ for $y(\zeta)$ in the right side of the last inequality we get

$$\iint_{G_1} |(f \circ y)(\zeta)|^2 \omega[y(\zeta)] |y_{\bar{\zeta}}(\zeta)|^2 d\sigma_\zeta \le \frac{\|f\|^2_{A^2(G_2,\omega)}}{1-k^2}.$$

□

Lemma 4. *Let g be an analytic function, bounded in G_1, non-vanishing in $G_1 \backslash \{0\}$ and having at $z = 0$ a zero of order $\nu \geq 2$. Then*

$$a_n\left(F'_m, g\right) = \begin{cases} 1, & m = n-1; \\ 0, & m \neq n-1. \end{cases}$$

Proof. Since $y(z)$ is identical on L, using Green's formulae and the Cauchy integral theorem, we have

$$\begin{aligned} a_n(F'_m, g) &= \frac{1}{\pi} \iint\limits_{C\overline{D}} \frac{F'_m\left[1/y\left(\psi\left(w\right)\right), g\right] \overline{\psi'}(w)}{w^{n+\nu+1} [y(\psi(w))]^2 g\left[\psi\left(w\right)\right]} y_{\overline{\zeta}}\left[\psi\left(w\right)\right] d\sigma_w \\ &= \frac{1}{\pi} \iint\limits_{C\overline{D}} -\frac{\partial}{\partial \overline{w}} \left(\frac{F_m\left[1/y\left(\psi\left(w\right)\right), g\right]}{g\left[\psi\left(w\right)\right] w^{n+\nu+1}} \right) d\sigma_w \\ &= \frac{1}{2\pi i} \int\limits_{|w|=1} \frac{F_m\left[1/\psi\left(w\right), g\right]}{g\left[\psi\left(w\right)\right] w^{n+\nu+1}} \, dw \\ &= \frac{1}{2\pi i} \int\limits_{|w|=R>1} \frac{F_m\left[1/\psi\left(w\right), g\right]}{g\left[\psi\left(w\right)\right] w^{n+\nu+1}} \, dw. \end{aligned}$$

Since, by (1)

$$F_m\left(1/z, g\right) = g(z) \varphi^{m+\nu+1}(z) + E_m(z, g),$$

where $E_m(z, g)$ is analytic in G_1 and $E_m(0, g) = const$, we get

$$\begin{aligned} a_n(F'_m, g) &= \frac{1}{2\pi i} \int\limits_{|w|=R>1} w^{m-n} dw + \frac{1}{2\pi i} \int\limits_{|w|=R>1} \frac{E_m\left[\psi\left(w\right), g\right]}{g\left[\psi\left(w\right)\right] w^{n+\nu+1}} dw \\ &= \frac{1}{2\pi i} \int\limits_{|w|=R>1} w^{m-n} dw = \begin{cases} 1, & m = n-1; \\ 0, & m \neq n-1. \end{cases} \end{aligned}$$

□

Consider the expansion

$$\varphi^m(z) = F_m(1/z) + Q_m(z), \qquad m = 1, 2, \ldots.$$

It is easily to verify that $F_m(1/z)$ is a polynomial of order m with respect to $1/z$. The following lemma holds.

Lemma 5. *For every natural numbers n, the following estimation holds*

$$\sum_{m=n+2}^{\infty} \frac{\left\| F'_{m,z} \right\|^2_{A^2(G_2)}}{m R^{2m}} \leq \frac{\pi}{R^{2(n+1)} (R^2 - 1)}, \qquad m = 1, 2, \ldots.$$

Proof. Let $S_m(G_2)$ be the area of the image of G_2 under $F_m(1/z)$ on the Riemann surface of $F_m(1/z)$. Since

$$[F_m(1/z) \circ \psi(w)] = w^m + \sum_{\upsilon=1}^{\infty} b_\upsilon w^{-\upsilon}, \qquad |w| > 1$$

(see [12, p. 255]) by means of a theorem due to Lebedev-Millin (given in [12, p. 170]), we have

$$S_m(G_2) = \pi \left(m - \sum_{\upsilon=1}^{\infty} \upsilon \left| b_\upsilon \right|^2 \right) \le m\pi. \tag{14}$$

On the other hand

$$S_m(G_2) = \iint_{G_2} \left| F'_{m,z} \right|^2 d\sigma_z = \left\| F'_{m,z} \right\|^2_{A^2(G_2)}. \tag{15}$$

From (14) and (15), it follows that

$$\sum_{m=n+2}^{\infty} \frac{\|F'_{m,z}\|^2_{A^2(G_2)}}{mR^{2m}} \le \pi \sum_{m=n+2}^{\infty} \frac{1}{R^{2m}} = \frac{\pi}{R^{2(n+1)}(R^2-1)}.$$

□

In general, we can not reduce

$$\frac{\pi}{R^{2(n+1)}(R^2-1)}$$

in the inequality above. In fact, if we consider the unit disc D, then $F_m(1/z) = 1/z^m$ and

$$\sum_{m=n+2}^{\infty} \frac{\|F'_{m,z}\|^2_{A^2(G_2)}}{mR^{2m}} = \frac{\pi}{R^{2(n+1)}(R^2-1)}.$$

3. Proof of the new results

Proof of Theorem 1. Let $f \in A^2(G_2, \omega)$, $\omega \in W^2(G_2)$. First of all we prove that $f \in A^1(G_2)$. Taking into account that g has at $z = 0$ a zero of order

$\nu \geq 2$, and using the relations (5) and (4) we get

$$\iint\limits_{G_2} |(g \circ y)(z)|^2 \, d\sigma_z = \iint\limits_{G_1} |g(z)|^2 \left(|y_{\overline{z}}|^2 - |y_z|^2\right) d\sigma_z$$
$$\leq \iint\limits_{G_1} |g(z)|^2 \, |y_{\overline{z}}|^2 \, d\sigma_z = \iint\limits_{G_{2,R} \setminus \overline{G_2}} |g(z)|^2 \, |y_{\overline{z}}|^2 \, d\sigma_z + \iint\limits_{CG_{2,R}} |g(z)|^2 \, |y_{\overline{z}}|^2 \, d\sigma_z$$
$$\leq c_4 \iint\limits_{G_{2,R} \setminus \overline{G_2}} |g(z)|^2 \, d\sigma_z + c_5 < \infty.$$

Hence, by virtue of Hölder's inequality

$$\left(\iint\limits_{G_2} |f(z)| \, d\sigma_z\right)^2 \leq \left(\iint\limits_{G_2} |f(z)|^2 \, \omega(z) d\sigma_z\right)\left(\iint\limits_{G_2} |(g \circ y)(z)|^2 \, d\sigma_z\right) < \infty.$$

Then by means of (7), (8) and Hölder's inequality we obtain

$$\begin{aligned} &\left| f(z) - \sum_{m=1}^{n} a_m(f,g) F'_m\left(1/z, g\right) \right|^2 \\ &\leq \frac{1}{\pi} \iint\limits_{C\overline{D}} \left| \frac{f\left[y(\psi(w))\right] \overline{\psi'}(w) y_{\overline{\zeta}}\left[\psi(w)\right]}{[y(\psi(w))]^2 g\left[\psi\left(w\right)\right]} \right|^2 d\sigma_w \\ &\qquad \times \iint\limits_{C\overline{D}} \left| \frac{g\left[\psi\left(w\right)\right] z^2 \psi'(w)}{\left[\psi\left(w\right) - z\right]^2} + \sum_{m=1}^{n} \frac{F'_m\left(1/z, g\right)}{w^{m+\nu+1}} \right|^2 d\sigma_w \\ &= \frac{1}{\pi} J_1 \cdot J_2 \end{aligned} \tag{16}$$

for every $z \in G_2$.

Since

$$\max_{z \in \overline{G_1}} |y\left(z\right)| \geq const > 0,$$

by virtue of Lemma 3 we have

$$\begin{aligned} J_1 = \iint\limits_{G_1} \left| \frac{f\left[y\left(z\right)\right] y_{\overline{z}}(z)}{\left[y\left(z\right)\right]^2 g(z)} \right|^2 d\sigma_z &\leq c_6 \iint\limits_{G_1} \left| f\left[y\left(z\right)\right]\right|^2 \omega\left[y\left(z\right)\right] |y_{\overline{z}}(z)|^2 \, d\sigma_z \\ &\leq c_6 \frac{\|f\|^2_{A^2(G_2,\omega)}}{1-k^2} < \infty, \end{aligned} \tag{17}$$

where the constant c_6 depends only on L. We now estimate the integral J_2. Let $1 < r < R < \infty$. In view of (2)

$$\begin{aligned}
&\iint\limits_{r<|w|<R} \left| \frac{z^2\psi'(w)g\left[\psi\left(w\right)\right]}{\left[\psi\left(w\right)-z\right]^2} + \sum_{m=1}^{n} \frac{F_m'\left(1/z,g\right)}{w^{m+\nu+1}} \right|^2 d\sigma_w \\
&= \iint\limits_{r<|w|<R} \left| \sum_{m=n+1}^{\infty} \frac{F_m'\left(1/z,g\right)}{w^{m+\nu+1}} \right|^2 d\sigma_w \\
&= \pi \sum_{m=n+1}^{\infty} \frac{1}{m+\nu} \left(\frac{1}{r^{2(m+\nu)}} - \frac{1}{R^{2(m+\nu)}} \right) \left| F_m'\left(1/z,g\right) \right|^2 \\
&\le \pi \sum_{m=n+1}^{\infty} \frac{\left| F_m'\left(1/z,g\right) \right|^2}{m+\nu},
\end{aligned}$$

and by letting $r \to 1$ and $R \to \infty$, we get

$$J_2 \le \pi \sum_{m=n+1}^{\infty} \frac{\left| F_m'\left(1/z,g\right) \right|^2}{m+\nu}. \tag{18}$$

Therefore, by (16), (17) and (18), the following estimate holds

$$\left| f(z) - \sum_{m=1}^{n} a_m(f,g) F_m'\left(1/z,g\right) \right|^2 \le c_7 \sum_{m=n+1}^{\infty} \frac{\left| F_m'\left(1/z,g\right) \right|^2}{m+\nu},$$

and then Lemma 2 completes the proof. □

Proof of Theorem 2. Let

$$\widetilde{S_n}\left(1/z\right) := \sum_{m=1}^{n+1} b_m F_m'\left(1/z,g\right)$$

be the nth partial sum of

$$\sum_{m=1}^{\infty} b_m F_m'\left(1/z,g\right).$$

Using Lemma 4 we get

$$\lim_{n\to\infty} \frac{1}{\pi} \iint\limits_{C\overline{D}} \frac{(\widetilde{S_n}\circ y)[\psi(w)]\overline{\psi'}(w)}{w^{m+\nu+1} g[\psi(w)][y(\psi(w))]^2} y_{\overline{\zeta}}\left[\psi\left(w\right)\right] d\sigma_w = b_m, \quad m = 1, 2, \ldots. \tag{19}$$

On the other hand, by using Hölder's inequality and Lemma 3 we have

$$
\begin{aligned}
&|a_m(f,g) - b_m| \\
&\leq \frac{1}{\pi}\left| \iint\limits_{C\overline{D}} \frac{\left[(f\circ y)\left(\psi(w)\right) - (\widetilde{S_n}\circ y)\left(\psi(w)\right)\right]\overline{\psi'}(w)}{w^{m+\nu+1} g\left(\psi(w)\right)\left[y(\psi(w))\right]^2} y_{\overline{\zeta}}\left(\psi\left(w\right)\right) d\sigma_w \right| \\
&\quad + \left| \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(\widetilde{S_n}\circ y)[\psi(w)]\overline{\psi'}(w)}{w^{m+\nu+1} g\left(\psi(w)\right)\left[y(\psi(w))\right]^2} y_{\overline{\zeta}}\left(\psi\left(w\right)\right) d\sigma_w - b_m \right| \\
&\leq \frac{1}{\pi}\left(\iint\limits_{C\overline{D}} \frac{d\sigma_w}{|w|^{2(m+\nu+1)}} \right)^{1/2} \\
&\times \left(\iint\limits_{C\overline{D}} \frac{\left|(f\circ y)\left(\psi(w)\right) - (\widetilde{S_n}\circ y)\left(\psi(w)\right)\right|^2 |\psi'(w)|^2 \left|y_{\overline{\zeta}}\left(\psi\left(w\right)\right)\right|^2}{\left|g\left(\psi(w)\right)\right|^2 \left|y(\psi(w))\right|^4} d\sigma_w \right)^{1/2} \\
&\quad + \left| \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(\widetilde{S_n}\circ y)[\psi(w)]\overline{\psi'}(w)}{w^{m+\nu+1} g\left(\psi(w)\right)\left[y(\psi(w))\right]^2} y_{\overline{\zeta}}\left(\psi\left(w\right)\right) d\sigma_w - b_m \right| \\
&\leq \frac{c_8}{\sqrt{m+\nu}} \left(\iint\limits_{G_1} \left| \frac{\left[\left(f - \widetilde{S_n}\right)\circ y\right](\zeta)}{g(\zeta)} \right|^2 \left| y_{\overline{\zeta}}\left(\zeta\right)\right|^2 d\sigma_\zeta \right)^{1/2} \\
&\quad + \left| \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(\widetilde{S_n}\circ y)[\psi(w)]\overline{\psi'}(w)}{w^{m+\nu+1} g\left(\psi(w)\right)\left[y(\psi(w))\right]^2} y_{\overline{\zeta}}\left(\psi\left(w\right)\right) d\sigma_w - b_m \right| \\
&\leq \frac{c_8 \left\| f - \widetilde{S_n} \right\|_{A^2(G_2,\omega)}}{\sqrt{(m+\nu)\left(1-k^2\right)}} \\
&\quad + \left| \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{(\widetilde{S_n}\circ y)[\psi(w)]\overline{\psi'}(w)}{w^{m+\nu+1} g\left(\psi(w)\right)\left[y(\psi(w))\right]^2} y_{\overline{\zeta}}\left(\psi\left(w\right)\right) d\sigma_w - b_m \right| \qquad (20)
\end{aligned}
$$

for every natural number n. Since $\lim\limits_{n\to\infty} \left\| f - \widetilde{S_n} \right\|_{A^2(G_2,\omega)} = 0$, (19) and (20) show that $a_m(f,g) = b_m$, $m = 1, 2, \ldots$. □

Proof of Theorem 3. Let P_n^* be the best approximant polynomial to $f \in A^2(G_{2,R})$ in the norm $\|\cdot\|_{A^2\left(G_{2,R}\right)}$, i.e.,

$$\|f - P_n^*\|_{A^2\left(G_{2,R}\right)} = E_n(f, G_{2,R}).$$

In a manner similar to the proof of Theorem 1 we can prove that the sequence $\{S_n\}$ of the partial sums $S_n(f, 1/z)$, $n = 1, 2, \ldots$, converges uniformly to $f \in A^2(G_{2,R})$ on compact subsets of $G_{2,R}$, which implies that

$$|f(z) - S_n(f, 1/z)| = \left| \sum_{m=n+2}^{\infty} a_m(f) F'_m(1/z) \right|$$

$$= \frac{1}{\pi} \left| \sum_{m=n+2}^{\infty} \iint_{|w|>R} \frac{((f - P_n^*) \circ y_R)(\psi(w)) \overline{\psi'}(w) y_{R_{\bar{\zeta}}}(\psi(w))}{[y_R(\psi(w))]^2} \cdot \frac{F'_m(1/z)}{w^{m+1}} \, d\sigma_w \right|$$

for every $z \in G_2$. Applying now Hölder's inequality and Lemma 3, we obtain

$$|f(z) - S_n(f, 1/z)|^2 \leq \frac{c_9 E_n^2(f, G_{2,R})}{\pi \left(1 - k_R^2\right)} \sum_{m=n+2}^{\infty} \frac{|F'_m(1/z)|^2}{m R^{2m}}.$$

Multiplying both sides of this inequality by $1/|z|^4$ we have

$$|f(z) - S_n(f, 1/z)|^2 \frac{1}{|z|^4} \leq \frac{c_9 E_n^2(f, G_{2,R})}{\pi \left(1 - k_R^2\right)} \sum_{m=n+2}^{\infty} \frac{\left|F'_{m,z}(1/z)\right|^2}{m R^{2m}}.$$

Now, integrating both sides over G_2 and using Lemma 5, we conclude that

$$\|f(z) - S_n(f, \cdot)\|^2_{A^2(G_2,\omega)} \leq \frac{c_9 E_n^2(f, G_{2,R})}{\left(1 - k_R^2\right)(R^2 - 1) R^{2(n+1)}},$$

i.e.,

$$\|f(z) - S_n(f, \cdot)\|_{A^2(G_2,\omega)} \leq \frac{c E_n(f, G_{2,R})}{\sqrt{\left(1 - k_R^2\right)(R^2 - 1)} R^{(n+1)}}.$$

for all natural numbers n. □

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