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PERTURBATING UPPER SEMI-FREDHOLM WITH STRICTLY SINGULAR OPERATORS

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Abstract. Given an upper semi-Fredholm operator L, following the work of P. Cassaza and N. Kalton [1] and others, we introduce and study a set of upper semi-Fredholm operators having the same index as L and which is invariant under strictly singular perturbations.

1. Introduction

Throughout this work X, Y and Z will be Banach spaces and $L \in \mathcal{L}(X,Y)$, where $\mathcal{L}(X,Y)$ is the Banach space of all bounded linear maps $T : X \to Y$. As usual, we take $\mathcal{L}(X) = \mathcal{L}(X,X)$.

Assume $a, b \in [0,1[, T \in \mathcal{L}(Y)$ and let I be the identity operator on Y. Generalizing a well known theorem of C. Neumann, it was proved by S. H. Hilding [4] in 1948 that if

$$\|(I-T)y\| \leq a\|y\| + b\|Ty\|, \ \forall\, y \in Y,$$

then $T : Y \to Y$ is an onto isomorphism. In 1999, P. Casazza and N. Kalton [1] discuss the more general case where X is a (closed) subspace of Y and $L, T \in \mathcal{L}(X,Y)$ satisfy

$$\|(L-T)x\| \leq a\|Lx\| + b\|Tx\|, \ \forall\, x \in X.$$

Under this assumption they proved that several properties of L, including that of being a Fredholm operator and the value of its index, carry over to T. In this paper we weaken the above inequality and analyze the consequences of assuming instead

$$\|(L-T)x\| \leq a\|Lx\| + \|Tx\| + \|Kx\|, \ \forall\, x \in X. \tag{1}$$

Here the "perturbation" $K : X \to Z$ is strictly singular and the operator $L : X \to Y$ is upper semi-Fredholm. This motivates the introduction and

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study of "perturbation sets" consisting of operators $T \in \mathcal{L}(X, Y)$ satisfying some cases of inequality (1). With this approach we obtain several of the results established in [1].

2. Strictly singular and upper semi-Fredholm operators

Let $T \in \mathcal{L}(X, Y)$. Its kernel will be denoted by $N(T)$ and its range by $R(T)$. If $V \subset X$ is a (linear) subspace, then T_V indicates the restriction of T to V. The operator T is *strictly singular* if when $V \subset X$ is a subspace, T is 1-1 on V and $(T_V)^{-1}$ is continuous, it follows that $\dim V < \infty$. We will denote the space consisting of all these operators by $S(X, Y)$; $S(X) \equiv S(X, X)$.

Let $K \in \mathcal{L}(X, Y)$. Then K is a *compact* operator if for any bounded sequence $\{x_n\} \subset X$, the sequence $\{Kx_n\}$ has a subsequence converging in Y. The space consisting of all these operators will be indicated by $K(X, Y)$; $K(X) \equiv K(X, X)$.

Clearly, if $K \in \mathcal{L}(X, Y)$ and $\dim R(K) < \infty$, then K is compact. Reciprocally, it is well known that if K is compact and $R(K)$ is closed, then $\dim R(K) < \infty$. This implies that compact operators are strictly singular. Thus all the statements concerning strictly singular operators apply, in particular, to those which are compact.

For the convenience of the reader we indicate the following fundamental property of operators whose range is not closed, together with a characterization of strictly singular operators. Both results were established by T. Kato [3, Ch. 3].

Theorem 1. *Let $T \in \mathcal{L}(X, Y)$.*

i) *If $R(T)$ is not closed, then given $\epsilon > 0$ there exists an infinite dimensional subspace $V \subset X$ such that $\|T_V\| \leq \epsilon$.*
ii) *The operator T is strictly singular if, and only if, given $\epsilon > 0$ and an infinite dimensional subspace $V \subset X$, there is an infinite dimensional subspace $W \subset V$ such that $\|T_W\| \leq \epsilon$.*

It is well known, and follows from the above result, that $S(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$

An operator $T \in \mathcal{L}(X, Y)$ is *upper semi-Fredholm* if $\dim N(T) < \infty$ and $R(T)$ is closed. The set consisting of all these operators will be indicated by $\Phi_+(X, Y)$; $\Phi_+(X) \equiv \Phi_+(X, X)$. Let $T \in \Phi_+(X, Y)$. Then its *index* is given by $i(T) = \alpha(T) - \beta(T)$, where $\alpha(T) \equiv \dim N(T)$ and $\beta(T) \equiv \dim Y/R(T)$. If also $\beta(T) < \infty$, then T is said to be a *Fredholm* operator.

Remark 1. The following proposition, with compact operators instead of strictly singular operators appears in [2, Lemma 1.3.12]. Apparently it is due to H. O. Tylli.

Proposition 1. *Let $L \in \mathcal{L}(X,Y)$. Then, $L \in \Phi_+(X,Y)$ if, and only if, there exists a number $C > 0$, a Banach space Z and a strictly singular operator $K \in \mathcal{L}(X,Z)$ such that*

$$\|x\| \leq C\|Lx\| + \|Kx\|, \ \forall\, x \in X. \tag{2}$$

Proof. Assume first $L \in \Phi_+(X,Y)$. Since $N \equiv N(L)$ is finite dimensional, there is a closed subspace $V \subset X$ such that $X = N \oplus V$. Let $P \in \mathcal{L}(X)$ be the corresponding projection onto N.

Noting L_V is 1-1 and $R(L_V) = R(L)$ is closed, from the open mapping theorem follows there is a number $C > 0$ such that

$$\|v\| \leq C\|L_V v\|, \ \forall\, v \in V. \tag{3}$$

Take $x \in X$. Since $x - Px \in V$, (2) implies

$$\|x\| \leq \|x - Px\| + \|Px\| \leq C\|L_V(x - Px)\| + \|Px\| = C\|Lx\| + \|Px\|.$$

Since $\dim R(P) < \infty$, it follows that P is compact and so it is strictly singular. Hence, the conclusion follows.

Suppose now that (2) holds. Thus

$$\|x\| \leq \|Kx\|, \ \forall\, x \in N(L).$$

This inequality states that K is 1-1 on $U = N(L)$ and $(K_U)^{-1}$ is continuous. Since $K \in S(X,Z)$, this implies $\dim N(L) < \infty$.

Assume now that $R(L)$ is not closed. Then, using Theorem 1, we find an infinite dimensional subspace $V \subset X$ such that $\|L_V\| < \dfrac{1}{C}$. By (2), this implies

$$(1 - C\|L_V\|)\,\|x\| \leq \|Kx\|, \ \forall\, x \in V.$$

Thus, K is not strictly singular. So, we conclude that $L \in \Phi_+(X,Y)$. □

Given Banach spaces Z_1, Z_2, we will consider in $Z_1 \times Z_2$ the norm given by $\|(z_1, z_2)\| \equiv \|z_1\| + \|z_2\|$. If $T_1 \in \mathcal{L}(X, Z_1), T_2 \in \mathcal{L}(X, Z_2)$, then the operator $(T_1, T_2) \in \mathcal{L}(X, Z_1 \times Z_2)$ is defined by $(T_1, T_2)x \equiv (T_1 x, T_2 x)$, $\forall\, x \in X$. Thus

$$\|(T_1, T_2)x\| = \|T_1 x\| + \|T_2 x\|, \ \forall\, x \in X.$$

and so $\|(T_1, T_2)\| \leq \|T_1\| + \|T_2\|$.

Lemma 1. *If $K_1 \in S(X, Z_1)$, $K_2 \in S(X, Z_2)$, it follows that $(K_1, K_2) \in S(X, Z_1 \times Z_2)$.*

Proof. Let $V \subset X$ be an infinite dimensional subspace and $\epsilon > 0$. Applying Theorem 1 twice it is possible to find an infinite dimensional subspace $W \subset V$ such that $\|(K_1)_W\| \leq \frac{\epsilon}{2}$, $\|(K_2)_W)\| \leq \frac{\epsilon}{2}$. This implies $\|(K_1, K_2)_W\| \leq \epsilon$. From Theorem 1 it follows now that $(K_1, K_2) \in S(X, Z_1 \times Z_2)$. □

Assume $\dim X < \infty$ and take $T \in \mathcal{L}(X,Y)$. Then $\dim R(T) < \infty$, and so $R(T)$ is closed. It follows that $T \in \Phi_+(X,Y)$. Hence, we have $\Phi_+(X,Y) = \mathcal{L}(X,Y)$. Take now $K \in \mathcal{L}(X,Y)$. Since $\dim R(K) < \infty$, then K is a compact operator. It follows that $S(X,Y) = \mathcal{L}(X,Y)$. So $\Phi_+(X,Y) \cap S(X,Y) = \mathcal{L}(X,Y)$.

When $\dim X = \infty$, the following result indicates a totally different situation.

Lemma 2. *If* $\dim X = \infty$, *then* $\Phi_+(X,Y) \cap S(X,Y) = \emptyset$.

Proof. Assume that $\dim X = \infty$ and $L \in \Phi_+(X,Y)$. We will prove that L is not strictly singular. Take $N = N(T)$ and choose as V a closed subspace $V \subset X$ and $C > 0$ as in (3). Clearly, $\|L_W\| \geq C^{-1}$, for any nonzero subspace $W \subset V$. Applying Theorem 1, we conclude that L is not strictly singular. □

Remark 2. Assume $\dim X = \infty$ and suppose X has closed infinite dimensional subspaces V and W such that $X = V \oplus W$. (Clearly, this situation occurs in the classical Banach spaces.) Let now $P \in \mathcal{L}(X)$ be the corresponding projection onto V. Then $P \notin \Phi_+(X,Y) \cup S(X,Y)$.

3. Perturbation sets

Now, after fixing $L \in \mathcal{L}(X,Y)$, we will consider three kinds of sets. First, we define $F_0(L)$ to consist of all those $T \in \mathcal{L}(X,Y)$ such that

$$\|(L-T)x\| \leq a\|Lx\| + \|Tx\|, \ \forall\, x \in X. \tag{4}$$

for some $a \in [0,1[$. Second, for a fixed Banach space Z and $K \in S(X,Z)$ we define $F_K(L)$ to consist of all those $T \in \mathcal{L}(X,Y)$ satisfying

$$\|(L-T)x\| \leq a\|Lx\| + \|Tx\| + \|Kx\|, \ \forall\, x \in X. \tag{5}$$

for some $a \in [0,1[$. Finally, we define

$$F(L) = \bigcup_K F_K(L). \tag{6}$$

Here the union is over all Banach spaces Z and $K \in S(X,Z)$, such that the inequality (5) holds. Clearly,

$$F_0(L) \subset F_K(L) \ \subset \ F(L), \ \forall\, L \in \mathcal{L}(X,Y). \tag{7}$$

In what follows, we will always have $K \in S(X,Z)$. Let $T \in F(L)$ and take a, Z and K such that (5) holds. Then

$$\|Lx\| \leq \|(L-T)x\| + \|Tx\| \leq a\|Lx\| + 2\|Tx\| + \|Kx\|, \ \forall\, x \in X.$$

Thus

$$(1-a)\|Lx\| \leq 2\|Tx\| + \|Kx\|, \ \forall\, x \in X. \tag{8}$$

Proposition 2. *We have*

i) $\lambda L \in F_0(L),\ \forall\, \lambda \in]0, \infty[$. *In particular,* $L \in F_0(L)$.

ii) *If* $T \in F_K(L)$, *then* $\lambda L + (1-\lambda)T \in F_K(L),\ \forall\, \lambda \in [0,1]$. *So,* $F_K(L)$ *is connected.*

Proof. i) Consider first the case $\lambda \in]0,1]$. Then

$$\|(L - \lambda L)x\| = (1-\lambda)\|Lx\|.$$

Hence, λL satisfies (4). Consider now $\lambda \in]1, \infty[$. Then

$$\|(L - \lambda L)x\| = (\lambda - 1)\|Lx\| \leq \frac{\lambda - 1}{\lambda}\, \|\lambda Lx\|,\ \forall\, x \in X.$$

And so λL satisfies (4).

ii) Consider $\lambda \in [0,1[$. Let $T \in F_K(L)$ and take $a \in [0,1[$ such that (5) holds. Thus

$$\begin{aligned}
&\|[L - (\lambda L + (1-\lambda)T]x\| \\
&\leq (1-\lambda)[a\|Lx\| + \|Tx\| + \|Kx\|] \\
&\leq (1-\lambda)a\|Lx\| + \|[\lambda L + (1-\lambda)T]x\| + \lambda\|Lx\| + \|Kx\|, \\
&\leq [(1-\lambda)a + \lambda]\|Lx\| + \|[\lambda L + (1-\lambda)T]x\| + \|Kx\|,\ \forall\, x \in X.
\end{aligned}$$

Since $(1-\lambda)a + \lambda \in [0,1[$, it follows that $\lambda L + (1-\lambda)T \in F_K(L)$. □

Theorem 2. *Let* $L \in \Phi_+(X,Y)$. *Then:*

i) *If* $T \in F(L)$, *then* $T + S \in F(L),\ \forall\, S \in S(X,Y)$.

ii) $F(L)$ *is connected.*

iii) $F(L) \subset \Phi_+(X,Y)$ *and the index is constant on* $F(L)$.

iv) $F(L)$ *is open.*

Proof. i) Let $T \in F(L)$ and take $\alpha \in [0,1[$, a Banach space Z and K such that (5) holds. Then

$$\begin{aligned}
\|[L - (T+S)]x\| &\leq \|(L-T)x\| + \|Sx\| \\
&\leq a\|Lx\| + \|Tx\| + \|Kx\| + \|Sx\| \\
&\leq a\|Lx\| + \|(T+S)x\| + \|Kx\| + \|2Sx\|,\ \forall\, x \in X.
\end{aligned}$$

Using Lemma 1, the conclusion follows.

ii) The conclusion follows directly from ii) in Proposition 2.

Using Proposition 1, let us next fix for L a Banach space Z_1, $K_1 \in S(X, Z_1)$ and a number $C > 0$ such that

$$\|x\| \leq C\|Lx\| + \|K_1 x\|,\ \forall\, x \in X. \tag{9}$$

Likewise, we consider $T \in F(L)$ and take $a \in [0,1[$, Z_2 a Banach space and $K_2 \in S(Y, Z_2)$ such that

$$\|(L-T)x\| \leq a\|Lx\| + \|Tx\| + \|K_2x\|, \ \forall\, x \in X.$$

iii) Proceeding as we did to obtain (8), from the above inequality follows

$$(1-a)\,\|Lx\| \leq 2\|Tx\| + \|K_2x\|, \ \forall\, x \in X. \tag{10}$$

Then, by (9) and (10) we have

$$\begin{aligned}(1-a)\|x\| &\leq C(1-a)\|Lx\| + (1-a)\|K_1x\| \\ &\leq 2C\|Tx\| + \|CK_2x\| + (1-a)\|K_1x\|, \ \forall\, x \in X.\end{aligned}$$

By Lemma 1, $K \equiv ((1-a)K_1, CK_2) \in S(X, Z_1 \times Z)$. Since $(1-a) > 0$, by Proposition 1 this implies $T \in \Phi_+(X,Y)$.

The index function is continuous on $\Phi_+(X,Y)$ and $F(L) \subset \Phi_+(X,Y)$ is connected. This implies the index is constant on $F(L)$.

iv) Take now $R \in \mathcal{L}(X,Y)$ and a number $r > 0$ such that $a + rC < 1$. Then, if $\|T - R\| \leq r$ and $x \in X$, by (9) we have

$$\begin{aligned}\|(L-R)x\| &\leq \|(L-T)x\| + \|(T-R)x\| \\ &\leq a\|Lx\| + \|Tx\| + \|K_2x\| + r\|x\| \\ &\leq (a+rC)\|Lx\| + \|Tx\| + \|K_2x\| + r\|K_1x\| \\ &\leq (a+rC)\|Lx\| + \|Tx\| + \|(rK_1, K_2)x\|.\end{aligned}$$

By Lemma 1, (rK_1, K_2) is strictly singular. Hence the conclusion follows. □

Remark 3. 1) Assume $\dim X < \infty$. Then $S(X,Y) = \mathcal{L}(X,Y)$ and i) of Theorem 2 implies $F(L) = \mathcal{L}(X,Y), \ \forall\, L \in \mathcal{L}(X,Y)$.

Assume now $\dim X = \infty$ and let $L \in S(X,Y)$. Take $T \in \mathcal{L}(X,Y)$. Then

$$\|(L-T)x\| \ \leq \ 0 + \|Tx\| + \|Lx\|, \ \forall\, x \in X,$$

It follows that $T \in F(L)$. So, we also have $F(L) = \mathcal{L}(X,Y), \forall\, L \in S(X,Y)$.

Let $I \in \mathcal{L}(X)$ be the identity operator on X. By iii) in Theorem 2 and lemma 2, it follows that $F(I) \cap S(X) = \emptyset$. Moreover, $-I \notin F(I)$.

2) Let $L \in \mathcal{L}(X,Y)$ be a Fredholm operator. Applying iii) in Theorem 2 it follows that $F(L)$ consists solely of Fredholm operators

The following result is Corollary 3 in [1]. An operator $T \in \mathcal{L}(X,Y)$ will be said to be an *isomorphism* if it is 1-1 and has closed range.

Corollary 1. *Let $L \in \mathcal{L}(X,Y)$ be an isomorphism.*

i) *If $aI - L$ is an isomorphism for all $a > 0$, then L is onto.*
ii) *If $aI - L$ is an isomorphism for all $a < 0$, then L is onto.*

Proof. i) Note $L \in \Phi_+(X,Y)$. Hence, by iii) and iv) in Theorem 2, there is some $r > 0$ such that if $T \in \mathcal{L}(X,Y)$ and $\|L - T\| < r$, then $T \in \Phi_+(X,Y)$ and $i(T) = i(L)$. Taking $a_0 > 0$ sufficiently small, it follows that $i(L) = i(L - a_0 I) = i(a_0 I - L)$. Similarly, taking $a > 0$ large enough, we have $i(aI - L) = i(I - \frac{L}{a}) = i(I) = 0$. On other hand, $A = \{aI - L : a > 0\} \subset \Phi_+(X,Y)$ is connected. By the continuity of the index, this implies the index is constant on A. Hence, $i(L) = 0$. Since $\alpha(L) = 0$, we conclude $\beta(L) = 0$.

The proof of ii) is analogous. □

The next result shows that strictly singular and upper semi-Fredholm operators have some kind of "complementary" properties.

Proposition 3. *Let $R \in \mathcal{L}(X,Y), T \in \mathcal{L}(X,Z)$ satisfy $\|Rx\| \leq \|Tx\|$, $\forall\, x \in X$.*

i) *If T is strictly singular, then R is also strictly singular.*
ii) *If R is upper semi-Fredholm, then T is also upper semi-Fredholm.*

Proof. i) Let $V \subset X$ be a subspace such that $\|Rx\| \geq C\|x\|$, $\forall\, x \in V$. Since $\|Tx\| \geq \|Rx\|$ and T is strictly singular, this implies that $\dim V < \infty$.

ii) The hypothesis implies that $T \in F(L)$. The conclusion follows now from i) in Theorem 2. □

Assume $L \in \Phi_+(X,Y), L \neq 0$. Then we can find a compact operator $K \in \mathcal{L}(X,Y)$ such that $\alpha(L+K) \neq \alpha(L)$. By i) in Theorem 2, we have $L + K \in \Phi_+(X,Y)$. This implies that $\alpha(\cdot)$ is not constant on $F_K(L)$. The next result, which extends Theorem 8 in [1], indicates this is not the case for $F_0(L)$, when $\alpha(L) = 0$.

Corollary 2. *Let $L \in \mathcal{L}(X,Y)$ be an isomorphism. If $T \in F_0(L)$, then T is also an isomorphism and $\beta(T) = \beta(L)$.*

Proof. From (8) (with $K = 0$) follows $N(T) \subset N(L)$ and so $\alpha(T) = 0$. Theorem 2 indicates that $T \in \Phi_+(X,Y)$ and $i(L) = i(T)$. This implies $\beta(L) = \beta(T)$. □

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