STABILITY OF $n$-TH ORDER FLETT’S POINTS AND LAGRANGE’S POINTS

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Abstract. In this article we show the stability of Flett’s points and Lagrange’s points in the sense of Hyers and Ulam.

In 1958 Flett [1] proved the following variant of Lagrange’s Mean Value Theorem:

**Theorem 1.** If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a) = f'(b)$ then there exists a point $\eta$ such that

$$f(\eta) - f(a) = f'(\eta)(\eta - a).$$

The problem of stability of functional equations was first posed by S. M. Ulam in 1960 [2]. He asked “when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation”? The first result concerned stability of homomorphisms was proved by Hyers [3]. In [4] Das, Riedel and Sahoo proved the Hyers-Ulam stability of Flett’s points i.e. such points $\eta$ for which Flett’s Mean Value Theorem holds true. Let

$$\mathcal{F} = \{f : [a, b] \to \mathbb{R} : f \text{ is continuously differentiable}, f(a) = 0, f'(a) = f'(b)\}.$$

Their result is the following:

**Theorem 2.** Let $f \in \mathcal{F}$ and $\eta$ be a Flett’s point of $f$ in $(a, b)$. Assume that there is a neighbourhood $N$ of $\eta$ in $(a, b)$ such that $\eta$ is the unique Flett’s point of $f$ in $N$. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every $h \in \mathcal{F}$ satisfying $|h(x) - f(x)| < \delta$ for all $x \in N$, there exists a point $\xi \in N$ such that $\xi$ is a Flett’s point of $h$ and $|\xi - \eta| < \varepsilon$.

The main tool in their work is the result due to Hyers and Ulam [5]:

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Theorem 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be \( n \)-times differentiable in a neighbourhood \( N \) of the point \( \eta \). Suppose that \( f^{(n)}(\eta) = 0 \) and \( f^{(n)} \) changes sign at \( \eta \). Then, for all \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for every function \( g : \mathbb{R} \to \mathbb{R} \) which is \( n \)-times differentiable in \( N \) and satisfies \( |f(x) - g(x)| < \delta \) for all \( x \in N \), there exists a point \( \xi \in N \) with \( g^{(n)}(\xi) = 0 \) and \( |\xi - \eta| < \varepsilon \).

In [6] we proved the following extension of Flett’s Mean Value Theorem:

Theorem 4. Let \( f : [a, b] \to \mathbb{R} \) be an \( n \) times differentiable function with \( f^{(n)}(a) = f^{(n)}(b) \). Then there exists a point \( \eta \in (a, b) \) such that

\[
f(\eta) - f(a) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} f^{(k)}(\eta)(\eta - a)^k.
\]

Let \( \mathcal{F}^{bc}_n = \{ f : [a, b] \to \mathbb{R} | f \text{ is } n \text{-times continuously differentiable,} \]
\[ f(a) = 0, \quad f^{(n)}(a) = f^{(n)}(b) \} \).

By Theorem 4 for any \( f \in \mathcal{F}^{bc}_n \), there is an intermediate point \( \eta \in (a, b) \) such that (1) holds. Such an intermediate point \( \eta \) will be called the \( n \)-th order Flett’s point of the function \( f \) with boundary condition.

Now we want to investigate the Hyers-Ulam stability of such points. Our result reads as follows:

Theorem 5. Let \( f \in \mathcal{F}^{bc}_n \) and \( \eta \) be a \( n \)-th order Flett’s point of \( f \) with boundary condition in \((a, b)\). Suppose that there exists a neighbourhood \( N \) of \( \eta \) in \((a, b)\) such that \( \eta \) is the unique \( n \)-th order Flett’s point of \( f \) with boundary condition in \( N \). Then for each \( \varepsilon > 0 \), there exists a positive constant \( \delta \) such that for every \( h \in \mathcal{F}^{bc}_n \) satisfying \( ||h - f||_{n-1} < \delta \) in \( N \), there exists a point \( \xi \in N \) such that \( \xi \) is a \( n \)-th order Flett’s point of \( h \) with boundary condition and \( |\xi - \eta| < \varepsilon \).

In the proof we will need the following lemma (cf.[6, pg.282]).

Lemma 2.1. [6] Let \( n \) be a nonnegative integer and \( f : [a, b] \to \mathbb{R} \) be \( n \) times differentiable. Define \( g : (a, b) \to \mathbb{R} \) by

\[
g(x) = \frac{f(x) - f(a)}{x - a}.
\]

Then \( g \) is \( n \) times differentiable for all \( x \in (a, b) \) and we have

\[
g^{(n)}(x) = (-1)^n n! \frac{f(x) - f(a)}{x - a} + \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} f^{(k)}(x)(x - a)^k.
\]
STABILITY OF $n$-TH ORDER FLETT’S POINTS AND LAGRANGE’S POINTS

that is

$$g^{(n)}(x) = \frac{f^{(n)}(x)}{x-a} - n \frac{g^{(n-1)}(x)}{x-a}, \quad x \in (a, b].$$

Moreover, if $f^{(n+1)}(a)$ exists, we have

$$\lim_{x \to a} g^{(n)}(x) = \frac{1}{n+1} f^{(n+1)}(a).$$

Proof of Theorem 5. The first part of the proof is similar to the one after Theorem 3 in [4].

Let $N = (\eta - r, \eta + r)$ for some $r > 0$ satisfy our assumptions and let $c = \eta - r - a, (c > 0)$.

We define new functions $F : (a, b] \to \mathbb{R}$ and $G_f : [a, b] \to \mathbb{R}$ by the following formulas

$$F(x) = \frac{f(x) - f(a)}{x-a},$$

$$G_f(x) = \begin{cases} F^{(n-1)}(x), & \text{if } x \in (a, b], \\ \frac{1}{n} f^{(n)}(a), & \text{if } x = a. \end{cases}$$

By Lemma 2.1 we see that $F$ is $n$-times differentiable and thus $G_f$ is well defined and continuous. We can assume that $f^{(n)}(a) = f^{(n)}(b) = 0$, otherwise, we replace the function $f$ by $n! f(x) - x^n f^{(n)}(a)$.

Now we will show that there exists a point $\eta \in (a, b)$ such that $G_f(\eta) = 0$.

It follows that $F^{(n)}(\eta) = 0$ which is equivalent to the equation (1) so $\eta$ is the $n$-th order Flett’s point of $f$ with boundary condition in $(a, b)$. It is easily seen that $G_f(a) = 0$.

Let us assume that $G_f(b) = 0$. Then by Rolle’s Theorem there exists an intermediate point $\eta \in (a, b)$ such that $G_f'(\eta) = 0$. In the other case we have $G_f(b) \neq 0$. We will work under the assumption $G_f(b) > 0$. From Lemma 2.1 we conclude that

$$G_f'(b) = \frac{f^{(n)}(b)}{b-a} - n \frac{G_f(b)}{b-a} < 0.$$

Hence there exists a point $x_1 < b$ such that

$$G_f(x_1) > G_f(b) > G_f(a) = 0.$$

Thus there exists a $x_0 \in (a, x_1)$ such that $G_f(x_0) = G_f(b)$. Now applying Rolle’s Theorem we get an intermediate point $\eta \in (a, b)$ such that $G_f'(\eta) = F^{(n)}(\eta) = 0$.

Let $\varepsilon > 0$ be given. Analogously, for any function $h \in F^{bc}_n$ we define new
functions $H : (a, b] \to \mathbb{R}$ and $G_h : [a, b] \to \mathbb{R}$ by

$$H(x) = \frac{h(x) - h(a)}{x-a},$$
$$G_h(x) = \begin{cases} H^{(n-1)}(x), & \text{if } x \in (a, b], \\ \frac{1}{n!}h^{(n)}(a), & \text{if } x = a. \end{cases}$$

Applying Theorem 3 we see that there exists a $\delta > 0$ such that for every function $R : [a, b] \to \mathbb{R}$ which is differentiable in $N$ we have

$$|R(x) - G_f(x)| < \delta$$

for every $x \in N$ and there exists a point $\xi \in N$ with $R'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. Let

$$\tilde{\delta} = (n-1)!(\frac{1}{c^n} + \frac{1}{c^{n-1}} + \frac{1}{2c^{n-2}} + \frac{1}{3c^{n-3}} + \cdots + \frac{1}{(n-2)!c^2} + \frac{1}{c}) > 0.$$

Now we show that if $||h - f||_{n-1} = \max_{x \in N}\{|f(x) - h(x)|, |f'(x) - h'(x)|, \ldots, |f^{n-1}(x) - h^{n-1}(x)|\} < \tilde{\delta}$ in $N$ for every $h \in \mathcal{F}_n$, then $|G_f(x) - G_h(x)| < \delta$ where $x \in N$. If $||h - f||_{n-1} < \tilde{\delta}$ then applying Lemma 2.1 we obtain for every $x \in N$

$$|G_f(x) - G_h(x)| \leq \frac{(n-1)!}{|x-a|^n} |f(x) - h(x)| + \frac{(n-1)!}{|x-a|^{n-1}} |f'(x) - h'(x)| + \frac{(n-1)!}{2|x-a|^{n-2}} |f''(x) - h''(x)| + \cdots + \frac{(n-1)!}{(n-1)!|x-a|^2} |f^{(n-1)}(x) - h^{(n-1)}(x)| + \frac{(n-1)!}{c^n} |f(x) - h(x)|$$

where the last inequality follows from the definition of $\tilde{\delta}$. From Theorem 3 we conclude that there exists a point $\xi \in N$ such that $G_h'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. From $G_h'(\xi) = 0$ it is easily seen that $\xi$ is a $n$-th order Flett’s point of $h$ with boundary condition. Now the proof of the theorem is complete. \qed
Davitt, Powers, Riedel and Sahoo [7] removed the boundary assumption on the derivative and they proved the following generalization of Flett’s Mean Value Theorem:

**Theorem 6.** Let \( f : [a, b] \to \mathbb{R} \) be differentiable on \([a, b]\). Then there exists a point \( \eta \in (a, b) \) such that

\[
f(\eta) - f(a) = (\eta - a)f'(\eta) - \frac{1}{2} \frac{f'(b) - f'(a)}{b - a} (\eta - a)^2.
\]

In [6] we proved the following “Taylor-like” extension of Flett’s MVT.

**Theorem 7.** Let \( f : [a, b] \to \mathbb{R} \) be \( n \) times differentiable. Then there exists a point \( \eta \in (a, b) \) such that

\[
f(\eta) - f(a) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} f^{(k)}(\eta)(\eta - a)^k + (-1)^n \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a} (\eta - a)^{n+1}. \tag{3}
\]

We call \( \mathcal{F}_n = \{ f : [a, b] \to \mathbb{R} \mid f \text{ is } n\text{-times continuously differentiable, } f(a) = 0 \} \).

From Theorem 7 we know that for any \( f \in \mathcal{F}_n \) there exists a point \( \eta \in (a, b) \) such that formula (3) holds. Such an intermediate point we call the \( n \)th order Flett’s point of the function \( f \) in \((a, b)\).

Now we prove the stability of \( n \)th order Flett’s points.

**Theorem 8.** Let \( f \in \mathcal{F}_n \) and \( \eta \) be \( n \)th order Flett’s point of \( f \) in \((a, b)\). Suppose that there exists a neighborhood \( N \) of \( \eta \) in \((a, b)\) such that \( \eta \) is the unique \( n \)th order Flett’s point of \( f \) in \( N \). Then for each \( \varepsilon > 0 \), there exist positive constants \( \delta, \delta_a, \delta_b \) such that for every \( h \in \mathcal{F}_n \) satisfying \( \|h - f\|_{n-1} < \delta \) in \( N \), \( |h^{(n)}(a) - f^{(n)}(a)| < \delta_a \), \( |h^{(n)}(b) - f^{(n)}(b)| < \delta_b \), there exists a point \( \xi \in N \) such that \( \xi \) is a \( n \)th order Flett’s point of \( h \) and \( |\xi - \eta| < \varepsilon \).

**Proof.** Let \( \varepsilon > 0 \) be given and assume that \( N = (\eta - r, \eta + r) \) for some \( r > 0 \) satisfies assumptions of the theorem.

Let \( c = \eta - r - a \). We define new function \( \varphi : [a, b] \to \mathbb{R} \) by the formula

\[
\varphi(x) = f(x) - \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a} (x - a)^{n+1}.
\]

In view of our assumptions on \( f \) it follows that \( \varphi \) is \( n \)-times differentiable. We can show that \( \varphi^{(n)}(a) = \varphi^{(n)}(b) \). It is easily seen that if \( \eta \) is a \( n \)-th order Flett’s point of \( f \) in \((a, b)\) then \( \eta \) is the \( n \)-th order Flett’s point of \( \varphi \) in \((a, b)\) with boundary condition. So we have that if \( f \in \mathcal{F}_n \) then \( \varphi \in \mathcal{F}_n^{bc} \).
By Theorem 5 we know that there is a $\delta > 0$ such that for every function $R \in F_{nc}^\beta$ such that $\| R - \varphi \|_{n-1} < \delta$ in $N$ we get that there exists a $\xi \in N$ such that $\xi$ is the $n$-th order Flett’s point of $R$ with boundary condition in $N$ and $|\xi - \eta| < \delta$. Let

$$\tilde{\delta} = \frac{\delta}{3}, \delta_a = \delta_b = \frac{\delta}{3\sum_{k=1}^{n-1} \frac{c^k}{(k+1)!}}.$$

We define new function $\psi : [a, b] \rightarrow \mathbb{R}$, by the formula

$$\psi(x) = h(x) - \frac{1}{(n+1)!} \frac{h^{(n)}(b) - h^{(n)}(a)}{b - a} (x - a)^{n+1}.$$

We show that if

$$||f - h||_{n-1} < \tilde{\delta}, \ |h^{(n)}(a) - f^{(n)}(a)| < \delta_a, \ |h^{(n)}(b) - f^{(n)}(b)| < \delta_b \quad (4)$$

then $||\varphi - \psi||_{n-1} < \delta$ in $N$. Assume that (4) are satisfied. Then we have

$$||\varphi - \psi||_{n-1} \leq |f(x) - h(x)| + |f'(x) - h'(x)| + |f''(x) - h''(x)|$$

$$+ |f^{(n-1)}(x) - h^{(n-1)}(x)| + (|f^{(n)}(a) - h^{(n)}(a)| + |f^{(n)}(b) - h^{(n)}(b)|)$$

$$\cdot \left( |x - a|^n \frac{|x - a|^{n-1}}{n! |b - a|} + \cdots + \frac{|x - a|^2}{2! |b - a|} \right) < ||f - h||_{n-1}$$

$$+ (|f^{(n)}(a) - h^{(n)}(a)| + |f^{(n)}(b) - h^{(n)}(b)|) \left( \frac{c^{n-1}}{n!} + \frac{c^{n-2}}{(n-1)!} + \cdots + \frac{c}{2!} \right) < \delta$$

Analogously we get that if $\eta$ is a $n$-th order Flett’s point of $\psi$ in $(a, b)$ with boundary condition then $\eta$ is the $n$-th order Flett’s point of $h$ in $(a, b)$ which completes the proof. □

The similar result can be obtained with the Lagrange Mean Value Theorem.

**Theorem 9.** Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a point $\eta \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(\eta). \quad (5)$$

For every differentiable function $f$ there exists an intermediate point $\eta$ such that (5) holds. Such a point $\eta$ will be called the Langrange’s point of the function $f$.

Let us define

$$\mathcal{L} = \{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuously differentiable, } f(a) = 0 \}.$$

We will prove the following theorem:
Theorem 10. Let $f \in \mathcal{L}$ and $\eta$ be Lagrange’s point of $f$ in $(a, b)$. Suppose that there exists a neighbourhood $N$ of $\eta$ in $(a, b)$ such that $\eta$ is the unique Lagrange’s point of $f$ in $N$. Then for each $\varepsilon > 0$, there exist positive constants $\delta_0$ and $\delta_b$ such that for every $h \in \mathcal{L}$ satisfying $|h(x) - f(x)| < \delta_0$, $|h(b) - f(b)| < \delta_b$ for all $x \in N$, there exists a point $\xi \in N$ such that $\xi$ is Lagrange’s point of $h$ and $|\xi - \eta| < \varepsilon$.

Proof. Assume that $N = (\eta - r, \eta + r)$ for some $r > 0$ and $c = \eta - r - a$ ($c > 0$). Let us define new function $G_f : [a, b] \to \mathbb{R}$ by

$$G_f(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a).$$

Since $G_f$ is continuously differentiable and $G_f(a) = G_f(b) = f(a)$ it follows that there exists a point $\eta \in (a, b)$ such that $G_f'(\eta) = 0$.

Let $h : [a, b] \to \mathbb{R}$ have the asserted properties. Analogously we define $G_h : [a, b] \to \mathbb{R}$ by the following formula

$$G_h(x) = h(x) - \frac{h(b) - h(a)}{b - a} (x - a).$$

Let $\varepsilon > 0$ be given. From Theorem 3 it follows that there is a $\delta > 0$ such that for every function $R : [a, b] \to \mathbb{R}$ which is differentiable in $N$ satisfying $|R(x) - G_f(x)| < \delta$ in $N$ we get that there exists a $\xi$ such that $R'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.

Let us define $\delta_0 = \delta_b = \frac{\delta}{2}$. Now if $|h(x) - f(x)| < \delta_0$, $|h(b) - f(b)| < \delta_b$ then we have for all $x \in N$

$$|G_h(x) - G_f(x)| \leq |h(x) - f(x)| + \frac{|h(b) - f(b)|}{|b - a|} |x - a| + \frac{|h(a) - f(a)|}{|b - a|} |x - a| < \delta.$$

This concludes the proof. 

References


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