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A REMARK ON VETRIVEL'S EXISTENCE THEOREM ON KY FAN'S BEST APPROXIMANT

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ABSTRACT. Vetrivel (1996) proved an existence theorem on Ky fan's best approximant for multifunction with open inverse values. The object of this paper is to present an improved and extended version of this result.

1. INTRODUCTION

Let $\mathcal{M}$ be a nonempty compact convex subset of a Hausdorff locally convex topological vector space E. In 1969, Fan [1] established a best approximation theorem which says that for any continuous function $f : \mathcal{M} \to E$ there exists a continuous seminorm p on E and a point z in $\mathcal{M}$ such that

$$p\big(f(z) - z\big) = \inf \big\{p\big(f(z) - x\big) : x \in M\big\}.$$

Since then various aspects of this theorem have been studied by a number of authors, cf [4] and the references therein.

In 1996, Vetrivel [5] proved an existence theorem on Ky fan's best approximant for multifunction with open inverse values in the setting of Hausdorff locally convex topological vector spaces.

Theorem 1.1. *Let $\mathcal{M}$ be a nonempty compact convex subset of Hausdorff locally convex topological space E. Suppose that $\mathcal{T} : \mathcal{M} \to 2^E$ is a multifunction such that*

(i) *$\mathcal{T}^{-1}(y)$ is open for all $y \in E$;*
(ii) *for every open set U in $\mathcal{M}$, the set $\cap\{\mathcal{T}u : u \in U\}$ is empty or contractible; and*
(iii) *$\mathcal{T}(\mathcal{M})$ is contractible.*

Then, there exists an element $x_0 \in \mathcal{M}$ and $y_0 \in \mathcal{T}(x_0)$ such that

$$p(x_0 - y_0) = \inf \big\{p(y_0 - z) : z \in M\big\}$$

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In this paper, our purpose is to improve and extend the above result. For this purpose, we use the concept given by Tarafdar and Watson [6]. The results which we use to prove the result are due to Horvath [2] and Lassonde [3].

2. Preliminaries

Let us recall the following :

Definition 2.1. Let X and Y be non-empty sets. The collection of all non-empty subsets of X is denoted by 2^X. A multifunction or set-valued function from X to Y is defined to be a function that assigns to each elements of X a non-empty subset of Y. If $\mathcal{T}$ is a multifunction from X to Y, then it is designated as $\mathcal{T} : X \to 2^Y$, and for every $x \in X$, $\mathcal{T}x$ is called a value of $\mathcal{T}$. For $A \subseteq X$, the image of A under $\mathcal{T}$, denoted by $\mathcal{T}(A)$, is defined as

$$\mathcal{T}(A) = \bigcup_{x \in A} \mathcal{T}x$$

For $B \subseteq Y$, the preimage or inverse image of B under $\mathcal{T}$, denoted by $\mathcal{T}^{-1}(B)$, is defined as

$$\mathcal{T}^{-1}(B) = \{x \in X : \mathcal{T}x \cap B \neq \phi\}$$

If $y \in Y$, then $\mathcal{T}^{-1}(y)$ is called a inverse value of $\mathcal{T}$. If it is open, then it is called open inverse value.

Definition 2.2. A multivalued function $\mathcal{T} : X \to 2^Y$ is upper semicontinuous (usc)(lower semicontinuous(lsc)) if $\mathcal{T}^{-1}(B) = \{x \in X : \mathcal{T}x \cap B \neq \phi\}$ is closed(open) for each closed (open) subset B of Y. If $\mathcal{T}$ is both usc and lsc, then it is continuous .

A multifunction $\mathcal{T} : X \to 2^Y$ is said to be a compact multifunction, if $\mathcal{T}(X)$ is contained in a compact subset of Y.

It is known that if $\mathcal{T} : X \to 2^Y$ is an upper semicontinuous multifunction with compact values, then $\mathcal{T}(K)$ is compact in Y whenever K is compact subset of X.

Definition 2.3. A single valued function g from a topological space X to another topological space Y is said to be proper if $g^{-1}(K)$ is compact in X whenever K is compact in Y. It is remarked that if g is continuous and X is a compact space, then the map g is proper.

Throughout, E denotes a Hausdorff locally convex topological vector space and p be a continuous semi-norm on it.

Definition 2.4. Let $\mathcal{M}$ be a convex subset of E and $g : \mathcal{M} \to \mathcal{M}$ a continuous map. Then g is said to be

i) *almost affine*, if

$$p(gy - z) \leq \lambda p(gx_1 - z) + (1 - \lambda)p(gx_2 - z),$$

(ii) *almost quasi-convex* , if

$$p(gy - z) \leq max\{p(gx_1 - z), p(gx_2 - z)\},$$

where $y = \lambda x_1 + (1 - \lambda)x_2, x_1, x_2 \in \mathcal{M}, \lambda \in [0, 1]$ and $z \in E$.

Definition 2.5. [7]. Let $\mathcal{M}$ be a convex subset of E. Let $\mathcal{T} : \mathcal{M} \to 2^E$. A mapping g from $\mathcal{M}$ to E is said to be *almost quasi-convex with respect to* $\mathcal{T}$, if

$$p(gy - \mathcal{T}z) \leq max\{p(gx_1 - \mathcal{T}z), p(gx_2 - \mathcal{T}z)\},$$

where $y = \lambda x_1 + (1 - \lambda)x_2, x_1, x_2 \in \mathcal{M}, \lambda \in [0, 1]$ and $z \in \mathcal{M}$.

Definition 2.6. A single valued function $g : \mathcal{M} \to E$ is said to be $p-$continuous if $p[g(x_\alpha) - g(x)] \to 0$ for each x in M and every net $\{x_\alpha\}$ in $\mathcal{M}$ converging to E.

Definition 2.7. Let $\mathcal{M} \subseteq E$. Let $x \in E$. An element $y \in \mathcal{M}$ is called a best $\mathcal{M}-$approximant to $x \in E$, if

$$p(x - y) = d_p(x, \mathcal{M}) = \inf\{p(x - z) : z \in \mathcal{M}\}.$$

The set of best $\mathcal{M}$-approximants to x with respect to the seminorm p is denoted by $P_{\mathcal{M}}(x)$ and is defined as

$$P_{\mathcal{M}}(x) = \{z \in \mathcal{M} : p(x - z) = d_p(x, \mathcal{M})\}.$$

It is well known that if $\mathcal{M}$ is a nonempty compact convex subset of E, for each $x \in E$, $P_{\mathcal{M}}(x)$ is a nonempty compact convex subset of $\mathcal{M}$ and the function defined by $x \to P_{\mathcal{M}}(x)$ is an upper semicontinuous multifunction.

For $N \in \mathbb{N}$, let $< N >$ be the set of all nonempty subsets of $\{0,1,2,\dots,N\}$. Let $\Delta_n = \text{co}\{e_0, e_1, e_2, \dots, e_n\}$ be the standard simplex of dimensional n, where $\{e_0, e_1, e_2, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^{N+1}$ and for $J \in N$, let $\Delta_J = \text{co}\{e_j : j \in J\}$.

A topological space X is said to be contractible, if the identity mapping I_X of X is homotopic to a constant function. Note that any nonempty convex or star-shaped subset of a topological space is contractible [6].

Horvath [2] proved the following:

Theorem 2.8. [2]. *Let X be a topological space. For any nonempty subset J of $\{0, 1, \dots, n\}$, let Γ_J be a nonempty contractible subset of X. If $\phi \neq J \subset J' \subset \{0, 1, \dots, n\}$ implies $\Gamma_J \subset \Gamma_{J'}$, then there exists a single valued continuous function $f : \Delta_n \to X$ such that $g[\Delta_J] \subseteq \Gamma_J$ for all nonempty subset J of $\{0, 1, \dots, n\}$.*

Also, we need the following fixed point theorem due to Lassonde [3]:

Theorem 2.9. [3]. *Let $\mathcal{T} : \Delta_n \to \Delta_n$ be a multifunction such that $\mathcal{T} = \mathcal{T}_n \circ \mathcal{T}_{n-1} \circ \cdots \circ \mathcal{T}_1 \circ \mathcal{T}_0$, $\Delta_n \to^{\mathcal{T}_0} X_1 \to^{\mathcal{T}_1} X_2 \to^{\mathcal{T}_2} \cdots \to^{\mathcal{T}_n} X_{n+1} = \Delta_n$, where each $\mathcal{T}_i$ is either a single-valued continuous function (in which case X_{i+1} is assumed to be a Hausdorff topological space) or an upper semicontinuous multifunction with $\mathcal{T}_i(x)$, a nonempty compact convex subset of X_{i+1} (in which case X_{i+1} is a convex subset of a Hausdorff topological vector space). Then, there exists a point $x_0 \in \Delta_n$ such that $x_0 \in \mathcal{T}(x_0)$.*

3. Main result

First, we prove our main result.

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty compact convex subset of a Hausdorff locally convex topological space E. Suppose that $\mathcal{T} : \mathcal{M} \to 2^E$ is a multifunction and*

(i) *$\mathcal{T}^{-1}(y)$ contains an open set $\mathcal{O}_y$(which may be empty) such that $\bigcup_{y\in\mathcal{T}(\mathcal{M})} \mathcal{O}_y = \mathcal{M}$;*
(ii) *for every open set $\mathcal{V}$ in $\mathcal{M}$, the set $\cap\{\mathcal{T}v : v \in \mathcal{V}\}$ is empty or contractible;*
(iii) *$\mathcal{T}(\mathcal{M})$ is contractible;*
(iv) *$g : \mathcal{M} \to \mathcal{M}$ is a $p-$continuous, proper, almost quasi convex with respect to $\mathcal{T}$ and surjective single valued map.*

Then, there exists an element $x_0 \in \mathcal{M}$ such that

$$d_p(gx_0, \mathcal{T}x_0) = d_p(\mathcal{T}x_0, \mathcal{M})$$

Proof. We first show that there exist an $n-$simplex Δ_n and two functions $f : \Delta_n \to \mathcal{T}(\mathcal{M})$ and $h : \mathcal{M} \to \Delta_n$ such that $f(h(x)) \in \mathcal{T}(x)$ for all $x \in \mathcal{M}$.

Since $\mathcal{M}$ is compact and $\bigcup_{y\in\mathcal{T}(\mathcal{M})} \mathcal{O}_y = \mathcal{M}$, there exists a finite subset $\{y_0, y_1, y_2, \ldots, y_n\} \subset \mathcal{T}(\mathcal{M})$ such that $\mathcal{M} = \bigcup_{i=0}^n \mathcal{O}_{y_i}$. Now, for each nonempty subset J of $N = \{0, 1, 2, \ldots, n\}$, define

$$\Gamma_J = \begin{cases} \cap\{\mathcal{T}(x) : x \in \bigcap_{j\in J} \mathcal{O}_{y_j}\}, & \text{if } \bigcap_{j\in J} \mathcal{O}_{y_j} \neq \phi, \\ \mathcal{T}(\mathcal{M}), & \text{otherwise} \end{cases}$$

Evidently, if $x \in \bigcap_{j\in J} \mathcal{O}_{y_j}$, then $\{y_j : j \in J\} \subset \mathcal{T}(x)$. By($ii$), each Γ_J is nonempty contractible and it is clear that $\Gamma_J \subseteq \Gamma_{J'}$, whenever $\phi \neq J \subset J' \subset N$.

By Theorem 2.8, there exists a single valued continuous function $f : \Delta_n \to \mathcal{T}(\mathcal{M})$ such that $f[\Delta_J] \subseteq \Gamma_J$, for all $\phi \neq J \subset N$. Let $\{h_0, h_1, \ldots, h_n\}$ be a continuous partition of unity subordinated to the open covering $\{\mathcal{O}_{y_i}\}_{i\in N}$ i.e., for each $i \in N$, $h_i : \mathcal{M} \to [0, 1]$ is continuous; $\{x \in \mathcal{M} : h_i(x) \neq 0\} \subset \mathcal{O}_{y_i}$ such that $\sum_{i=0}^n h_i(x) = 1$ for all $x \in \mathcal{M}$.

Define $h : \mathcal{M} \to \Delta_n$ by

$$h(x) = \big(h_0(x), h_1(x), h_2(x), \dots, h_n(x)\big) \text{ for all } x \in \mathcal{M}.$$

Then, h is continuous. Then, $h(x) \subset \Delta_{J(x)}$ for all $x \in \mathcal{M}$, where $J(x) : \{j \in N : h_j(x) \neq 0\}$. Therefore, we have

$$f\big(h(x_0)\big) \in f(\Delta_{J(x)}) \subseteq \Gamma_{J(x)} \subseteq \mathcal{T}(x), \text{ for all } x \in \mathcal{M}. \tag{3.1}$$

Let $\mathcal{G} = g^{-1}P_A : E \to \mathcal{M}$. Since g is a almost quasi convex with respect to $\mathcal{T}$, $\mathcal{G}$ is a convex valued multi-function.

In fact, let $x_1, x_2 \in \mathcal{G}(x)$ and $\lambda \in [0,1]$. Since g is almost quasi-convex with respect to $\mathcal{T}$, it follows that

$$p\big(g(\lambda x_1 + (1-\lambda)x_2) - \mathcal{T}x\big) \leq \max\big\{p(g(x_1) - \mathcal{T}x), p(g(y_2) - \mathcal{T}x)\big\} = d_p(\mathcal{T}x, \mathcal{M}) \tag{3.2}$$

Since g is onto and $\mathcal{M}$ is convex,

$$p\left(g(\lambda x_1 + (1-\lambda)x_2) - \mathcal{T}x\right) \geq d_p(\mathcal{T}x, \mathcal{M}) \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$g\big(\lambda x_1 + (1-\lambda)x_2\big) \in P_{\mathcal{M}}\big(\mathcal{T}(x)\big)$$

and hence

$$\lambda x_1 + (1-\lambda)x_2 \in g^{-1}\big(P_{\mathcal{M}}(\mathcal{T}(x))\big)$$

Thus $\mathcal{G}(x)$ is convex for each $x \in \mathcal{M}$.

Moreover $\mathcal{G}x$ is compact as g^{-1} sends compact sets to compact sets and $P_{\mathcal{M}}(x)$ is compact. Also, $\mathcal{G}$ is a compact multi-function because both g^{-1} and $P_{\mathcal{M}}$ send compact sets onto compact sets.

It remains to show that $\mathcal{G}$ is upper semicontinuous. To prove this, we show that $\mathcal{G}^{-1}(D)$ is closed for any closed subset D of $\mathcal{M}$. Let D be any closed subset of $\mathcal{M}$ and $\{z_\alpha\}$ be any sequence in $\mathcal{G}^{-1}(D)$ and $z_\alpha \to z_0$ for some $z_0 \in \mathcal{M}$. Since $\mathcal{G}(z_\alpha) \cap D \neq \phi$ for each α, let $w_\alpha \in \mathcal{G}(z_\alpha) \cap D$ which implies $g(w_\alpha) \in P_{\mathcal{M}}(z_\alpha)$; i.e.,

$$d_p\big(g(w_\alpha), z_\alpha\big) = d_p(z_\alpha, \mathcal{M}).$$

Since $z_\alpha \to z_0$, we have $d_p(z_\alpha, \mathcal{M}) \to d_p(z_0, \mathcal{M})$. Since, $P_{\mathcal{M}}(\mathcal{T}(\mathcal{M}))$ is compact and hence by the hypothesis, $g^{-1}(P_{\mathcal{M}}(\overline{\text{co}}\,\mathcal{T}(\mathcal{M}))$ is compact. Now, since

$$w_\alpha \in \mathcal{G}(z_\alpha) \subseteq \mathcal{G}\big(\overline{\text{co}}\,\mathcal{T}(\mathcal{M})\big) = (g^{-1} \circ P_{\mathcal{M}})\big(\overline{\text{co}}\,\mathcal{T}(\mathcal{M})\big)$$

which is compact set, has a convergent subnet. Without loss of generality, we can assume $w_\alpha \to w_0$. Since g is a $p-$continuous, $p\,[g(w_\alpha) - g(w_0)] \to 0$. Now, we have

$$\begin{aligned} p\,[g(w_0) - z_0] &\leq p\,[g(w_0) - g(w_\alpha)] + p\,[g(w_\alpha) - z_\alpha] + p\,(z_\alpha - z_0) \\ &= p\,[g(w_0) - g(w_\alpha)] + d_p(z_\alpha, \mathcal{M}) + p\,(z_\alpha - z_0) \end{aligned}$$

Taking the limit, we see that

$$p\,[g(w_0) - z_0] = d_p(z_0, \mathcal{M}).$$

Thus, $g(w_0) \in P_{\mathcal{M}}(z_0)$ which implies $w_0 \in \mathcal{G}(z_0) \cap D$; i.e., $z_0 \in \mathcal{G}^{-1}(D)$ and so $\mathcal{G}^{-1}(D)$ is closed and hence $\mathcal{G}$ is upper semicontinuous.

Applying Theorem 2.9 to the multifunction $h\mathcal{G}f : \Delta_n \to \Delta_n$, there exists an element $s_0 \in \Delta_n$ such that $s_0 \in h\mathcal{G}f(s_0)$. So, $s_0 \in h(x_0)$ where $x_0 \in \mathcal{M}$ and $g(x_0) \in P_{\mathcal{M}} f s_0$. Hence, from (3.1) $f(s_0) = (fh)(x_0) \in \mathcal{T}(x_0)$.

Let $y_0 = fs_0$, then we have $gx_0 \in P_{\mathcal{M}} y_0$, $d_p(gx_0, y_0) = d_p(y_0, \mathcal{M})$. So, $d_p(gx_0, \mathcal{T}x_0) \le d_p(gx_0, y_0) = d_p(y_0, \mathcal{M})$.

But, since $y_0 \in \mathcal{T}x_0$, $d_p(y_0, \mathcal{M}) = d_p(\mathcal{T}x_0, \mathcal{M})$. So, $d_p(gx_0, Tx_0) \le d_p(\mathcal{T}x_0, \mathcal{M})$.

Also, it is evident that $d_p(\mathcal{T}x_0, \mathcal{M}) \le d_p(gx_0, \mathcal{T}x_0)$. Thus, $d_p(gx_0, \mathcal{T}x_0) = d_p(\mathcal{T}x_0, \mathcal{M})$.

This completes the proof. □

In the Theorem 3.1, if $g = I$, identity mapping, we then get the following consequence:

Corollary 3.2. *Let $\mathcal{M}$ and $\mathcal{T}$ are as in the Theorem 3.1. Then, there exists an element $x_0 \in \mathcal{M}$ such that*

$$d_p(x_0, \mathcal{T}x_0) = d_p(\mathcal{T}x_0, \mathcal{M}).$$

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