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STABILITY OF n-TH ORDER FLETT'S POINTS AND LAGRANGE'S POINTS

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ABSTRACT. In this article we show the stability of Flett's points and Lagrange's points in the sense of Hyers and Ulam.

In 1958 Flett [1] proved the following variant of Lagrange's Mean Value Theorem:

Theorem 1. *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a) = f'(b)$ then there exists a point η such that*

$$f(\eta) - f(a) = f'(\eta)(\eta - a).$$

The problem of stability of functional equations was first posed by S. M. Ulam in 1960 [2]. He asked "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation"? The first result concerned stability of homomorphisms was proved by Hyers [3]. In [4] Das, Riedel and Sahoo proved the Hyers-Ulam stability of Flett's points i.e. such points η for which Flett's Mean Value Theorem holds true. Let

$$\mathcal{F} = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuously differentiable}, \quad f(a) = 0, f'(a) = f'(b)\}.$$

Their result is the following:

Theorem 2. *Let $f \in \mathcal{F}$ and η be a Flett's point of f in (a, b). Assume that there is a neighbourhood N of η in (a, b) such that η is the unique Flett's point of f in N. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every $h \in \mathcal{F}$ satisfying $|h(x) - f(x)| < \delta$ for all $x \in N$, there exists a point $\xi \in N$ such that ξ is a Flett's point of h and $|\xi - \eta| < \varepsilon$.*

The main tool in their work is the result due to Hyers and Ulam [5]:

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Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable in a neighbourhood N of the point η. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}$ changes sign at η. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for all $x \in N$, there exists a point $\xi \in N$ with $g^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

In [6] we proved the following extension of Flett's Mean Value Theorem:

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be an n times differentiable function with $f^{(n)}(a) = f^{(n)}(b)$. Then there exists a point $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} f^{(k)}(\eta)(\eta - a)^k. \tag{1}$$

Let

$$\mathcal{F}_n^{bc} = \{f : [a, b] \to \mathbb{R} \mid f \text{ is } n\text{-times continuously differentiable}, \quad f(a) = 0,\ f^{(n)}(a) = f^{(n)}(b)\}.$$

By Theorem 4 for any $f \in \mathcal{F}_n^{bc}$, there is an intermediate point $\eta \in (a, b)$ such that (1) holds. Such an intermediate point η will be called the n-th order Flett's point of the function f with boundary condition.

Now we want to investigate the Hyers-Ulam stability of such points. Our result reads as follows:

Theorem 5. *Let $f \in \mathcal{F}_n^{bc}$ and η be a n-th order Flett's point of f with boundary condition in (a, b). Suppose that there exists a neighbourhood N of η in (a, b) such that η is the unique n-th order Flett's point of f with boundary condition in N. Then for each $\varepsilon > 0$, there exists a positive constant $\tilde{\delta}$ such that for every $h \in \mathcal{F}_n^{bc}$ satisfying $||h - f||_{n-1} < \tilde{\delta}$ in N, there exists a point $\xi \in N$ such that ξ is a n- th order Flett's point of h with boundary condition and $|\xi - \eta| < \varepsilon$.*

In the proof we will need the following lemma (cf.[6, pg.282]).

Lemma 2.1. [6] *Let n be a nonnegative integer and $f : [a, b] \to \mathbb{R}$ be n times differentiable. Define $g : (a, b] \to \mathbb{R}$ by*

$$g(x) = \frac{f(x) - f(a)}{x - a}.$$

Then g is n times differentiable for all $x \in (a, b]$ and we have

$$g^{(n)}(x) = (-1)^n n! \frac{f(x) - f(a) + \sum_{k=1}^{n} (-1)^k \frac{1}{k!} f^{(k)}(x)(x - a)^k}{(x - a)^{n+1}},$$

that is

$$g^{(n)}(x) = \frac{f^{(n)}(x)}{x-a} - n\frac{g^{(n-1)}(x)}{x-a}, \; x \in (a,b].$$

Moreover, if $f^{(n+1)}(a)$ *exists, we have*

$$\lim_{x \to a} g^{(n)}(x) = \frac{1}{n+1} f^{(n+1)}(a).$$

Proof of Theorem 5. The first part of the proof is similar to the one after Theorem 3 in [4].

Let $N = (\eta - r, \eta + r)$ for some $r > 0$ satisfy our assumptions and let $c = \eta - r - a$, $(c > 0)$.

We define new functions $F : (a,b] \to \mathbb{R}$ and $G_f : [a,b] \to \mathbb{R}$ by the following formulas

$$F(x) = \frac{f(x) - f(a)}{x-a},$$

$$G_f(x) = \begin{cases} F^{(n-1)}(x), & \text{if } x \in (a,b], \\ \frac{1}{n} f^{(n)}(a), & \text{if } x = a. \end{cases}$$

By Lemma 2.1 we see that F is n-times differentiable and thus G_f is well defined and continuous. We can assume that $f^{(n)}(a) = f^{(n)}(b) = 0$, otherwise, we replace the function f by $n!f(x) - x^n f^{(n)}(a)$.

Now we will show that there exists a point $\eta \in (a,b)$ such that $G'_f(\eta) = 0$. It follows that $F^{(n)}(\eta) = 0$ which is equivalent to the equation (1) so η is the n-th order Flett's point of f with boundary condition in (a,b). It is easily seen that $G_f(a) = 0$.

Let us assume that $G_f(b) = 0$. Then by Rolle's Theorem there exists an intermediate point $\eta \in (a,b)$ such that $G'_f(\eta) = 0$. In the other case we have $G_f(b) \neq 0$. We will work under the assumption $G_f(b) > 0$. From Lemma 2.1 we conclude that

$$G'_f(b) = \frac{f^{(n)}(b)}{b-a} - n\frac{G_f(b)}{b-a} < 0.$$

Hence there exists a point $x_1 < b$ such that

$$G_f(x_1) > G_f(b) > G_f(a) = 0.$$

Thus there exists a $x_0 \in (a, x_1)$ such that $G_f(x_0) = G_f(b)$. Now applying Rolle's Theorem we get an intermediate point $\eta \in (a,b)$ such that $G'_f(\eta) = F^{(n)}(\eta) = 0$.

Let $\varepsilon > 0$ be given. Analogously, for any function $h \in \mathcal{F}_n^{bc}$ we define new functions $H : (a, b] \to \mathbb{R}$ and $G_h : [a, b] \to \mathbb{R}$ by

$$H(x) = \frac{h(x) - h(a)}{x - a},$$
$$G_h(x) = \begin{cases} H^{(n-1)}(x), & \text{if } x \in (a, b], \\ \frac{1}{n} h^{(n)}(a), & \text{if } x = a. \end{cases}$$

Applying Theorem 3 we see that there exists a $\delta > 0$ such that for every function $R : [a, b] \to \mathbb{R}$ which is differentiable in N we have

$$|R(x) - G_f(x)| < \delta \tag{2}$$

for every $x \in N$ and there exists a point $\xi \in N$ with $R'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. Let

$$\tilde{\delta} = \frac{\delta}{(n-1)!(\frac{1}{c^n} + \frac{1}{c^{n-1}} + \frac{1}{2!c^{n-2}} + \frac{1}{3!c^{n-3}} + \cdots + \frac{1}{(n-2)!c^2} + \frac{1}{c})} > 0.$$

Now we show that if $||h - f||_{n-1} = \max_{x \in N}\{|f(x) - h(x)|, |f'(x) - h'(x)|, \dots, |f^{n-1}(x) - h^{n-1}(x)|\} < \tilde{\delta}$ in N for every $h \in \mathcal{F}_n^{bc}$ then $|G_f(x) - G_h(x)| < \delta$ where $x \in N$. If $||h - f||_{n-1} < \tilde{\delta}$ then applying Lemma 2.1 we obtain for every $x \in N$

$$\begin{aligned} |G_f(x) - G_h(x)| &\le \frac{(n-1)!}{|x-a|^n}|f(x) - h(x)| + \frac{(n-1)!}{|x-a|^{n-1}}|f'(x) - h'(x)| \\ &+ \frac{(n-1)!}{2!|x-a|^{n-2}}|f''(x) - h''(x)| + \frac{(n-1)!}{3!|x-a|^{n-3}}|f'''(x) - h'''(x)| + \dots \\ &+ \frac{(n-1)!}{(n-1)!|x-a|}|f^{(n-1)}(x) - h^{(n-1)}(x)| < \frac{(n-1)!}{c^n}|f(x) - h(x)| \\ &+ \frac{(n-1)!}{c^{n-1}}|f'(x) - h'(x)| + \frac{(n-1)!}{2!c^{n-2}}|f''(x) - h''(x)| \\ &+ \frac{(n-1)!}{3!c^{n-3}}|f'''(x) - h'''(x)| + \cdots + \frac{(n-1)!}{(n-1)!c}|f^{(n-1)}(x) - h^{(n-1)}(x)| < \\ &||f - h||_{n-1}(n-1)!\left(\frac{1}{c^n} + \frac{1}{c^{n-1}} + \frac{1}{2!c^{n-2}} + \frac{1}{3!c^{n-3}} + \cdots + \frac{1}{(n-2)!c^2} + \frac{1}{c}\right) \\ &< \delta \end{aligned}$$

where the last inequality follows from the definition of $\tilde{\delta}$. From Theorem 3 we conclude that there exists a point $\xi \in N$ such that $G_h'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. From $G_h'(\xi) = 0$ it is easily seen that ξ is a n-th order Flett's point of h with boundary condition. Now the proof of the theorem is complete. □

Davitt, Powers, Riedel and Sahoo [7] removed the boundary assumption on the derivative and they proved the following generalization of Flett's Mean Value Theorem:

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Then there exists a point $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = (\eta - a)f'(\eta) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(\eta - a)^2.$$

In [6] we proved the following "Taylor-like" extension of Flett's MVT.

Theorem 7. *Let $f : [a, b] \to \mathbb{R}$ be n times differentiable. Then there exists a point $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!}f^{(k)}(\eta)(\eta - a)^k + (-1)^n\frac{1}{(n+1)!}\,\frac{f^{(n)}(b) - f^{(n)}(a)}{b - a}(\eta - a)^{n+1}. \quad (3)$$

We call

$$\mathcal{F}_n = \{f : [a, b] \to \mathbb{R} \mid f \text{ is } n\text{-times continously differentiable, } f(a) = 0\}.$$

From Theorem 7 we know that for any $f \in \mathcal{F}_n$ there exists a point $\eta \in (a, b)$ such that formula (3) holds. Such an intermediate point we call the n-th order Flett's point of the function f in (a, b).

Now we prove the stability of n-th order Flett's points.

Theorem 8. *Let $f \in \mathcal{F}_n$ and η be n-th order Flett's point of f in (a, b). Suppose that there exists a neighborhood N of η in (a, b) such that η is the unique n-th order Flett's point of f in N. Then for each $\varepsilon > 0$, there exist positive constants $\tilde{\delta}$, δ_a, δ_b such that for every $h \in \mathcal{F}_n$ satisfying $||h-f||_{n-1} < \tilde{\delta}$ in N, $|h^{(n)}(a) - f^{(n)}(a)| < \delta_a$, $|h^{(n)}(b) - f^{(n)}(b)| < \delta_b$, there exists a point $\xi \in N$ such that ξ is a n-th order Flett's point of h and $|\xi - \eta| < \varepsilon$.*

Proof. Let $\varepsilon > 0$ be given and assume that $N = (\eta - r, \eta + r)$ for some $r > 0$ satisfies assumptions of the theorem.

Let $c = \eta - r - a$. We define new function $\varphi : [a, b] \to \mathbb{R}$ by the formula

$$\varphi(x) = f(x) - \frac{1}{(n+1)!}\,\frac{f^{(n)}(b) - f^{(n)}(a)}{b - a}(x - a)^{n+1}.$$

In view of our assumptions on f it follows that φ is n-times differentiable. We can show that $\varphi^{(n)}(a) = \varphi^{(n)}(b)$. It is easily seen that if η is a n-th order Flett's point of f in (a, b) then η is the n-th order Flett's point of φ in (a, b) with boundary condition. So we have that if $f \in \mathcal{F}_n$ then $\varphi \in \mathcal{F}_n^{bc}$.

By Theorem 5 we know that there is a $\delta > 0$ such that for every function $R \in \mathcal{F}_n^{bc}$ such that $||R - \varphi||_{n-1} < \delta$ in N we get that there exists a $\xi \in N$ such that ξ is the n-th order Flett's point of R with boundary condition in N and $|\xi - \eta| < \delta$. Let

$$\tilde{\delta} = \frac{\delta}{3}, \ \ \delta_a = \delta_b = \frac{\delta}{3\sum_{k=1}^{n-1} \frac{c^k}{(k+1)!}}.$$

We define new function $\psi : [a, b] \to \mathbb{R}$, by the formula

$$\psi(x) = h(x) - \frac{1}{(n+1)!}\,\frac{h^{(n)}(b) - h^{(n)}(a)}{b-a}\,(x-a)^{n+1}.$$

We show that if

$$||f - h||_{n-1} < \tilde{\delta}, \quad |h^{(n)}(a) - f^{(n)}(a)| < \delta_a, \quad |h^{(n)}(b) - f^{(n)}(b)| < \delta_b \quad (4)$$

then $||\varphi - \psi||_{n-1} < \delta$ in N. Assume that (4) are satisfied. Then we have

$$\begin{aligned} &||\varphi - \psi||_{n-1} \le |f(x) - h(x)| + |f'(x) - h'(x)| + |f''(x) - h''(x)| \\ &+ |f^{(n-1)}(x) - h^{(n-1)}(x)| + (|f^n(a) - h^n(a)| + |f^n(b) - h^n(b)|) \\ &\cdot \left(\frac{|x-a|^n}{n!|b-a|} + \frac{|x-a|^{n-1}}{(n-1)!|b-a|} + \cdots + \frac{|x-a|^2}{2!|b-a|}\right) < ||f-h||_{n-1} \\ &+ (|f^n(a) - h^n(a)| + |f^n(b) - h^n(b)|)\left(\frac{c^{n-1}}{n!} + \frac{c^{n-2}}{(n-1)!} + \cdots + \frac{c}{2!}\right) < \delta \end{aligned}$$

Analogously we get that if η is a n-th order Flett's point of ψ in (a, b) with boundary condition then η is the n-th order Flett's point of h in (a, b) which completes the proof. □

The similar result can be obtained with the Lagrange Mean Value Theorem.

Theorem 9. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\eta \in (a, b)$ such that*

$$f(b) - f(a) = (b-a)f'(\eta). \quad (5)$$

For every differentiable function f there exists an intermediate point η such that (5) holds. Such a point η will be called the Langrange's point of the function f.

Let us define

$$\mathcal{L} = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuously differentiable}, f(a) = 0\}.$$

We will prove the following theorem:

Theorem 10. *Let $f \in \mathcal{L}$ and η be Lagrange's point of f in (a, b). Suppose that there exists a neighbourhood N of η in (a, b) such that η is the unique Lagrange's point of f in N. Then for each $\varepsilon > 0$, there exist positive constants δ_0 and δ_b such that for every $h \in \mathcal{L}$ satisfying $|h(x) - f(x)| < \delta_0$, $|h(b) - f(b)| < \delta_b$ for all $x \in N$, there exists a point $\xi \in N$ such that ξ is Lagrange's point of h and $|\xi - \eta| < \varepsilon$.*

Proof. Assume that $N = (\eta - r, \eta + r)$ for some $r > 0$ and $c = \eta - r - a$ $(c > 0)$. Let us define new function $G_f : [a, b] \to \mathbb{R}$ by

$$G_f(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Since G_f is continuously differentiable and $G_f(a) = G_f(b) = f(a)$ it follows that there exists a point $\eta \in (a, b)$ such that $G'_f(\eta) = 0$.

Let $h : [a, b] \to \mathbb{R}$ have the asserted properties. Analogously we define $G_h : [a, b] \to \mathbb{R}$ by the following formula

$$G_h(x) = h(x) - \frac{h(b) - h(a)}{b - a}(x - a).$$

Let $\varepsilon > 0$ be given. From Theorem 3 it follows that there is a $\delta > 0$ such that for every function $R : [a, b] \to \mathbb{R}$ which is differentiable in N satisfying $|R(x) - G_f(x)| < \delta$ in N we get that there exists a ξ such that $R'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.

Let us define $\delta_0 = \delta_b = \frac{\delta}{2}$. Now if $|h(x) - f(x)| < \delta_0$, $|h(b) - f(b)| < \delta_b$ then we have for all $x \in N$

$$|G_h(x) - G_f(x)| \le |h(x) - f(x)| + \frac{|h(b) - f(b)|}{|b - a|}|x - a| + \frac{|h(a) - f(a)|}{|b - a|}|x - a| < \delta.$$

This concludes the proof. □

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